Content Dictionaries for Algebraic Topology

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Abstract

Kenzo is a Symbolic Computation System devoted to Algebraic Topology that works with the main mathematical structures used in this discipline. In this poster, we present an OpenMath Content Dictionary for each mathematical structure in Algebraic Topology the Kenzo system works with. Besides, how using them to interoperate with a particular Theorem Prover and to obtain certified calculations from Kenzo is explained.

Introduction

Kenzo [2] is a Common Lisp system devoted to Symbolic Computation in Algebraic Topology. It was developed in 1997 under the direction of F. Sergeraert and has been successful, in the sense that it has been capable of computing homology groups unreachable by any other means.

Up to now, the mathematical structures Kenzo works with have not been represented in OpenMath [1]. In order to communicate Kenzo with other systems, like GAP [3] or ACL2 [6], we have tackled the task of developing these OpenMath Content Dictionaries. Previous works in these directions are [7] and [4].

Kenzo Content Dictionaries

- The Kenzo system works with the main mathematical structures used in Simplicial Algebraic Topology, [5].
- For each one of these mathematical structures, an OpenMath Content Dictionary has been defined.

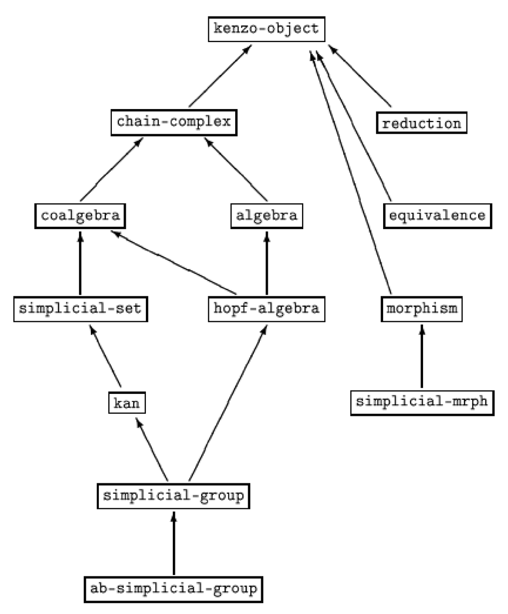

Figure: Class diagram of the Kenzo system.

- All the mathematical structures Kenzo works with are graded structures.
- Each graded structure is represented in Kenzo by means of the invariant of its underlying set.

```
inv: U nat  -> bool
     x  n   -> True  if x ∈ K^n
               False if x ∉ K^n
```

Specification of a Mathematical Structure $\langle \Sigma, Prop \rangle$ $\longrightarrow$ **Specification of a Mathematical Structure *Representation*** $\langle \Sigma \cup \{inv\}, Prop \cup \{Prop_{inv}\} \rangle$

A case study: Simplicial Sets

Definition

A simplicial set K, [5], is a disjoint union $K = \bigcup_{q \geq 0} K^q$, where the K^q are sets, together with functions

$$\partial_i^q : K^q \to K^{q-1},\ q > 0,\ i = 0, \ldots, q,$$
$$\eta_i^q : K^q \to K^{q+1},\ q \geq 0,\ i = 0, \ldots, q,$$

subject to relations

$$\begin{aligned}
\partial_i^{q-1}\partial_j^q &= \partial_{j-1}^{q-1}\partial_i^q, & i < j \\
\eta_i^{q+1}\eta_j^q &= \eta_j^{q+1}\eta_{i-1}^q, & i > j \\
\partial_i^{q+1}\eta_j^q &= \eta_{j-1}^{q-1}\partial_i^q, & i < j \\
\partial_i^{q+1}\eta_i^q &= \partial_{i+1}^{q+1}\eta_i^q, & \text{identity} \\
\partial_i^{q+1}\eta_j^q &= \eta_j^{q-1}\partial_{i-1}^q, & i > j+1
\end{aligned}$$

Mathematical Representation vs Content Dictionary

- Signature

```
inv  : u  nat      -> bool
face : u  nat  nat -> u
deg  : u  nat  nat -> u

u ≡ Universe of Lisp Objects
```

$\rightsquigarrow$

```
...
<Signature name="simplicial-set">
    <OMOBJ xmlns="http://www.openmath.org/OpenMath">
        <OMA>
            <OMS name="mapsto" cd="sts"/>
            <OMA id="inv">
                <OMS cd="sts" name="mapsto"/>
                <OMV name="Simplicial-Set-Element"/>
                <OMV name="PositiveInteger"/>
                <OMS cd="setname2" name="boolean"/>
            </OMA>
            <OMA id="face">
                <OMS cd="sts" name="mapsto"/>
                <OMV name="Simplicial-Set-Element"/>
                <OMV name="PositiveInteger"/>
                <OMV name="PositiveInteger"/>
                <OMV name="Simplicial-Set-Element"/>
            </OMA>
            <OMA id="degeneracy">
                <OMS cd="sts" name="mapsto"/>
                <OMV name="Simplicial-Set-Element"/>
                <OMV name="PositiveInteger"/>
                <OMV name="PositiveInteger"/>
                <OMV name="Simplicial-Set-Element"/>
            </OMA>
            <OMV name="Simplicial-Set"/>
        </OMA>
    </OMOBJ>
</Signature>
...
```

- Properties

$x \in K^q \implies \partial_i^q(x) \in K^{q-1}$ $\rightsquigarrow$

$\ldots$

```
<CMP> The face operator is invariant </CMP>
<FMP>
...
<OMA>
    <OMS cd="logic1" name="implies"/>
    <OMA>
        <OMV name="inv"/>
        <OMV name="x"/>
        <OMV name="q"/>
    </OMA>
    <OMA>
        <OMV name="inv"/>
        <OMA>
            <OMV name="face"/>
            <OMV name="x"/>
            <OMV name="i"/>
            <OMV name="q"/>
        </OMA>
        <OMA>
            <OMS cd="arith1" name="minus"/>
            <OMV name="q"/>
            <OMI>1</OMI>
        </OMA>
    </OMA>
</OMA>
...
</FMP>
...
```

- Example

$K = \bigcup_{q \geq 0} \{*\}$ $\rightsquigarrow$

```
<example>
...
    <OMBIND>
        <OMS name="face"/>
        <OMBVAR>
            <OMV name="x"/>
            <OMV name="i"/>
            <OMV name="q"/>
        </OMBVAR>
        <OMS cd="list" name="nil"/>
    </OMBIND>
...
</example>
...
```

Predefined Objects

- Some particular mathematical structures are predefined objects in the Kenzo system.
- These objects have been included in the corresponding Content Dictionary.
- e.g. A *sphere* of dimension n is a Simplicial Set with a base point and only one non degeneracy simplex in dimension n, whose faces are the degenerations of the base point.

Mathematical Representation vs **Content Dictionary**

- Signature:

```
sphere: nat -> Simplicial-Set
```

$\rightsquigarrow$

```
...
<Signature name="sphere">
    <OMOBJ  xmlns="http://www.openmath.org/OpenMath">
        <OMA>
            <OMS name="mapsto" cd="sts"/>
            <OMV name="PositiveInteger"/>
            <OMV name="Simplicial-Set"/>
        </OMA>
    </OMOBJ>
</Signature>
...
```

- Properties:

$sphere(n) \Rightarrow n \in \mathbb{N} \wedge 1 \leq n \leq 14$ $\rightsquigarrow$

```
...
<FMP>
    <OMOBJ  xmlns="http://www.openmath.org/OpenMath">
        <OMA>
            <OMS cd="logic1" name="implies"/>
            <OMA>
                <OMS cd="Simplicial-Sets" name="sphere"/>
                <OMV name="n"/>
            </OMA>
            <OMA>
                <OMS cd="logic1" name="and"/>
                <OMA>
                    <OMS cd="set1" name="in"/>
                    <OMV name="n"/>
                    <OMS name="N" cd="setname1"/>
                </OMA>
                <OMA>
                    <OMS cd="relation1" name="leq"/>
                    <OMI>1</OMI>
                    <OMV name="n"/>
                </OMA>
                <OMA>
                    <OMS cd="relation1" name="leq"/>
                    <OMV name="n"/>
                    <OMI>14</OMI>
                </OMA>
            </OMA>
        </OMA>
    </OMOBJ>
</FMP>
...
```

- Example:

S^3 $\rightsquigarrow$

```
...
<Example>
    The sphere of dimension 3
    <OMOBJ  xmlns="http://www.openmath.org/OpenMath">
        <OMA>
            <OMS cd="Simplicial-Sets" name="sphere"/>
            <OMI>3</OMI>
        </OMA>
    </OMOBJ>
</Example>
...
```

From a Kenzo CD to an ACL2 encapsulate

- Goal: Integration of Kenzo with ACL2 [6] to increase the reliability of the Kenzo system.
- Based on the encapsulate notion:
  - list of function signatures,
  - properties of the encapsulated functions,
  - a witness for the set of functions.
- Interpreter from Kenzo CDs to ACL2 encapsulates

ACL2 encapsulate from Simplicial Sets CD

```
(encapsulate

 (((inv * *) => *)
 ((face * * *) => *)
 ((degeneracy * * *) => *)
  )

 (local (defun inv (x q)
          (declare (IGNORE q))
          (equal x nil)))

 (local (defun face (x i q)
          (declare (IGNORE x i q))
          nil))

 (local (defun deg (x i q)
          (declare (IGNORE x i q))
          nil))

 ; ... lines skipped

 (defthm prop5
     (implies (and (inv x q) (< i j))
              (equal (face (deg x j q) i (+ q 1))
                     (deg (face x i q) (- j 1) (- q 1)))))

 )
```

Conclusions

In this paper, we have presented some OpenMath Content Dictionaries where the main mathematical structures used in Simplicial Algebraic Topology have been defined. The definitions given in these Content Dictionaries include the axiomatic parts and have been used, for example, to interoperate with deduction systems. In this way, a gate has been opened not only to communicate with other systems which work with the same mathematical structures but also to prove the correctness of some constructions or calculations carried out by the Kenzo system. For instance, when Kenzo builds an object belonging to the simplicial-set class, that it is really a simplicial set can be proved. In this way, some calculations with certificates can be carried out.

References

[1] Buswell S., Caprotti O., Carlisle D.P., Dewar M.C., Gaëtano M., Kohlhase M. *OpenMath* Version 2.0, 2004. http://www.openmath.org/.

[2] Dousson X., Sergeraert F., Siret Y., *The Kenzo program*, Institut Fourier, Grenoble, 1999. http://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo/.

[3] GAP - Groups, Algorithms, Programming - a System for Computational Discrete Algebra. http://www.gap-system.org/.

[4] Heras J., Pascual V., Rubio J., *Using Open Mathematical Documents to interface Computer Algebra and Proof Assistant systems.* To appear in Proceedings of MKM 2009.

[5] Hilton P. J., Wylie S., *Homology Theory*, Cambridge University Press, (1967).

[6] Kaufmann M., Manolios P., Moore J., *Computer-Aided Reasoning: An Approach. Kluwer Academic Press*, Boston (2000).

[7] Romero A., Ellis G., Rubio J., *Interoperating between Computer Algebra systems: computing homology of groups with Kenzo and GAP.* To appear in Proceedings of ISSAC 2009.