CEDYA+CMA
XXV CONGRESO DE ECUACIONES DIFERENCIALES Y APLICACIONES
XV CONGRESO DE MATEMÁTICA APLICADA
DEL 26 AL 30 DE JUNIO DE 2017

A new parameter plane for Chebyshev's method*

J. M. Gutiérrez† and J. L. Varona*

Abstract— The aim of this work is to provide analytic and graphic arguments to explain the dynamic behavior of Chebyshev's method applied to cubic polynomials. In fact, we plot the parameter plane related to this method and we compare it with other previously know, as the parameter planes of Newton's or Halley's methods. We are interested in finding "bad polynomials" for which the iterative method present convergence to points that are not roots of the involved polynomial. Actually, we show the existence of polynomials for which Chebyshev's method has superattracting n-cycles or the existence of polynomials for which Chebyshev's method has extraneous fixed points. The first fact is shared with other root-finding methods, such as Newton's or Halley's, but not the second one.

Keywords: Chebyshev's method, parameter plane, dynamic study.

1 Introduction

Chebyshev's method is a well-known iterative method for numerically solving nonlinear equations $f(z) = 0$. It is defined recursively by $z_{n+1} = C_f(z_n)$, for a given $z_0 \in \mathbb{C}$, where

$$C_f(z) = z - \left(1 + \frac{1}{2}\frac{f(z)f''(z)}{f'(z)^2}\right)\frac{f(z)}{f'(z)}. \tag{1}$$

This method and its convergence properties for solving nonlinear equations (not only in the complex plane, also in the real line or even in Banach spaces) have been widely studied by different authors (see [5] for instance). Throughout this paper we introduce the operator L_f that maps a function f into the quotient

$$L_f : f \mapsto \frac{ff''}{(f')^2}. \tag{2}$$

The aim of this work is to provide analytic and graphic arguments to explain the dynamic behavior of Chebyshev's method applied to cubic polynomials. In fact, taking into account some ideas given by Varona [9], we can plot the known as parameter plane (see [8] for more details).

To be more precise, we follow the orbits of the two free critical points that appear when Chebyshev's method is applied to polynomials in the form

$$p_\lambda(z) = (z^2 - 1)(z - \lambda), \quad \lambda \in \mathbb{C}. \tag{3}$$

We color the λ-plane depending on the convergence of these two free critical points to any of the three roots of the polynomial p_λ, leading to $2^3 = 8$ possible color schemes.

We are interested in finding "bad polynomials" for which the iterative method present convergence to points that are not roots of the involved polynomial. Actually, we show the existence of parameters λ that give rise to superattracting n-cycles, for each $n \in \mathbb{N}$, $n \geq 2$, or to extraneous fixed points. The first fact is shared with other root-finding methods, such as Newton's or Halley's, but not the second one, as it was pointed out by García-Olivo et al [6].

* **PUBLISHED IN THE "Libro de comunicaciones definitivas presentadas en CEDYA + CMA 2017" (ISBN: 978-84-944402-1-2) AS:** J. M. Gutiérrez and J. L. Varona, A new parameter plane for Chebyshev's method, *Proceedings XXV CEDYA + XV CMA*, pp. 350–354, Cartagena, 2017.

† Departamento de Matemáticas y Computación, Universidad de La Rioja, Complejo Científico-Tecnológico, C/ Madre de Dios 53, 26006-Logroño (SPAIN). Email: jmguti@unirioja.es, jvarona@unirioja.es

The iteration of the Chebyshev's map (1) when f is a polynomial in the form (3) gives rise to a rational map defined in the extended complex plane. Therefore, the theory and concepts related with the iteration of rational maps [2] can be applied in this situation. We assume that concepts such as Fatou and Julia sets, attracting, repelling or indifferent fixed points, cycles, critical points, etcetera are understood for the reader and we do not proceed to explain them. In particular, the following classical theorem plays a key role in our analysis.

THEOREM 1 ***(Fatou-Julia).*** *Every attracting cycle of a rational map attracts at least one critical point.*

Before undertaking our study, we mention some important features related with Chebyshev's method. All of them are straightforward calculations.

1. **(Scaling theorem).** Let $A(z) = \alpha z + \beta$ with $\alpha \neq 0$ be an affine map and let $q(z) = p(A(z))$. Then Chebyshev's iteration map C_p is topologically conjugate to C_q, namely $A \circ C_q \circ A^{-1} = C_p$.

2. Let $\nu(z) = \bar{z}$ the usual complex conjugation. Suppose that $p(z) = \prod(z - r_i)$ and let us define $q(z) = \prod(z - \bar{r}_i)$. Then C_p is topologically conjugate to C_q, namely $\nu \circ C_p = C_q \circ \nu$.

3. Simple roots of p are superattracing fixed points of C_p. Even more, if z^* is a simple root of p, then $C_p'(z^*) = C_p''(z^*) = 0$, so the method is cubically convergent.

4. Chebyshev's method has linear convergence for roots z^* with multiplicity $m > 1$. In this case
$$C_p'(z^*) = \frac{(m-1)(2m-1)}{2m^2} \in (0, 1).$$

5. Chebyshev's method has extraneous fixed points, i.e., fixed points of C_p that are not roots of p. These points are solutions of
$$L_p(z) = -2.$$
In addition, they are attracting if
$$\left| 1 - \frac{L_p(z)^2 L_{p'}(z)}{2} \right| < 1$$
and superattracting if $L_{p'}(z) = 3$. Note that, according (2), $L_{p'}(z) = p'(z)p'''(z)/p''(z)^2$.

6. Taking into account
$$C_p'(z) = \frac{(3 - L_{p'}(z))L_p(z)^2}{2},$$
the critical points of C_p are the roots of p or the solutions of $L_{p'}(z) = 3$. The last ones are called free critical points.

2 Chebyshev's method applied to cubics

The dynamical study of Chebyshev's method applied to quadratic polynomials have been carried out in other previous works, as [3], [4] or [7]. The cubic case has been also considered in [6] for instance. Now we present a new perspective, following the steps given by Roberts and Horgan-Kobelski for Newton's or Halley's methods [8]. So, we are interested in study the behavior of Chebyshev's method applied to cubic polynomials with at least two different roots. As a first step, we highlight that the use of the Scaling theorem reduces (see [1]) the problem to the study of Chebyshev's method applied to the one-parameter family of polynomials (3). That is, for any cubic polynomial $q(z)$ with at least two distinct roots, there exists a parameter $\lambda \in \mathbb{C}$ such that Chebyshev's map for q, C_q, is topologically conjugate to C_{p_λ}, with p_λ defined in (3).

Chebyshev's map for this kind of polynomials is a seventh degree rational function
$$C_{p_\lambda}(z) = \frac{H_\lambda(z)}{(3z^2 - 2\lambda z - 1)^3},$$
where $H_\lambda(z) = 15z^7 - 26\lambda z^6 + 15\lambda^2 z^5 - 6z^5 - 3\lambda^3 z^4 - 9\lambda z^4 + 18\lambda^2 z^3 - z^3 - 6\lambda^3 z^2 + 12\lambda z^2 - 9\lambda^2 z + \lambda^3 - \lambda$, with two free critical points:
$$\rho_\pm = \frac{5\lambda \pm \sqrt{-5(\lambda^2 + 3)}}{15}.$$

The strategy is to color the complex plane (that in this context is known as *parameter plane*) depending on the behavior of the orbits of the two critical points $\rho_\pm$ under the iteration of C_{p_λ}. As there are two free critical points and three roots, we can find $3^2 = 9$ possible convergence schemes, as shown in Table 1. For instance, λ is colored in blue when the orbit of the critical point ρ_- converges to the root -1 and the orbit of the critical point ρ_+ converges to the root 1. Figures 1 and 2 show a part of the parameter plane of Chebyshev's method applied to cubic polynomials. Actually, Figure 2 reveals a symmetry about the real axis, a question that could be deduced from the properties of the complex conjugation.

Table 1: Coloring scheme for Chebyshev's method applied to polynomials $(z^2 - 1)(z - \lambda)$, $\lambda \in \mathbb{C}$, according to the convergence of the two free critical points ρ_- and ρ_+.

Color	$(\rho_-, \rho_+) \to$	Color	$(\rho_-, \rho_+) \to$
Yellow	$(-1, -1)$	Blue	$(-1, 1)$
Green	$(1, -1)$	Red	(λ, λ)
Brown	$(-1, \lambda)$	Pink	$(\lambda, -1)$
Orange	$(1, 1)$	Cyan	$(\lambda, 1)$
Purple	$(1, \lambda)$		

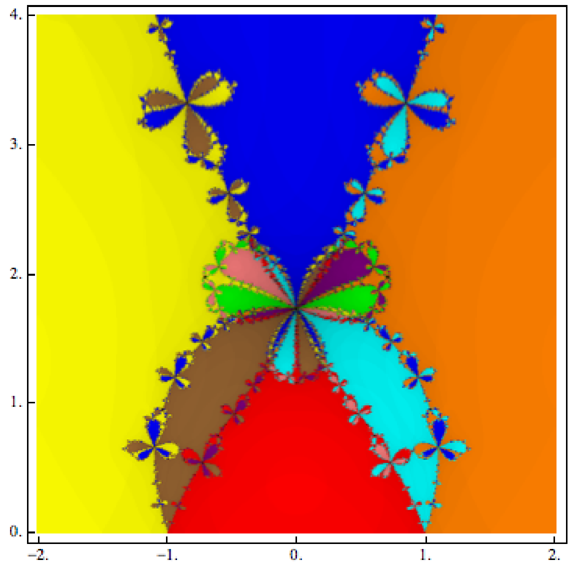

Figure 1: A square $[-2, 2] \times [0, 4]$ of the parameter plane for Chebyshev's method colored according the color scheme indicated in Table 1.

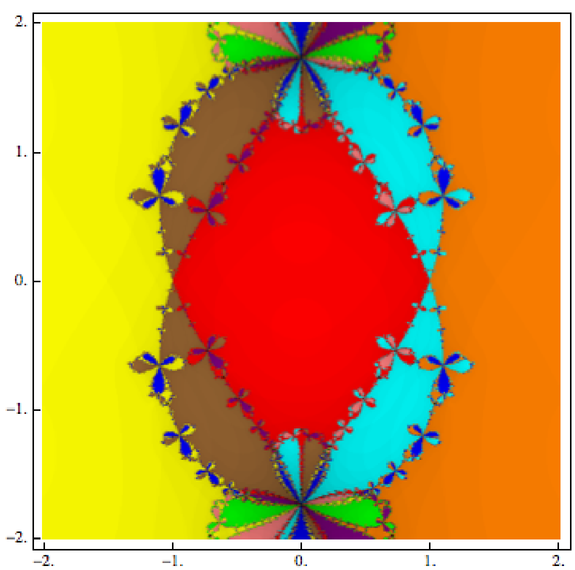

Figure 2: A square $[-2, 2] \times [-2, 2]$ of the parameter plane for Chebyshev's showing the symmetry about the real axis.

As in the case of other iterative methods, such as Newton's or Halley's methods, we can find black zones in the parameter plane corresponding to points of no convergence to any of the roots. In the case of Newton's or Halley's methods, these black zones are due to the existence of attracting n-cycles. But in the case of Chebyshev's method this black zones are originated by:

- The presence of attracting extraneous fixed points, that do not exist in Newton's or Halley's method.
- The presence of attracting n-cycles.

It is complicated to find these black holes in the parameter plane just by visual inspection. However after a few calculations, we can find some of them. These black holes (see Figures 3–6) correspond to values of the parameter λ for which Chebyshev's method does not converge to any of the roots of the polynomial p_λ. That is, the method fails from the root-finding point of view.

Superattracting extraneous fixed points for Chebyshev's method are solutions of the system of nonlinear equations

$$\begin{cases} L_p(z) = 2 \\ L_{p'}(z) = 3 \end{cases}$$

that satisfy $p''(z) \neq 0$. For polynomials defined in (3) this system is

$$\begin{cases} 12z^4 - 16z^3\lambda + 5z^2\lambda^2 - 9z^2 + 8z\lambda - \lambda^2 + 1 = 0 \\ 15z^2 - 10z\lambda + 2\lambda^2 + 1 = 0 \end{cases}$$

with $\lambda \neq 3z$. We obtain six different solutions:

$$\lambda = \pm\frac{2\sqrt{3}}{3}i \approx \pm 1.154701i,$$

$$\lambda = \frac{\pm 5 \pm 8\sqrt{3}i}{7} \approx \pm 0.714286 \pm 1.979487i.$$

Figures 3 and 4 show the existence of Mandelbrot-like sets near two of these values of λ.

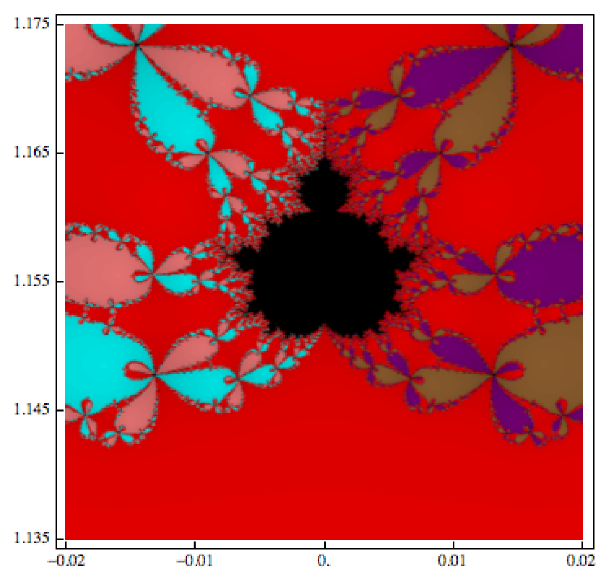

Figure 3: A rectangle $[-0.02, 0.02] \times [1.135, 1.175]$ of the parameter plane containing a black hole originated by the superattracting extraneous fixed point $\frac{2\sqrt{3}}{3}i$.

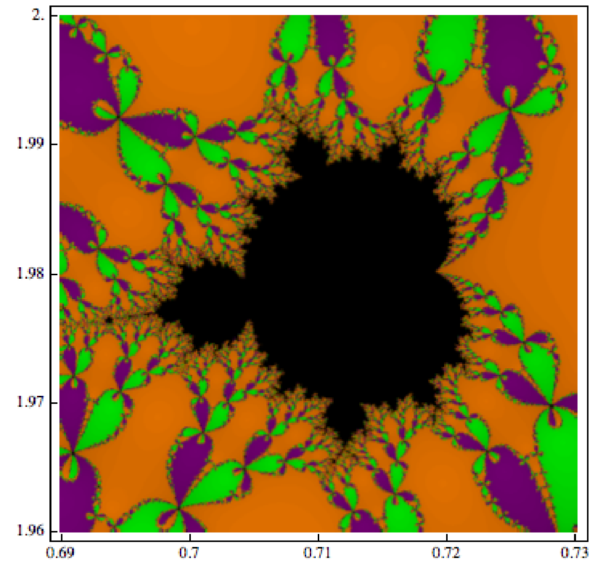

Figure 4: A rectangle $[0.69, 0.73] \times [1.96, 2]$ of the parameter plane containing a black hole originated by the superattracting extraneous fixed point $\frac{5+8\sqrt{3}i}{7}$.

Now, let us analyze the existence of superattracting n-cycles for Chebyshev's method. To do this, we restrict our study to the imaginary axis, i.e. to Chebyshev's method applied to

$$p_\lambda(z) = (z^2 - 1)(z - \lambda), \quad \lambda = \beta i, \quad \beta \in \mathbb{R}^+.$$

Note that Chebyshev's method leaves the imaginary axis invariant, so it is enough to study the imaginary part of $C_{p_{\beta i}}(iy)$, $y \in \mathbb{R}$:

$$R_\beta(y) = \frac{q(\beta, y)}{(3y^2 - 2\beta y + 1)^3}, \tag{4}$$

$$\begin{aligned} q(\beta, y) = 15y^7 - 26y^6\beta + 15y^5\beta^2 + 6y^5 - 3y^4\beta^3 + 9y^4\beta \\ - 18y^3\beta^2 - y^3 + 6y^2\beta^3 + 12y^2\beta - 9y\beta^2 + \beta^3 + \beta. \end{aligned}$$

The map R_β defined in (4) has the following properties:

- $R_\beta(\beta) = \beta$ for all $\beta \geq 0$.
- R_β has free critical points iff R_β has no asymptotes.
- R_β has two poles: $\frac{1}{3}\left(\beta \pm \sqrt{\beta^2 - 3}\right)$.
- R_β has two free critical points: $\frac{1}{15}\left(5\beta \pm \sqrt{5(3 - \beta^2)}\right)$.

The orbits of the biggest free critical point

$$\rho_+ = \frac{1}{15}\left(5\beta + \sqrt{5(3 - \beta^2)}\right)$$

can give rise to superattracting n-cycles. Actually, for each $n \geq 2$, we define the function

$$g_n(\beta) = R_\beta^n(\rho_+) - \rho_+. \tag{5}$$

A root of $g_n(\beta) = 0$ originates a superattracting n-cycle of R_β and, consequently, a superattracting n-cycle for Chebychev's method.

Table 2 contains some solutions, β_n, of the equation $g_n(\beta) = 0$ for different values of n. An increasing sequence of values for β_n is obtained. This sequence converges to $\sqrt{3} \approx 1.73205$.

If we define $\lambda_n = \beta_n i$, then Chebyshev's method applied to $(z^2 - 1)(z - \lambda_n)$ has a superattracting n-cycle. Consequently, the values in Table 2 gives raise to a "channel" of superattracting n-cycles in the parameter plane. Figure 5 shows this channel, and Figure 6 shows a magnification around $\beta_2 i$.

Table 2: Some solutions of the equation $g_n(\beta) = 0$.

n	2	3	4	5
β_n	1.2865	1.3401	1.3894	1.4377

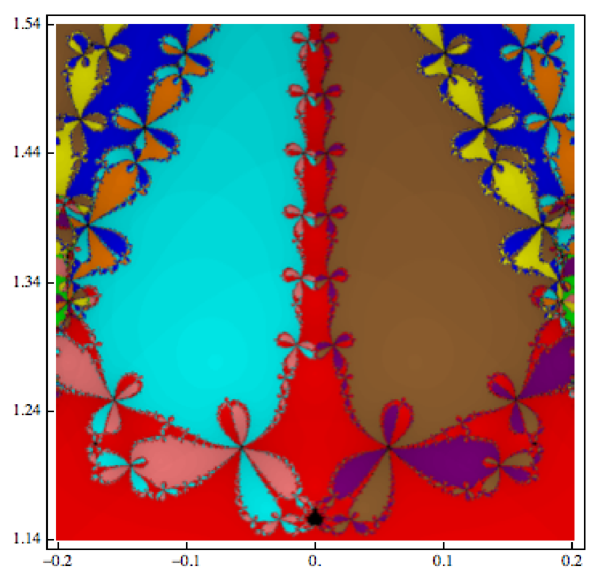

Figure 5: A channel of black holes created by the presence of superattracting n-cycles, in the rectangle $[-0.2, 0.2] \times [1.14, 1.54]$ of the parameter plane.

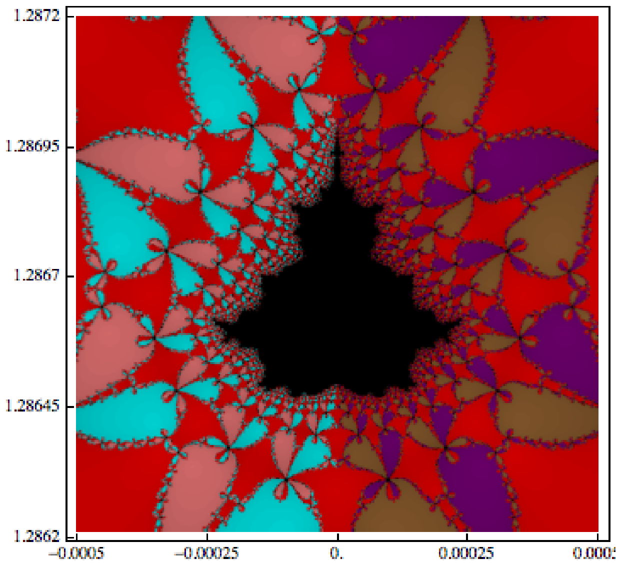

Figure 6: A detail in the rectangle $[-0.0005, 0.0005] \times [1.2862, 1.2872]$ showing a black hole originated by a superattracting 2-cycle.

Finally, Figures 7 and 8 show the graphs or R_β for $\beta = 1.2865$ and $\beta = 1.3401$ respectively. These graphics show a typical graph of function R_β defined in (4), for $\beta \in (0, \sqrt{3})$, with no asymptotes, a maximum at ρ_- and a a minimum at ρ_+. When $\beta \to \sqrt{3}$, the two free critical points collapse in the value $\sqrt{3}/3$ that becomes an asymptote of the corresponding function R_β.

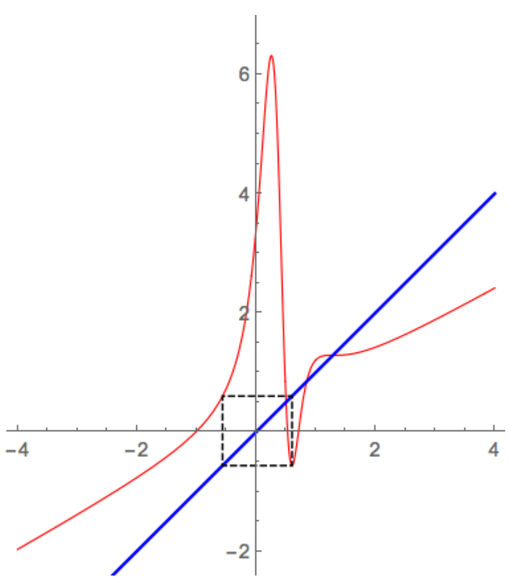

Figure 7: Graph or R_β for $\beta = \beta_2 = 1.2865$ together with the web diagram proving the existence of a superattracting 2-cycle containing the free critical point ρ_+.

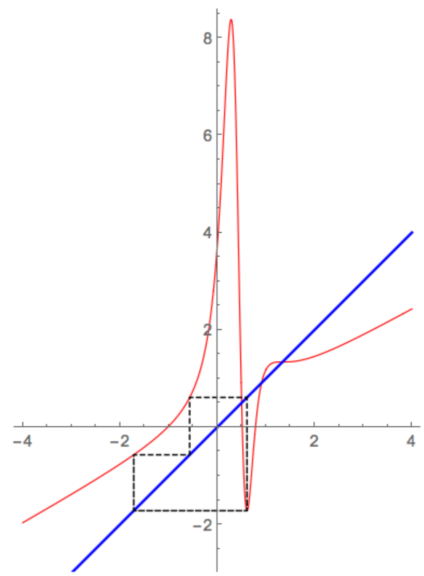

Figure 8: Graph or R_β for $\beta = \beta_3 = 1.3401$ together with the web diagram proving the existence of a superattracting 3-cycle containing the free critical point ρ_+.

Acknowledgements

This research was partially supported by Ministerio de Economía y Competitividad under grants MTM2014-52016-C2-1-P and MTM2015-65888-C4-4-P.

References

[1] S. Amat, S. Busquier and S. Plaza, *Review of some iterative root-finding methods from a dynamical point of view*. SCIENTIA, Series A: Math. Sciences **10** (2004), 3–35.

[2] A. F. Beardon, *Iteration of rational functions*. Springer-Verlag, NY, 1991.

[3] A. Cordero, J. R. Torregrosa and P. Vindel, *Dynamics of a family of Chebyshev-Halley type methods*. Appl. Math. Comput. **216** (2013), 8568–8583.

[4] A. Cordero, J. R. Torregrosa and P. Vindel, *Bulbs of period two in the family of Chebyshev-Halley iterative methods on quadratic polynomials*. Abstr. Appl. Anal. (2013), Article ID 536910, 1–10.

[5] M. A. Diloné, M. García-Olivo and J. M. Gutiérrez, *A note on the semilocal convergence of Chebyshev's method*. Bull. Austral. Math. Soc. **88** (2013), 98–105.

[6] M. García-Olivo, J. M. Gutiérrez and Á. A. Magreñán, *A complex dynamical approach of Chebyshev's method*. SeMA Journal **71** (2015), no. 1, 57–68.

[7] K. Kneisl, *Julia sets for the super-Newton method, Cauchy's method and Halley's method*. Chaos **11** (2001), 359–370.

[8] G. E. Roberts and J. Horgan-Kobelski, *Newton's versus Halley's methods: a dynamical systems approach*. Int. J. Bifur. Chaos Appl. Sci. Eng. **24** (2004), 3459–3475.

[9] J. L. Varona, *Graphic and numerical comparison between iterative methods*. Math. Intelligencer **24** (2002), no. 1, 37–46.