# Integrating multiple sources to answer questions in Algebraic Topology

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# Introduction

- Different sources of information:

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  - Consult papers or textbooks

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- Different sources of information:
  - Consult papers or textbooks
  - Computations with CAS

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  - Focus to a subset of Algebraic Topology:

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- Different sources of information:
  - Consult papers or textbooks
  - Computations with CAS
  - Check results against tables
  - Verify conjectures with a proof assistant tool
- Aim:
  - Mechanize the management of these sources
  - Focus to a subset of Algebraic Topology:
    - Two Computer Algebra Systems
    - A Theorem Prover
    - A Rule Based System

# Table of Contents


1. General View of the system
2. Interoperability among modules
3. Putting all together
4. Conclusions and Further Work

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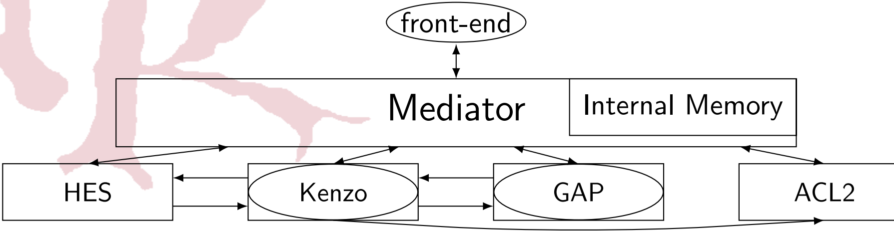
# Architecture of the system



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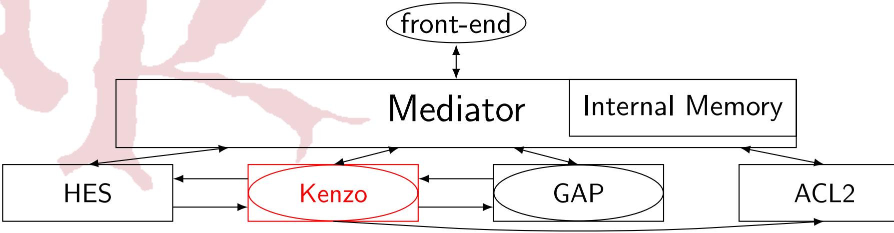
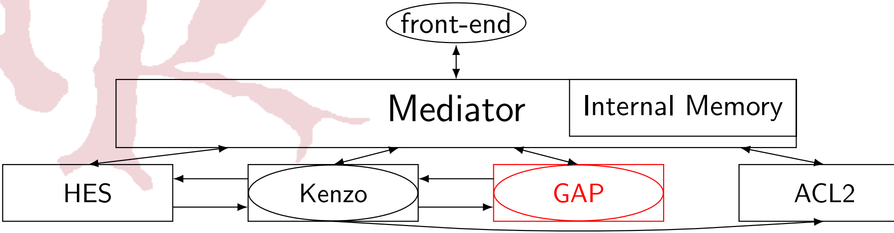


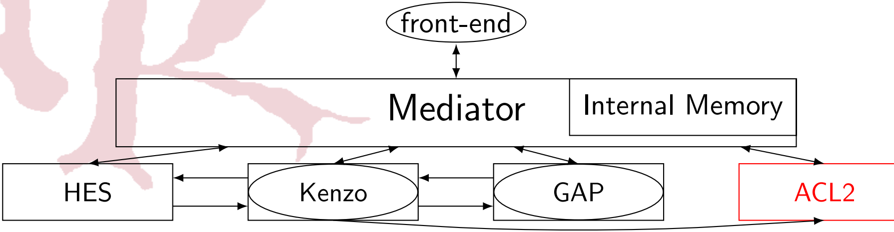

- Kenzo:
  - Symbolic Computation System devoted to Algebraic Topology
  - Homology groups unreachable by any other means
  - Core for computations related to homology groups of spaces

# Architecture of the system




- GAP/HAP:
  - GAP: Computer Algebra System devoted to Computational Group Theory
  - HAP: Homological Algebra library for using with GAP
  - Core for computations related to group homology

# Architecture of the system




- ACL2:
  - A Computational Logic for an Applicative Common Lisp
  - Programming language, logic and theorem prover
  - Kernel for verifying the correctness of statements

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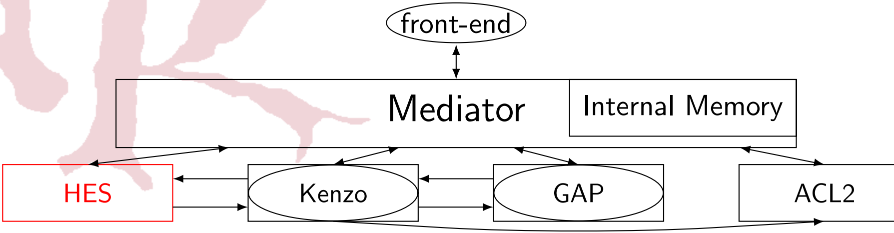
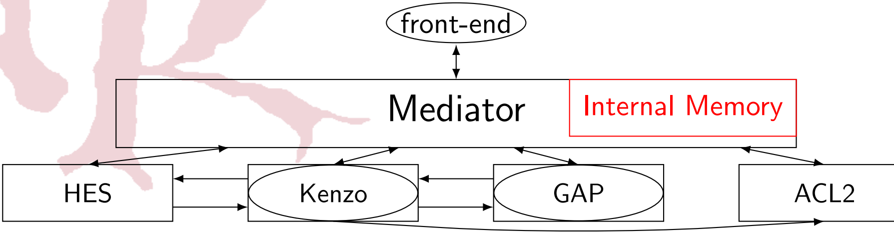



- Homotopy Expert System:
  - A new rule-based expert system
  - Computation of homotopy groups of spaces

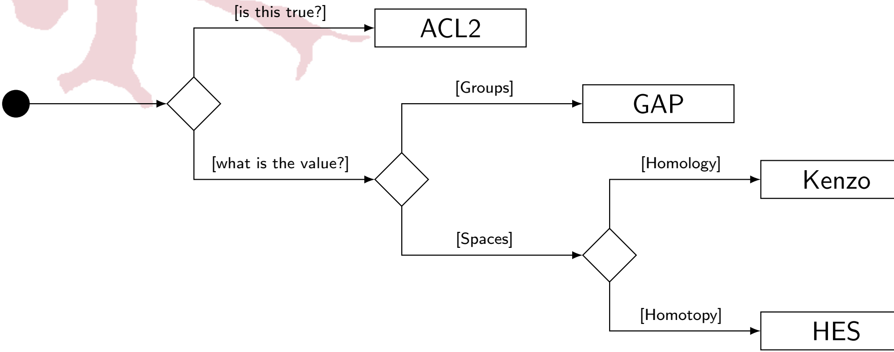
# Architecture of the system




- Internal Memory:
  - Memoization
  - Objects decorated with several annotations:
    - type of the object
    - reduction degree
    - ...

# Decision procedure

Information guides the mediator to decide which component to use.



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   - Kenzo $\leftrightarrow$ GAP
   - Kenzo $\longrightarrow$ ACL2
   - Kenzo $\leftrightarrow$ HES
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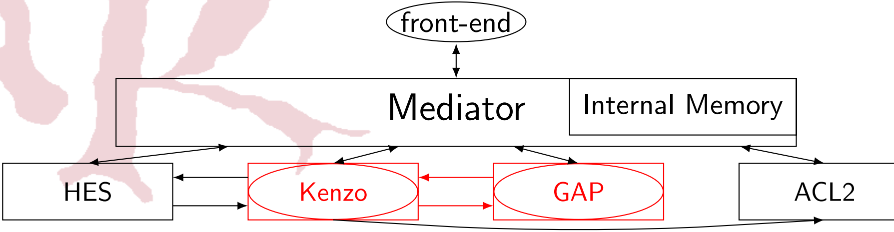
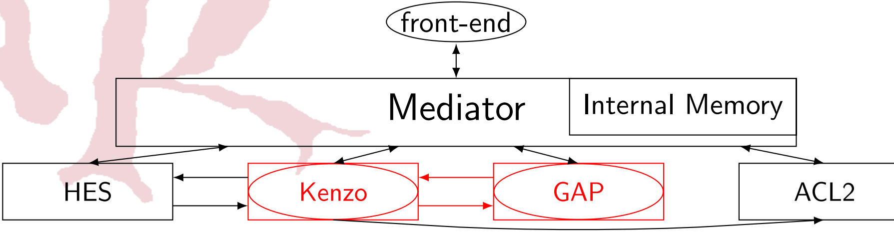
# Kenzo $\leftrightarrow$ GAP




- Aim:
  - Compute homology groups of Eilenberg-MacLane spaces $K(\pi, n)$

# Kenzo ↔ GAP

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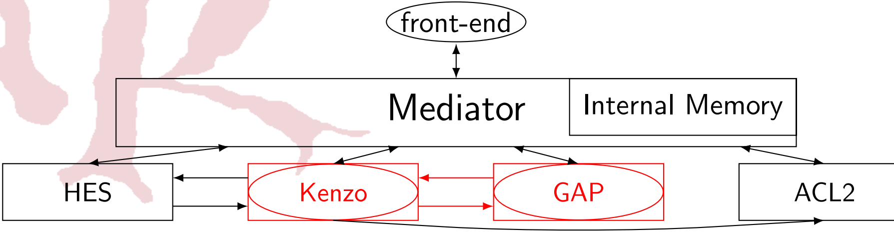


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- Kenzo $\longrightarrow$ GAP
  - Requests intermediary computations
- Kenzo $\longleftarrow$ GAP
  - Sends results

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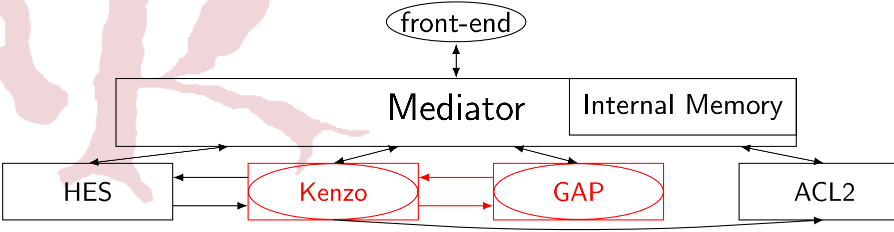


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A. Romero, G. Ellis and J. Rubio. *Interoperating between Computer Algebra Systems: computing homology groups with Kenzo and GAP*. Proceedings of International Symposium on Symbolic and Algebraic Computation, pages 303-310, 2009.

# Kenzo $\leftrightarrow$ GAP

Previous approach:

1. load the necessary packages and files in GAP and Kenzo,
2. build the cyclic group $G$ in GAP,
3. build a resolution for $G$ using the HAP package,
4. export from GAP the resolution in a file using an OpenMath format,
5. import the resolution to Kenzo,
6. build the cyclic group in Kenzo,
7. assign the resolution to the corresponding cyclic group in Kenzo,
8. build the space $K(G, 1)$ in Kenzo, using the imported resolution and finally
9. compute $H_n(K(G, 1))$ in Kenzo

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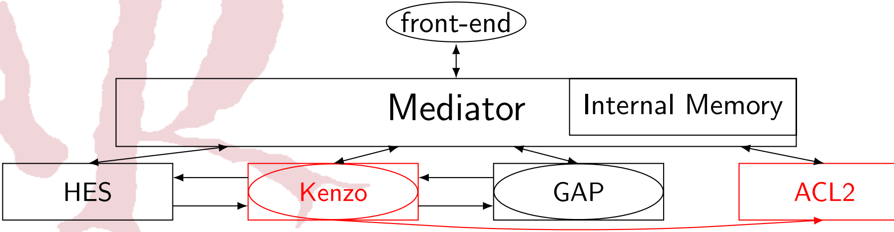
- Automatization of intermediary steps
- Communication Kenzo ↔ GAP
  - SCSCP protocol

# Kenzo $\longrightarrow$ ACL2




- Aim: Prove truthfulness of Kenzo statements

```
> (cyclicgroup 5)
[K1 Abelian-Group]
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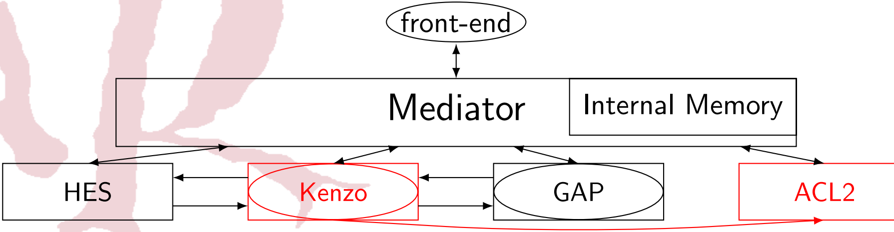


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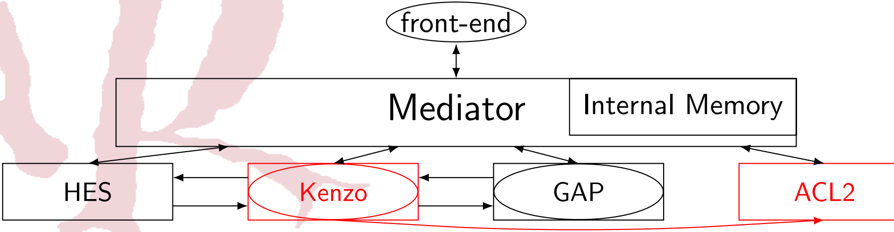


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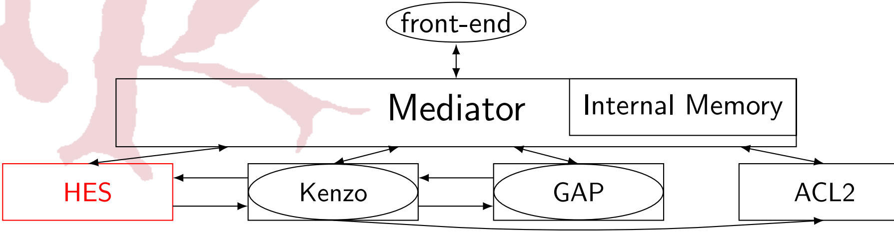
- Communication by means of OMDoc documents

# Homotopy Expert System

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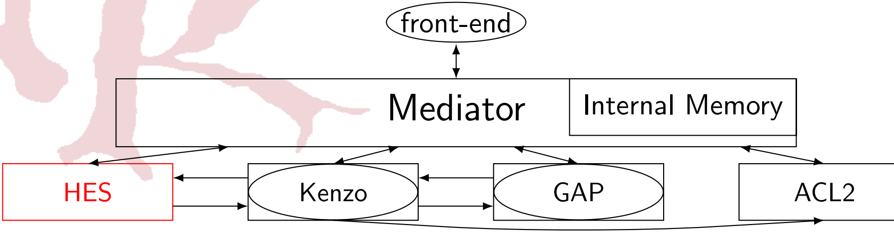


- Homotopy Expert System:
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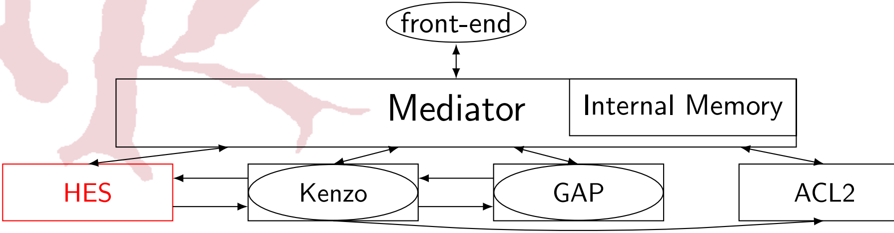


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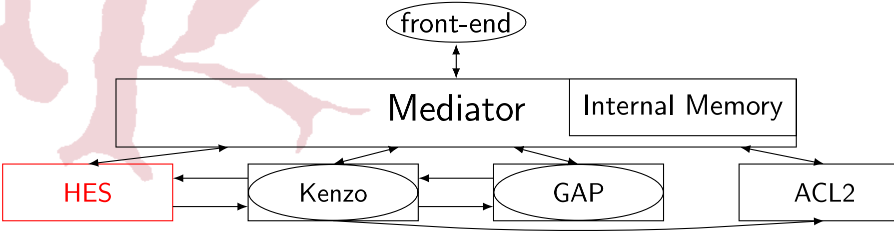


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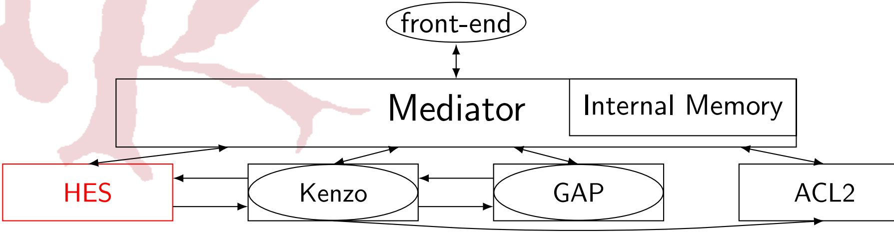


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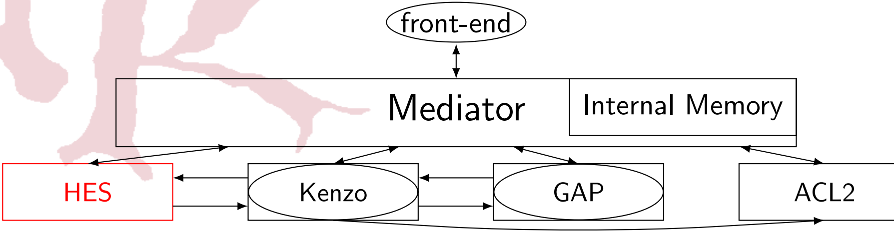


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  - Two kinds of facts: static and dynamic
  - Rules: RuleML and OMDoc files
  - Explanation facility module

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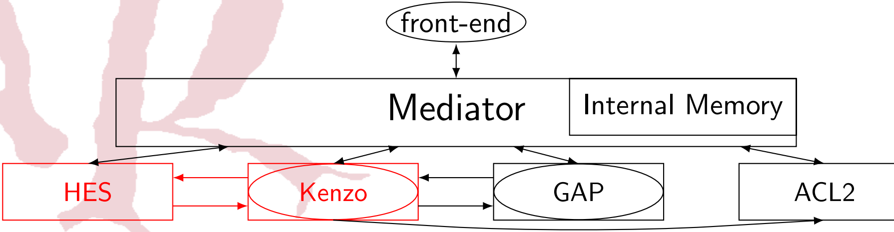


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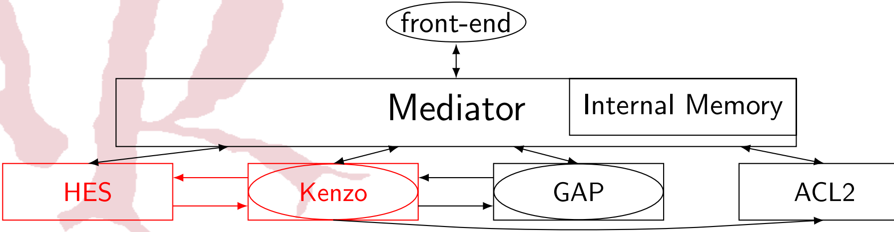


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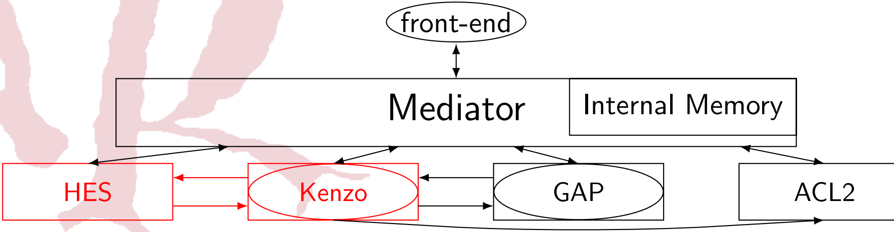


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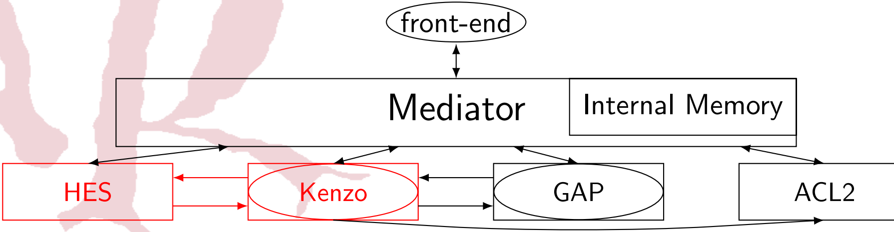


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- Example:
  - Compute $\pi_n(S^3)$ for $n = 1, \ldots, 6$

# Kenzo $\leftrightarrow$ HES

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    - **IF** $X$ is a 1-connected space
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      **THEN** $\pi_s(X) = H_s(X)$

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  - "last appeal" tool

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  - "last appeal" tool
  - high complexity

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# Putting all together

- Demo:
    - Computations with Kenzo
    - Computations with GAP
    - Computations with HES
    - Computations with Kenzo + GAP
    - Computations with Kenzo + HES
    - Certifications with ACL2

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  - Improve interaction with ACL2

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