Some notes about simplicial complexes and homology

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Simplicial Complexes

Definition

Let V be an ordered set, called the vertex set.
A simplex over V is any finite subset of V.

Definition

Let α and β be simplices over V, we say α is a face of β if α is a subset of β.

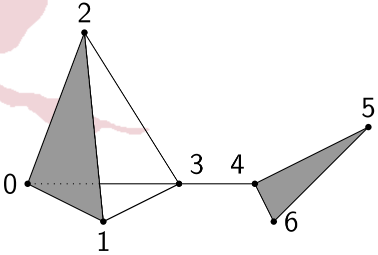
Definition

An ordered (abstract) simplicial complex over V is a set of simplices $\mathcal{K}$ over V satisfying the property:

$$\forall \alpha \in \mathcal{K}, \text{ if } \beta \subseteq \alpha \Rightarrow \beta \in \mathcal{K}$$

Let $\mathcal{K}$ be a simplicial complex. Then the set $S_n(\mathcal{K})$ of n-simplices of $\mathcal{K}$ is the set made of the simplices of cardinality $n+1$.

Simplicial Complexes



$$V = (0,1,2,3,4,5,6)$$
$$\mathcal{K} = \{\emptyset, (0), (1), (2), (3), (4), (5), (6),$$

$$(0,1), (0,2), (0,3), (1,2), (1,3), (2,3), (3,4), (4,5), (4,6), (5,6),$$
$$(0,1,2), (4,5,6)\}$$

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Incidence Matrices

Definition

Let X and Y be two ordered finite sets of simplices, we call incidence matrix to a matrix $m \times n$ where

$$m = \sharp|X| \;\wedge\; n = \sharp|Y|$$

$$M = \begin{array}{c} \\ X[1] \\ \vdots \\ X[m] \end{array} \begin{array}{c} \begin{array}{ccc} Y[1] & \cdots & Y[n] \end{array} \\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \end{array}$$

where

$$a_{i,j} = \begin{cases} 1 & \textit{if } X[i] \textit{ is a face of } Y[j] \\ 0 & \textit{if } X[i] \textit{ is not a face of } Y[j] \end{cases}$$

Incidence Matrices

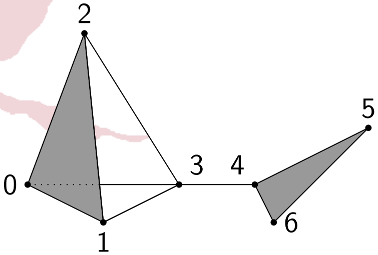
Definition

Let C be a finite set of simplices, A be the set of n-simplices of C with an order between its elements and B the set of $(n-1)$-simplices of C with an order between its elements.
We call incidence matrix of dimension n ($n \geq 1$), to a matrix $p \times q$ where

$$p = \sharp|B| \ \wedge \ q = \sharp|A|$$

$$M_{i,j} = \begin{cases} 1 & \textit{if } B_i \textit{ is a face of } A_j \\ 0 & \textit{if } B_i \textit{ is not a face of } A_j \end{cases}$$

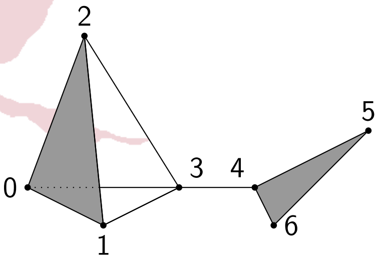
Incidence Matrices of Simplicial Complexes



$M_1 =$

	{0, 1}	{0, 2}	{0, 3}	{1, 2}	{1, 3}	{2, 3}	{3, 4}	{4, 5}	{4, 6}	{5, 6}
{0}	1	1	1	0	0	0	0	0	0	0
{1}	1	0	0	1	1	0	0	0	0	0
{2}	0	1	0	1	0	1	0	0	0	0
{3}	0	0	1	0	1	1	1	0	0	0
{4}	0	0	0	0	0	0	1	1	1	0
{5}	0	0	0	0	0	0	0	1	0	1
{6}	0	0	0	0	0	0	0	0	1	1

Incidence Matrices of Simplicial Complexes



$$M_2 = \begin{array}{c} \\ \{0,1\} \\ \{0,2\} \\ \{0,3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{3,4\} \\ \{4,5\} \\ \{4,6\} \\ \{5,6\} \end{array} \begin{array}{c} \begin{array}{cc} \{0,1,2\} & \{4,5,6\} \end{array} \\ \left(\begin{array}{cc} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{array} \right) \end{array}$$

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Computing homology groups from Smith Normal Form

Let X be a simplicial complex and M_n, M_{n+1} be the incidence matrices of X of dimension n and $n+1$.

If we compute the Smith Normal Form of both matrices we obtain two matrices of the form:

$$SNF(M_n) = \begin{pmatrix} a_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_k & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \qquad SNF(M_{n+1}) \begin{pmatrix} b_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_m & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

Then $H_n(X) = f - k - m$ where f is the number of simplexes of X of dimension n

Butterfly Example

Let us compute H_0 of the butterfly simplicial complex
So, we need M_0 and M_1:

- in this case M_0 is the void matrix; so $k = 0$;
- we compute the Smith Normal Form of M_1:

$$SNF(M_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

  so, $m = 6$;
- in addition, there are 7 0-simplexes

Therefore, $H_0(X) = 7 - 6 - 0 = 1 \rightarrow \mathbb{Z}$

This result must be interpreted as stating that the butterfly simplicial complex only has one connected component

Butterfly Example continued

Let us compute H_1 of the butterfly simplicial complex

So, we need M_1 and M_2:

- we have computed in the previous slide the Smith Normal Form of M_1: $k = 6$;
- we compute the Smith Normal Form of M_2:

$$SNF(M_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};$$

  so, $m = 2$;
- in addition, there are 10 1-simplexes

Therefore, $H_1(X) = 10 - 6 - 2 = 2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}$

This result must be interpreted as stating that the butterfly simplicial complex has two "holes" in the topological sense

You can think that there is three holes in the butterfly example, but one of them is the composition of the others

A more detailed explanation about this fact is given in Page 6 of `http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/Genova-Lecture-Notes.pdf`

Butterfly Example continued

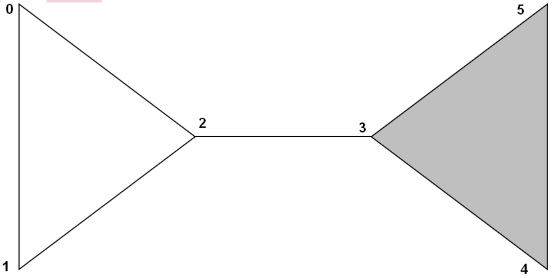
Let us compute H_2 of the butterfly simplicial complex
So, we need M_2 and M_3:

- we have computed in the previous slide the Smith Normal Form of M_2: $k = 2$;
- M_3 is a void matrix; so, $m = 0$,
- in addition, there are 2 2-simplexes

Therefore, $H_2(X) = 2 - 2 - 0 = 0$
This result must be interpreted as stating that the butterfly simplicial complex has not "voids" in the topological sense

The rest of matrices are void, then the homology groups $H_n(X)$ with $n \geq 3$ are null

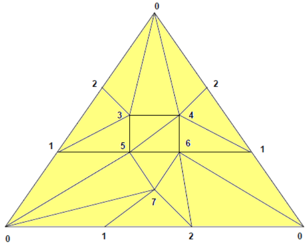
Other Examples



$\mathcal{K}_1 =$
$\{\emptyset, (0), (1), (2), (3), (4), (5), (0,1), (0,2), (1,2), (2,3), (3,4), (3,5), (4,5), (3,4,5)\}$

$H_0(K_1) = \mathbb{Z}, H_1(K_1) = \mathbb{Z}$

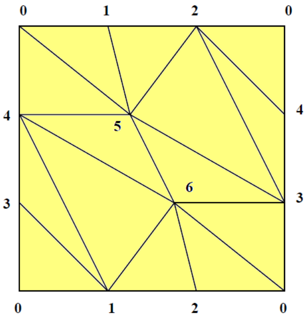
Other Examples



$V = (0, 1, 2, 3, 4, 5, 6, 7)$
$\mathcal{K}_2 = \{\emptyset, (0), (1), (2), (3), (4), (5), (6), (7),$
$(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7)$
$(2, 3), (2, 4), (2, 6), (2, 7), (3, 4), (3, 5), (4, 5), (4, 6), (5, 6), (5, 7), (6, 7)$
$(0, 1, 5), (0, 1, 6), (0, 1, 7), (0, 2, 3), (0, 2, 4), (0, 2, 6), (0, 3, 4), (0, 5, 7), (1, 2, 3), (1, 2, 4),$
$(1, 2, 7), (1, 3, 5), (1, 4, 6), (2, 6, 7), (3, 4, 5), (4, 5, 6), (5, 6, 7)\}$

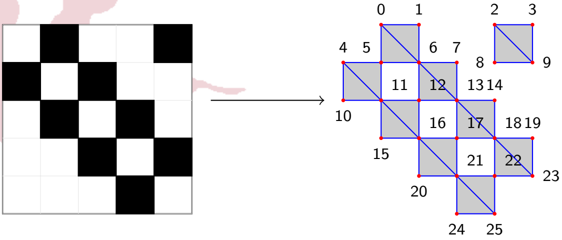
$H_0(\mathcal{K}_2) = \mathbb{Z}, H_1(\mathcal{K}_2) = 0, H_2(\mathcal{K}_2) = 0$

Other Examples



$H_0(\mathcal{K}_3) = \mathbb{Z}$, $H_1(\mathcal{K}_3) = \mathbb{Z} \oplus \mathbb{Z}$, $H_2(\mathcal{K}_3) = \mathbb{Z}$

Other Examples



$V = (0, 1, 2, \ldots, 24, 25)$
$\mathcal{K} =$ vertices $\cup$ edges $\cup$ triangles

$H_0(\mathcal{K}_4) = \mathbb{Z}$, $H_1(\mathcal{K}_4) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, $H_2(\mathcal{K}_4) = 0$