Incidence Matrices of Simplicial Complex in SSreflect[1]

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From “General” Topology to Homological Algebra

Topological Space

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\text{Topological Space} \arrow[d, "\text{triangulation}"] \\
\text{Simplicial Complex} \arrow[d, "\text{computing}"] \\
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\end{tikzcd}$$

From “General” Topology to Homological Algebra

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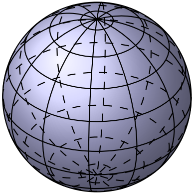
\text{Topological Space} \arrow[d, "\text{triangulation}"] \\
\text{Simplicial Complex} \arrow[d, "\text{computing}"] \\
\text{Homology} \arrow[uu, bend left, "\text{properties}"]
\end{tikzcd}
$$

An example

Topological Space



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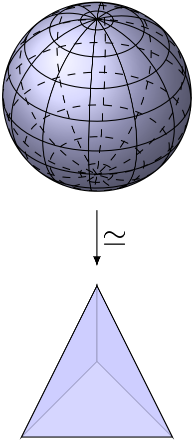
Topological Space



Simplicial Complex:

- 0-simplices: vertices (4 vertices)
- 1-simplices: edges (6 edges)
- 2-simplices: triangles (4 triangles)

An example

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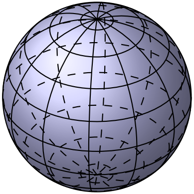
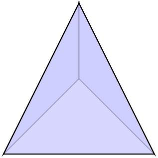


$\downarrow \simeq$



Homology groups

$$
\begin{aligned}
H_0 &= \mathbb{Z}\\
H_1 &= 0\\
H_2 &= \mathbb{Z}\\
H_3 &= 0\\
&\dots
\end{aligned}
$$

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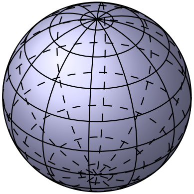
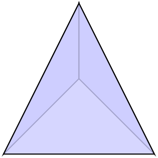


$\simeq$



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Simplicial Complexes

Definition:

Let V be a set, called the vertex set, a *simplex* over V is any finite subset of V.

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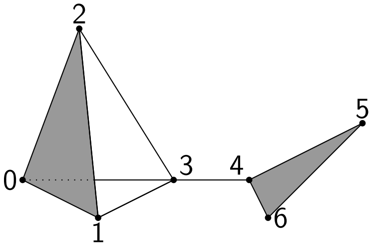
Definition:

An *(abstract) simplicial complex* over V is a set of simplices C over V satisfying the property:

$$\forall \alpha \in C,\ \textit{if}\ \beta \subseteq \alpha \Rightarrow \beta \in C$$

Simplicial Complexes

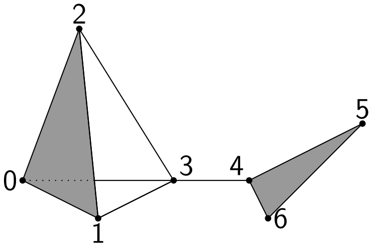


$C = \{\emptyset, \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\},$
$\{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{4,5\}, \{4,6\}, \{5,6\},$
$\{0,1,2\}, \{4,5,6\}\}$

Simplicial Complexes

Definition:

The *facets* of a simplicial complex C are the maximal simplices of the simplicial complex.



The facets are: $\{\{1,3\},\{3,4\},\{0,3\},\{2,3\},\{0,1,2\},\{4,5,6\}\}$

Incidence Matrices

Definition

Let X and Y be two enumerated finite sets and r be a relationship between the elements of X and the elements of Y, we call *incidence matrix*

$$M = \begin{array}{c} \\ X[1] \\ \vdots \\ X[m] \end{array} \begin{array}{c} \begin{array}{ccc} Y[1] & \cdots & Y[n] \end{array} \\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \end{array}$$

where

$$a_{i,j} = \begin{cases} 1 & \text{si } X[i] \text{ is related to } Y[j] \\ 0 & \text{si } X[i] \text{ is not related to } Y[j] \end{cases}$$

Incidence Matrices of Simplicial Complexes

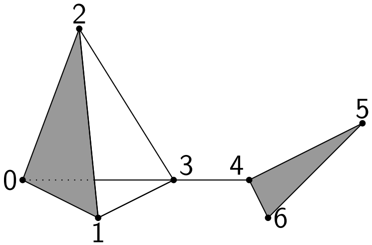
Definition

Let C be a simplicial complex, A the set of n-simplices of C and B the set of $(n-1)$-simplices of C.
We call *incidence matrix* of dimension n ($n \geq 1$), M_n of the simplicial complex C, to a matrix $p \times q$ where

$$p = \sharp|B| \ \wedge \ q = \sharp|A|$$

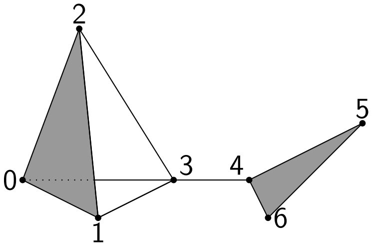
$$M_{i,j} = \begin{cases} 1 & \text{si } B_i \subset A_j \\ 0 & \text{si } B_i \not\subset A_j \end{cases}$$

Incidence Matrices of Simplicial Complexes



$$
\begin{array}{c c}
 & \begin{array}{cccccccccc} \{0,1\} & \{0,2\} & \{0,3\} & \{1,2\} & \{1,3\} & \{2,3\} & \{3,4\} & \{4,5\} & \{4,6\} & \{5,6\} \end{array} \\
\begin{array}{c} \{0\} \\ \{1\} \\ \{2\} \\ \{3\} \\ \{4\} \\ \{5\} \\ \{6\} \end{array} &
\left(\begin{array}{cccccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right)
\end{array}
$$

Incidence Matrices of Simplicial Complexes



$$
\begin{array}{c}
\\
\{0,1\} \\ \{0,2\} \\ \{0,3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{3,4\} \\ \{4,5\} \\ \{4,6\} \\ \{5,6\}
\end{array}
\begin{array}{c}
\begin{array}{cc} \{0,1,2\} & \{4,5,6\} \end{array} \\
\begin{pmatrix}
1 & 0 \\
1 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 1 \\
0 & 1
\end{pmatrix}
\end{array}
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Incidence Matrices of Simplicial Complexes

Importance of the I.M. of a S.C.

The incidence matrices of simplicial complexes are used to compute the homology of the simplicial complex

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Problem

Theorem: Product of two consecutive incidence matrices in $\mathbb{Z}_2$

Let C be a simplicial complex and n a number natural such that $n \geq 2$, then the product of the incidence matrix of dimension $n-1$, denoted by M_{n-1}, and the incidence matrix of dimension n, denoted by M_n, is equal to the null matrix.

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Sketch of the proof.

- Let C_n be the set of n-simplices of C
- Let C_{n-1} be the set of $(n-1)$-simplices of C
- Let C_{n-2} be the set of $(n-2)$-simplices of C

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$$M_{n-1} = \begin{array}{c c} & \begin{array}{ccc} C_{n-1}[1] & \cdots & C_{n-1}[r1] \end{array} \\ \begin{array}{c} C_{n-2}[1] \\ \vdots \\ C_{n-2}[r2] \end{array} & \begin{pmatrix} a_{1,1} & \cdots & a_{1,r1} \\ \vdots & \ddots & \vdots \\ a_{r2,1} & \cdots & a_{r2,r1} \end{pmatrix} \end{array} \quad M_n = \begin{array}{c c} & \begin{array}{ccc} C_n[1] & \cdots & C_n[r3] \end{array} \\ \begin{array}{c} C_{n-1}[1] \\ \vdots \\ C_{n-1}[r1] \end{array} & \begin{pmatrix} b_{1,1} & \cdots & b_{1,r1} \\ \vdots & \ddots & \vdots \\ b_{r1,1} & \cdots & b_{r1,r3} \end{pmatrix} \end{array}$$

Problem

$$M_{n-1} \times M_n = \begin{pmatrix} c_{1,1} & \cdots & c_{1,r3} \\ \vdots & \ddots & \vdots \\ c_{r2,1} & \cdots & c_{r2,r3} \end{pmatrix}$$

where

$$c_{i,\ j} = \sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j}$$

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$$c_{i,\ j} = \sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j}$$

we need to prove that

$$\forall i, j, c_{i,\ j} = 0$$

in order to prove that $M_{n-1} \times M_n = 0$

Problem

Lemma

Under the previous conditions, $\forall i, j, c_{i,\ j} = 0$

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Proof.

$$\sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j} = \begin{array}{l} \displaystyle\sum_{j0 | M_{n-2}[i] \subset M_{n-1}[j0] \wedge M_{n-1}[j0] \subset M_n[j]} a_{i,\ j0} \times b_{j0,\ j} + \\ \displaystyle\sum_{j0 | M_{n-2}[i] \not\subset M_{n-1}[j0] \wedge M_{n-1}[j0] \subset M_n[j]} a_{i,\ j0} \times b_{j0,\ j} + \\ \displaystyle\sum_{j0 | M_{n-2}[i] \subset M_{n-1}[j0] \wedge M_{n-1}[j0] \not\subset M_n[j]} a_{i,\ j0} \times b_{j0,\ j} + \\ \displaystyle\sum_{j0 | M_{n-2}[i] \not\subset M_{n-1}[j0] \wedge M_{n-1}[j0] \not\subset M_n[j]} a_{i,\ j0} \times b_{j0,\ j} \end{array}$$

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Proof.

$$\begin{aligned}
\sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j} &= \Big(\sum_{j0 \mid M_{n-2}[i] \subset M_{n-1}[j0] \wedge M_{n-1}[j0] \subset M_n[j]} 1\Big) + 0 + 0 + 0 \\
&= \sharp|\{j0 \mid M_{n-2}[i] \subset M_{n-1}[j0] \wedge M_{n-1}[j0] \subset M_n[j]\}|
\end{aligned}$$

Problem

Lemma

Under the previous conditions, let $T \in C_n$ and $x \in C_{n-2}$ if $x \subset T$ then,

$$\sharp|\{y \in C_{n-1}|(x \subset y) \wedge (y \subset T)\}| = 2$$

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Sketch of the proof.

- $T \in C_n \Rightarrow T = \{a_0, \dots, a_n\}$
- $x \in C_{n-2} \wedge x \subset T \Rightarrow x = \{a_0, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n\}$
- $y \in C_{n-1} \wedge y \subset T \Rightarrow y = \{a_0, \dots, \widehat{a_r}, \dots, a_n\}$
- $y \in C_{n-1} \wedge y \subset T \wedge x \subset y \Rightarrow y = \{a_0, \dots, \widehat{a_r}, \dots, a_n\}$ with r={i,j}

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Then

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□