Embedding theorems for anisotropic Lipschitz spaces

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Abstract

Anisotropic Lipschitz spaces are considered. For these spaces we obtain sharp embeddings in Besov and Lorentz spaces. The methods used are based on estimates of iterative rearrangements. We find a unified approach that arises from the estimation of functions defined as minimum of a given system of functions. The case of L^1-norm also is covered.

1 Introduction

In this paper we prove embedding theorems for anisotropic Lipschitz spaces. More precisely, we study integrability and smoothness properties of functions under certain conditions on its moduli of continuity.

In the study of anisotropic spaces, we have different estimates with respect to different variables. The final result will be sharp if we find an equilibrium between these estimates, that is, an optimal average estimate. Therefore it is an important problem to determine a right contribution for each variable in this average. To discuss this problem we first recall some basic definitions.

Denote by $W^r_{p;j}(\mathbb{R}^n)$ ($r \in \mathbb{N}$, $1 \le p < \infty$, $1 \le j \le n$) the Sobolev space with respect to the jth variable; i.e. the class of functions f in $L^p(\mathbb{R}^n)$ with

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partial generalized derivative $D_j^r f \in L^p(\mathbb{R}^n)$. Now, if $r_1, \dots, r_n \in \mathbb{N}$, then we set

$$W_p^{r_1,\dots,r_n}(\mathbb{R}^n) \equiv \cap_{j=1}^n W_{p;j}^{r_j}(\mathbb{R}^n)$$

and

$$\|f\|_{W_p^{r_1,\dots,r_n}} = \|f\|_p + \sum_{j=1}^n \|D_j^{r_j} f\|_p.$$

If f is a function on $\mathbb{R}^n$, $1 \le j \le n$, and $k \in \mathbb{N}$, then we denote

$$\Delta_j^k(h)f(x) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x + ihe_j)$$

(where $x \in \mathbb{R}^n$, $h \in \mathbb{R}$, and e_j is a basis vector). Let $f \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$. Then the function

$$\omega_j^k(f;\delta)_p = \sup_{0 \le h \le \delta} \|\Delta_j^k(h)f\|_p \quad (\delta \ge 0)$$

is called the *partial modulus of continuity* of f of order k in L^p with respect to x_j.

Let $r > 0$, and let k be the least integer such that $k > r$. We denote by $H_{p;j}^r(\mathbb{R}^n)$ the Nikol'skiĭ space of functions f in $L^p(\mathbb{R}^n)$ for which

$$\omega_j^k(f;\delta)_p = O(\delta^r).$$

Assume that $r_j > 0$, $(j = 1, \dots, n)$ and that k_j are the least integers such that $k_j > r_j$. Then the space $H_p^{r_1,\dots,r_n}(\mathbb{R}^n)$ is defined as $\cap_{j=1}^n H_{p;j}^{r_j}(\mathbb{R}^n)$, with the norm

$$\|f\|_{H_p^{r_1,\dots,r_n}} = \|f\|_p + \sum_{j=1}^n \sup_{u>0} u^{-r_j} \omega_j^{k_j}(f;u)_p.$$

It is well known that an important characteristic of the spaces W is the harmonic mean

$$r = n \left(\sum_{j=1}^n \frac{1}{r_j} \right)^{-1} \tag{1.1}$$ deferre

(see [9, 10]). In particular, if $1 \le p < \frac{n}{r}$ and $q^* = \frac{np}{n-rp}$, then

$$W_p^{r_1,\dots,r_n} \hookrightarrow L^q$$

if and only if $p \le q \le q^*$. That is, the integrability properties of functions in $W_p^{r_1,\ldots,r_n}$ are completely determined by r, and the contribution of the variable x_k is proportional to $1/r_k$ in a sense.

A similar situation holds for Nikol'skiĭ spaces, although in this case the embedding with the limit exponent fails (see [15, 2]).

However, the behaviour of anisotropic Lipschitz spaces is completely different.

Let $1 \le p < \infty$, $r > 0$ and let $\bar{r}$ be the least integer such that $\bar{r} \ge r$. We say that $f \in L^p(\mathbb{R}^n)$ belongs to the *Lipschitz space* with respect to the jth variable $\Lambda^r_{p;j}(\mathbb{R}^n)$ if

$$\omega_j^{\bar{r}}(f;\delta)_p = O(\delta^r).$$

Let $r_j > 0$ $(j = 1, \ldots, n)$ and denote by $\bar{r}_j$ the least integers that $r_j \le \bar{r}_j$. The *anisotropic Lipschitz space* $\Lambda_p^{r_1,\ldots,r_n}(\mathbb{R}^n)$ is defined as $\cap_{j=1}^n \Lambda^{r_j}_{p;j}(\mathbb{R}^n)$. So,

$$\|f\|_{\Lambda_p^{r_1,\ldots,r_n}} \equiv \|f\|_p + \|f\|_{\lambda_p^{r_1,\ldots,r_n}},$$

where the seminorm is

$$\|f\|_{\lambda_p^{r_1,\ldots,r_n}} = \sum_{j=1}^n \sup_{\delta>0} \delta^{-r_j}\omega_j^{\bar{r}_j}(f;\delta)_p\,.$$

It is clear that

$$\Lambda^{r_j}_{p;j} = H^{r_j}_{p;j} \text{ if } r_j \notin \mathbb{N}.$$

Also, by Hardy-Littlewood theorem [15], if $r_j \in \mathbb{N}$, then

$$\Lambda^{r_j}_{p;j} = W^{r_j}_{p;j} \quad (p > 1).$$

For $r_j \in \mathbb{N}$ we have the *strict* embedding $\Lambda^{r_j}_{p;j} \subset H^{r_j}_{p;j}$.

Thus, Lipschitz spaces have partly character of Sobolev spaces and partly - the character of Nikol'skiĭ spaces. This mixed behaviour creates a main difficulty in their study.

The integrability properties of functions in Lipschitz space and Nikol'skiĭ space with the same indices can be completely different. It was proved in [5] (for $r_k \le 1$) that, in contrast with W and H spaces, the embedding $\Lambda_p^{r_1,\ldots,r_n} \hookrightarrow L^q$ is not uniquely determined by the value of the harmonic mean r (see (1.1)). Roughly speaking, this means that the contribution of the variable x_k is not proportional to $1/r_k$.

The proof in [5] (as well as alternative proofs given in [7, 8]) was based on estimates of rearrangements and special reasonings that led to a kind of equilibrium between these estimates.

One of the main objectives of this paper is to give a quantitative sharp expression for this type of equilibrium. We obtain the following results. First, basing on known estimates of rearrangements, we modify them to special type involving functions from the spaces $L^\theta(\mathbb{R}_+, dx/x)$, $\mathbb{R}_+ \equiv (0,\infty)$. The invariance of these spaces under changes of variables of power type plays an important role. Then, using the modified estimates, we consider the "minimum-function"

$$\rho(t) = \min_{1\le i\le n} \{t_i^{r_i}\phi_i(t_{l_i})\}, \quad t \in \mathbb{R}^n_+,\ \phi_i \in L^{\theta_i}(\mathbb{R}_+, dx/x),\ l_i \in \{1,\dots,n\}. \tag{1.2}$$

minimo

We prove a special weight estimate for this function. This result provides a unified approach to estimations of various norms. Using this approach, we prove sharp estimates of Lorentz norms as well as Besov norms for functions in Lipschitz spaces.

Let us give a more detailed description of the latter results.

As it was mentioned above, the first sharp results on embedding of Lipschitz spaces into L^q were obtained in [5] (for $r_k \le 1$) with the use of non-increasing rearrangements. Afterwards, Netrusov [13, 14] studied embeddings of the spaces $\Lambda_p^{r_1,\dots,r_n}$ for $p > 1$ and arbitrary $r_j > 0$. His approach was based on special integral representations. First, he proved sharp results on embedding into Lorentz spaces (an alternative proof of these results including the case $p = 1$ was given in [10] and was based on non-increasing rearrangements). Then, he considered the embedding into Besov spaces.

Assume that $1 \le p, \theta_j < \infty$ and $0 < r_j < \infty$ $(j = 1,\dots,n)$. The *anisotropic Besov space* $B^{r_1,\dots,r_n}_{p,\theta_1,\dots,\theta_n}(\mathbb{R}^n)$ is the class of functions $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{B^{r_1,\dots,r_n}_{p,\theta_1,\dots,\theta_n}} \equiv \|f\|_p + \sum_{j=1}^n \left(\int_0^\infty (t^{-r_j}\|\Delta_j^{k_j}(t)f\|_p)^{\theta_j}\frac{dt}{t}\right)^{1/\theta_j} < \infty$$

where $k_j \in \mathbb{N}$ and $k_j > r_j$. For each choice of the integers k_j one obtains equivalent norms; in addition, one can replace in the definition the norm of finite differences by the corresponding moduli of continuity ([15], Chapter 4 and [2], Chapter 4). For simplicity we denote $B^{r_1,\dots,r_n}_{p,\theta} \equiv B^{r_1,\dots,r_n}_{p,\theta,\dots,\theta}$.

Il'in [2, §18.12] obtained the following refinement of the classical Sobolev inequality: if $1 < p < q < \infty$, $r_j \in \mathbb{N}$, and $\varkappa \equiv 1 - n/r(1/p - 1/q) > 0$, then

$$W_p^{r_1,\dots,r_n}(\mathbb{R}^n) \hookrightarrow B_{q,p}^{\varkappa r_1,\dots,\varkappa r_n}(\mathbb{R}^n). \tag{1.3}$$ lipentero

In the case $p = n = 1$ this embedding fails. It was proved by Kolyada [6, 9] that the embedding (1.3) is true in the case $p = 1$, $n \geq 2$, too.

For the Lipschitz spaces, the following result was obtained by Netrusov [14]: if $1 < p < q < \infty$, $r_j > 0$ $(j = 1, \dots, n)$, and $\varkappa \equiv 1 - n/r(1/p - 1/q) > 0$, then

$$\Lambda_p^{r_1,\dots,r_n}(\mathbb{R}^n) \hookrightarrow B_{q,\gamma_1,\dots,\gamma_n}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n) \tag{1.4}$$ Lipabe

Although we do not specify here the values of parameters, it is important to point out that here γ_j take two values - one for all j such that $r_j \notin \mathbb{N}$ and other in the case $r_j \in \mathbb{N}$.

Let us emphasize that the methods of integral representations used in [14] fail in the case $p = 1$. In particular, the question on validity of the embedding for $p = 1$ was remained open.

In this paper (section §5) we prove the embedding (1.4) for $p \geq 1$. It is the most important application of our main estimates concerning integrability of functions of the type (1.2). Moreover, we prove estimates for stronger norms defined in terms of iterative rearrangements.

For a given function on $\mathbb{R}^n$, we obtain its iterative rearrangement, rearranging this function first with respect to one variable, then respect to another, and so on. It turns out that the iterative rearrangement is defined on $\mathbb{R}_+^n \equiv (0, \infty)^n$, it is non-increasing in each variable and equimeasurable with $|f|$. It is defined a Lorentz kind norm $\|.\|_{q,p;\mathcal{R}}$ in term of iterative rearrangements (see §2). It is important to stress that in the case $q > p$ this norm is stronger than the usual Lorentz norm $\|.\|_{q,p}$. Observe also that iterative rearrangements were used in embedding theorems in the works [5, 6, 8, 11, 16]. In particular, it was proved in [11] that for anisotropic Sobolev spaces a stronger version of Sobolev type inequality with the generalized Lorentz norm $\|.\|_{q,p;\mathcal{R}}$ is true.

Applying estimates of functions (1.2) we immediately obtain a similar result for Lipschitz spaces. That is, in Section §5 we prove a Sobolev type inequality

$$\|f\|_{q^*,s;\mathcal{R}} \leq c\|f\|_{\lambda_p^{r_1,\dots,r_n}}, \quad 1 \leq p < n/r,$$

which gives an extension of the results of Kolyada and Netrusov mentioned above.

Further, in Section §5 we prove one of our main results – Il'in's type inequality

$$\sum_{i=1}^{n}\left(\int_0^\infty \left[h^{-\alpha_i}\|\Delta_i^{\bar{r}_i}(h)f\|_{q,1;\mathcal{R}}\right]^{\gamma_i}\frac{dh}{h}\right)^{1/\gamma_i} \le c\|f\|_{\lambda_p^{r_1,\dots,r_n}}. \tag{1.5}$$

LipB

This immediately implies the embedding (1.4) for all $p \ge 1$. Let us emphasize that $p = 1$ is included. Moreover, comparing with (1.4), the left hand side of (1.5) contains the stronger Lorentz norm $\|.\|_{q,1;\mathcal{R}}$ instead of $\|.\|_q$. Note also that it is even possible to replace $\|.\|_{q,1;\mathcal{R}}$ by a stronger norm $\|.\|_{q,\xi;\mathcal{R}}$ for any $\xi > 0$.

As it was observed above, our approach is based on two tools. First, we use some modifications of estimates of rearrangements obtained in [11, 10]. Second, we apply estimates of functions of the type (1.2).

The paper is organized as follows. In Section 2 we consider the definition and basic properties of the iterative rearrangements. Section 3 is devoted to modify known estimates of rearrangements into a special type. Next, in Section 4 we get main lemmas that give us special weight estimates for functions of type (1.2). Finally, sharp embeddings for anisotropic Lipschitz spaces are proved in Section 5.

2 Non-Increasing rearrangements

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This section contains basic facts concerning rearrangements. We refer to ([11], §2).

Let $S_0(\mathbb{R}^n)$ be the class of measurable and almost everywhere finite functions f on $\mathbb{R}^n$ such that for each $y > 0$,

$$\lambda_f(y) \equiv |\{x \in \mathbb{R}^n : |f(x)| > y\}| < \infty.$$

A *non-increasing rearrangement* of a function $f \in S_0(\mathbb{R}^n)$ is a non-increasing function f^* on $\mathbb{R}_+ \equiv (0, +\infty)$ that is equimeasurable with $|f|$. The rearrangement f^* can be defined by the equality

$$f^*(t) = \sup_{|E|=t} \inf_{x\in E} |f(x)|, \qquad 0 < t < \infty.$$

Next, we consider the so called iterative rearrangements.

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Removing the variable x_k from the n-tuple x we obtain a $(n-1)$-dimensional vector denoted by $\hat{x}_k$.

We denote by $(\tau, \hat{x}_k)$ $(\tau \in \mathbb{R})$ the vector in $\mathbb{R}^n$ with first component τ and the remaining components equal to the $(n-1)$-dimensional vector $\hat{x}_k$.

Let $k \in \{1, \dots, n\}$ and $f \in S_0(\mathbb{R}^n)$. We obtain $\mathcal{R}_k f(t_1, \hat{x}_k)$ a.e. on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ by fixing $\hat{x}_k$ and "rearranging" f in non-increasing order as a function of the variable x_k only.

Let $\mathcal{P}_n$ be the collection of all permutations $\sigma = \{k_1, \dots, k_n\}$ of the set $\{1, \dots, n\}$. For each $\sigma \in \mathcal{P}_n$ we set $\mathcal{R}_\sigma f \equiv \mathcal{R}_{k_n} \cdots \mathcal{R}_{k_1} f$. It is easy to see that $\mathcal{R}_\sigma f$ decreases monotonically with respect to each variable and is equimeasurable with $|f|$ (for more details, see ([11], §2)).

It is easy to verify that

$$\mathcal{R}_\sigma f(t) \le f^*(t_1 \cdots t_n), \tag{2.1}$$ nir1

$$\mathcal{R}_\sigma(f+g)(t+s) \le \mathcal{R}_\sigma f(t) + \mathcal{R}_\sigma g(s) \quad (t, s \in \mathbb{R}^n_+).$$

Let $k \in \{1, \dots, n\}$, $t_1 \in \mathbb{R}_+$, $\hat{x}_k \in \mathbb{R}^{n-1}$. We consider the following averages:

$$\mathcal{R}^*_k f(t_1, \hat{x}_k) \equiv \frac{1}{t_1} \int_0^{t_1} \mathcal{R}_k f(u, \hat{x}_k) du,$$

$$\overline{\mathcal{R}}_k f(t_1, \hat{x}_k) \equiv \frac{1}{t_1} \int_{t_1}^{\infty} \mathcal{R}_k f(u, \hat{x}_k) du.$$

Now, for each $\sigma \in \mathcal{P}_n$ we set

$$\mathcal{R}^*_\sigma f(t) = \mathcal{R}^*_{k_n} \cdots \mathcal{R}^*_{k_1} f(t), \quad t \in \mathbb{R}^n_+.$$

It holds (see [11, §2])

$$\|\mathcal{R}^*_\sigma f\|_p \le c_p \|f\|_p, \quad 1 < p < \infty. \tag{2.2}$$ desestr

We denote also

$$\overline{\mathcal{R}}_\sigma f(t) = \overline{\mathcal{R}}_{k_n} \cdots \overline{\mathcal{R}}_{k_1} f(t), \quad t \in \mathbb{R}^n_+,$$

and for each $1 < \nu < \infty$ we set

$$\overline{\mathcal{R}}^{(\nu)}_\sigma f(t) \equiv (\overline{\mathcal{R}}_\sigma f^\nu(t))^{1/\nu}.$$

This operator was defined in [11] and it was used to prove embedding theorems. Its important property is that

$$\|\overline{\mathcal{R}}^{(\nu)}_\sigma f\|_1 \le c\|f\|_1. \tag{2.3}$$ desover

Assume that $0 < p, q < \infty$. A function $f \in S_0(\mathbb{R}^n)$ belongs to the Lorentz space $L^{q,p}(\mathbb{R}^n)$ if

$$\|f\|_{q,p} \equiv \left(\int_0^\infty (t^{1/q} f^*(t))^p \frac{dt}{t} \right)^{1/p} < \infty.$$

We have the inequality (see [1, p.217])

$$\|f\|_{q,s} \le c\|f\|_{q,p} \quad (0 < p \le s < \infty),$$

so that $L^{q,p} \subset L^{q,s}$ for $p < s$. In particular, for $0 < p \le q$,

$$L^{q,p} \subset L^{q,q} \equiv L^q.$$

In what follows we set

$$\pi(t) = \prod_{k=1}^n t_k, \qquad t \in \mathbb{R}^n_+.$$

Assume that $0 < q, p < \infty$ and let $\sigma \in \mathcal{P}_n$ $(n \ge 2)$. We denote by $L^{q,p}_{\mathcal{R}_\sigma}(\mathbb{R}^n)$ the class of functions $f \in S_0(\mathbb{R}^n)$ such that

$$\|f\|_{q,p;\mathcal{R}_\sigma} \equiv \left(\int_{\mathbb{R}^n_+} [\pi(t)^{1/q} \mathcal{R}_\sigma f(t)]^p \frac{dt}{\pi(t)} \right)^{1/p} < \infty$$

(see [3]). We also set

$$L^{q,p}_{\mathcal{R}}(\mathbb{R}^n) = \bigcap_{\sigma \in \mathcal{P}_n} L^{q,p}_{\mathcal{R}_\sigma}(\mathbb{R}^n), \quad \|f\|_{q,p;\mathcal{R}} = \sum_{\sigma \in \mathcal{P}_n} \|f\|_{q,p;\mathcal{R}_\sigma}.$$

It is easy to see that

$$\|f\|_{q,s;\mathcal{R}} \le c\|f\|_{q,p;\mathcal{R}} \quad (0 < p \le s < \infty). \tag{2.4}$$ ann1

If $q > p$, then for each $\sigma \in \mathcal{P}_n$ and each $f \in S_0(\mathbb{R}^n)$,

$$\|f\|_{q,p} \le c\|f\|_{q,p;\mathcal{R}_\sigma}$$

(see [17]). Thus,

$$L^{q,p}_{\mathcal{R}_\sigma} \subset L^{q,p} \quad (q > p).$$

Moreover, this is a proper embedding [17].

3 Estimates

estimates

From now on $n \in \mathbb{N}$. Let $0 < r_j < +\infty$ $(j = 1, \dots, n)$. We denote by $\bar{r}_j$ the least integer such that $r_j \le \bar{r}_j$.

Let $f \in L^p(\mathbb{R}^n)$ $(1 \le p < +\infty)$. For each $j = 1, \dots, n$ set

$$f_{j,h}(x) \equiv \Delta_j^{\bar{r}_j}(h) f(x).$$

In this section we consider some modifications of the estimates of the iterative rearrangements $\mathcal{R}_\sigma f$ and $\mathcal{R}_\sigma f_{j,h}$ obtained in [11] and [10].

For $1 \le p < \infty$ we denote $\mathcal{L}^p \equiv L^p(\mathbb{R}_+, du/u)$; set also $\mathcal{L}^\infty \equiv L^\infty(\mathbb{R}_+)$(see [4]).

reductor **Lemma 1.** *Let $n \in \mathbb{N}$, $1 \le p < \infty$. Assume that $F \in L^p(\mathbb{R}_+^n)$ is a non-negative function, non-increasing at each one of its variables. Then, for any $\delta > 0$ and any $j \in \{1, \dots, n\}$ there exists a non-negative function $\phi \equiv \phi_{\delta,j}$ on $\mathbb{R}_+$ such that*
i) $F(t) \le \pi(t)^{-1/p} \phi(t_j)$,
ii) $\|\phi\|_{\mathcal{L}^p} \le c(\delta) \|F\|_{L^p(\mathbb{R}_+^n)}$,
iii) $\phi(u) u^\delta \uparrow$ and $\phi(u) u^{-\delta} \downarrow$.

Proof. As F is non-increasing at each one of its variables, we use a weak type inequality

$$F(t) \le \pi(\hat{t}_j)^{-1/p} \left(\int_{\mathbb{R}_+^{n-1}} F(t)^p d\hat{t}_j \right)^{1/p} \equiv \pi(\hat{t}_j)^{-1/p} g(t_j). \qquad (3.1)$$ reductor1

Then g is non-negative and non-increasing in $\mathbb{R}_+$ and

$$\|g\|_{L^p(\mathbb{R}_+)} = \|F\|_{L^p(\mathbb{R}_+^n)}. \qquad (3.2)$$ reductor2

Applying Lemma 2.1 of [12] we obtain a function $\bar{g}$ on $\mathbb{R}_+$ such that

$$g \le \bar{g}, \quad \|\bar{g}\|_p \le c(\delta) \|g\|_p \quad \text{and} \quad \bar{g}(u) u^{1/p - \delta} \downarrow, \ \bar{g}(u) u^{1/p + \delta} \uparrow, \quad u > 0. \qquad (3.3)$$ reductor3

Denoting $\phi(u) \equiv \bar{g}(u) u^{1/p}$, by (3.1) and (3.3) we get i). Next, ii) follows from (3.2) and (3.3), and iii) follows from (3.3). $\square$

get **Lemma 2.** *Let $n \ge 2$, $j \in \{1, \dots, n\}$, $r_j \in \mathbb{N}$ and $1 \le p < \infty$. Let $f \in W^{r_j}_{p;j}(\mathbb{R}^n)$. We choose $\sigma \in \mathcal{P}_n$, $1 \le l \le n$ $(l \ne \sigma^{-1}(j))$, and $0 < \delta < 1$. Then there exists a non-negative function $\phi \equiv \phi_{j,l,\sigma,\delta}$ on $\mathbb{R}_+$ such that:*

$$\|\phi\|_{\mathcal{L}^p} \le c \|D_j^{r_j} f\|_p; \qquad (3.4)$$ e23

$$\phi(u)u^{\delta}\uparrow \text{ and } \phi(u)u^{-\delta}\downarrow \quad u>0; \tag{3.5}$$ $\boxed{\texttt{e24}}$

for any $K>1$

$$\mathcal{R}_{\sigma}f(t) \le 2^{r_j}\mathcal{R}_{\sigma'_j}f\left(Kt_{m_j}, \frac{\hat{t}_{m_j}}{2}\right) + c(K)\pi(t)^{-1/p}t_{m_j}^{r_j}\phi(t_l); \tag{3.6}$$ $\boxed{\texttt{e21}}$

$$\mathcal{R}_{\sigma}f_{j,h}(t) \le c\pi(t)^{-1/p}h^{r_j-\delta}t_{m_j}^{\delta}\phi(t_l) \quad \textit{for all } 0<h<t_{m_j}, \tag{3.7}$$ $\boxed{\texttt{e22}}$

where σ'_j *is obtained from* σ *by moving the* j*th index to the first position,* $m_j=\sigma^{-1}(j)$ *and* c*,* $c(K)$ *do not depend on* f*.*

Proof. Case 1. First we suppose that $p>1$. Denote $g_j \equiv D_j^{r_j}f$. From [11, (3.3) and (3.7)] we get

$$\mathcal{R}_{\sigma}f(t) \le 2^{r_j}\mathcal{R}_{\sigma'_j}f\left(Kt_{m_j}, \frac{\hat{t}_{m_j}}{2}\right) + c(K)t_{m_j}^{r_j}\mathcal{R}_{\sigma}^{*}g_j(\frac{t}{2}) \quad \forall K>1 \tag{3.8}$$ $\boxed{\texttt{e1}}$

where σ'_j is obtained from σ by moving the jth index to the first position.

Besides, by [11, (4.5)],

$$\mathcal{R}_{\sigma}f_{j,h}(t) \le ch^{r_j}\mathcal{R}_{\sigma}^{*}g_j(t). \tag{3.9}$$ $\boxed{\texttt{e2}}$

Now (see (2.2)) note that $\mathcal{R}_{\sigma}^{*}g_j(t)$ satisfies the conditions of Lemma 1. So, for δ and l we obtain a non-negative function ϕ such that (3.5) holds,

$$\mathcal{R}_{\sigma}^{*}g_j(t) \le \pi(t)^{-1/p}\phi(t_l), \tag{3.10}$$ $\boxed{\texttt{Rred}}$

and

$$\|\phi\|_{\mathcal{L}^p} \le c\|\mathcal{R}_{\sigma}^{*}g_j\|_p.$$

Then (3.4) follows from the last estimate and (2.2). Inequalities (3.6) and (3.7) are immediate consequences of (3.10), (3.8) and (3.9).

Case 2. Now we suppose that $p=1$. Set $\nu=1/(1-\delta)$. We have (see [11, (3.3) and (3.10)])

$$\mathcal{R}_{\sigma}f(t) \le 2^{r_j}\mathcal{R}_{\sigma'_j}f\left(Kt_{m_j}, \frac{\hat{t}_{m_j}}{2}\right) + c(K)t_{m_j}^{r_j-1}F_j(\frac{\hat{t}_{m_j}}{2}), \tag{3.11}$$ $\boxed{\texttt{e7}}$

where

$$F_j(\hat{t}_{m_j}) = \overline{\mathcal{R}}_{\hat{\sigma}_j}^{(\nu)}\psi_j(\hat{t}_{m_j}), \qquad \psi_j(\hat{x}_j) = \int_{\mathbb{R}} g_j(x)dx_j.$$

Besides, by [11, (4.11)]

$$\mathcal{R}_\sigma f_{j,h}(t) \le c h^{r_j-\delta} t_{m_j}^{\delta-1} F_j(\hat{t}_{m_j}). \tag{3.12}$$ e8

By (2.3) we have

$$\|F_j\|_{1,\mathbb{R}_+^{n-1}} \le c\|\psi_j\|_{1,\mathbb{R}^{n-1}} = \|g_j\|_1. \tag{3.13}$$ opernorma

So, for any $l \neq m_j$, $l \in \{1, \ldots, n\}$ and $0 < \delta < 1$ we apply Lemma 1 to F_j and obtain a function $\phi(t_l)$ satisfying (3.5). Besides,

$$F_j(\hat{t}_{m_j}) \le c\pi(\hat{t}_{m_j})^{-1}\phi(t_l).$$

Thus, by (3.11) and (3.12) we get (3.6) and (3.7). Finally,

$$\|\phi\|_{\mathcal{L}} \le c\|F_j\|_1,$$

and (3.13) imply (3.4). □

getprime **Lemma 3.** *Let $n \in \mathbb{N}$, $j \in \{1, \ldots, n\}$, $0 < r_j < \infty$ and $1 \le p < \infty$. Let $f \in L^p(\mathbb{R}^n)$. Then, for any $\sigma \in \mathcal{P}_n$ and any $K > 1$*

$$\mathcal{R}_\sigma f(t) \le 2^{\bar{r}_j} \mathcal{R}_{\sigma_j'} f\left(K t_{m_j}, \frac{\hat{t}_{m_j}}{2}\right) + c(K)\pi(t)^{-1/p}\omega_j^{\bar{r}_j}(f; t_{m_j})_p \tag{3.14}$$ e21prime

and

$$\mathcal{R}_\sigma f_{j,h}(t) \le \pi(t)^{-1/p}\omega_j^{\bar{r}_j}(f; h)_p, \tag{3.15}$$ e22prime

where σ_j' is obtained from σ by moving the jth index to the first position and $m_j = \sigma^{-1}(j)$.

Proof. By [11, (3.3)], we have for any $K > 1$

$$\mathcal{R}_\sigma f(t) \le 2^{\bar{r}_j} \mathcal{R}_{\sigma_j'} f\left(K t_{m_j}, \frac{\hat{t}_{m_j}}{2}\right) + \mathcal{R}_\sigma \Phi_j\left(\frac{t}{2}\right), \tag{3.16}$$ e13

where

$$\Phi_j(x) = \frac{1}{t_{m_j}} \int_0^{(\bar{r}_j+1)K t_{m_j}} |\Delta_j^{\bar{r}_j}(h) f(x)| dh.$$

Besides, by (2.1),

$$\mathcal{R}_\sigma \Phi_j\left(\frac{t}{2}\right) \le \Phi_j^*\left(\frac{\pi(t)}{2^n}\right). \tag{3.17}$$ e14

We choose a measurable set $E \subset \mathbb{R}^n$ such that $|E| \geq \frac{\pi(t)}{2^n}$ and $|\Phi_j(x)| \geq \Phi_j^*\left(\frac{\pi(t)}{2^n}\right)$ for all $x \in E$. Integrating over E, applying Fubini theorem and using Hölder's inequality, we get

$$\Phi_j^*\left(\frac{\pi(t)}{2^n}\right) \leq \frac{1}{|E|}\int_E \Phi_j(x)dx = \frac{1}{|E|t_{m_j}}\int_0^{(\bar{r}_j+1)Kt_{m_j}}\left(\int_E |\Delta_j^{\bar{r}_j}(h)f(x)|dx\right)dh \leq$$

$$\leq \frac{1}{|E|^{1/p}t_{m_j}}\int_0^{(\bar{r}_j+1)Kt_{m_j}} \|\Delta_j^{\bar{r}_j}(h)f\|_p dh \leq c(K)\pi(t)^{-1/p}\omega_j^{\bar{r}_j}(f;t_{m_j})_p. \quad (3.18)$$ e15

Now (3.16), (3.17) and (3.18) imply (3.14). Inequality (3.15) is immediate; indeed, we have

$$\mathcal{R}_\sigma f_{j,h}(t) \leq f_{j,h}^*(\pi(t)) \leq \pi(t)^{-1/p}\|f_{j,h}\|_p \leq \pi(t)^{-1/p}\omega_j^{\bar{r}_j}(f;h)_p.$$

□

estim1 *Remark* 1. If $f \in \left[\bigcap_{j:r_j\in\mathbb{N}} W_{p;j}^{r_j}(\mathbb{R}^n)\right] \cap \left[\bigcap_{j:r_j\notin\mathbb{N}} H_{p;j}^{r_j}(\mathbb{R}^n)\right]$, then we can simultaneously apply the estimates obtained in Lemmas 2 and 3. Let $\sigma \in \mathcal{P}_n$, $K > 1$, and $0 < \delta < 1$. If $r_j \in \mathbb{N}$, choose $l_j \neq \sigma^{-1}(j)$ and denote by ϕ_j the function $\phi \equiv \phi_{j,l_j,\sigma,\delta}$ defined in Lemma 2. If $r_j \notin \mathbb{N}$ denote $\Omega_j \equiv \sup_{u>0} u^{-r_j}\omega_j^{\bar{r}_j}(f;u)_p$. Now, combining (3.6) and (3.14)

$$\mathcal{R}_\sigma f(t) \leq 2^{\bar{r}'}\sum_{j=1}^n \mathcal{R}_{\sigma_j'} f\left(Kt_{m_j}, \frac{\hat{t}_{m_j}}{2}\right) + c(K)\pi(t)^{-1/p}\rho_\sigma(t), \quad (3.19)$$ e21cor

where $m_j = \sigma^{-1}(j)$, $\bar{r}' \equiv \max \bar{r}_j$ and

$$\rho_\sigma(t) = \min\{\min_{r_j\in\mathbb{N}}\{t_{m_j}^{r_j}\phi_j(t_{l_j})\}, \min_{r_j\notin\mathbb{N}}\{t_{m_j}^{r_j}\Omega_j\}\}, \quad (3.20)$$ ro

4 The main lemma

mainlema

In this section we prove main lemmas that form the base of our approach (see Lemmas 5, 7, and 8 below). It will be convenient to use the following auxiliary proposition.

lp **Lemma 4.** *Let $m \in \mathbb{N}$; $0 < \alpha_i < \infty$ $(i = 1, \ldots, m)$. Define $\alpha = (\sum_{i=1}^m \alpha_i^{-1})^{-1}$. Let $a, b > 0$ be such that $a/b < \alpha$. Set*

$$\rho(z) = \min\{\lambda, z_1^{\alpha_1}\lambda_1, \ldots, z_m^{\alpha_m}\lambda_m\} \quad (z \in \mathbb{R}_+^m),$$

where $\lambda, \lambda_1, \ldots, \lambda_m$ are positive constants. Then

$$\int_{\mathbb{R}_+^m} \rho(z)^b \pi(z)^{-a} \frac{dz}{\pi(z)} \leq c\lambda^{b-a/\alpha} \prod_{i=1}^{m} \lambda_i^{\frac{a}{\alpha_i}} \qquad (4.1) \quad \boxed{\texttt{lp1}}$$

where c is a constant that only depends on α_i, a, b.

Proof. Set $\rho_i(z_i) = \min\{\lambda, z_i^{\alpha_i}\lambda_i\}$ $i = 1, \ldots, m$. Denote by I the left hand side of (4.1). It is clear that

$$I \leq \int_{\mathbb{R}_+^m} \prod_{i=1}^{m} \rho_i(z_i)^{\frac{b\alpha}{\alpha_i}} z_i^{-a} \frac{dz}{\pi(z)} = \prod_{i=1}^{m} I_i, \qquad (4.2) \quad \boxed{\texttt{lp2}}$$

where

$$I_i = \int_0^\infty \rho_i(z_i)^{\frac{b\alpha}{\alpha_i}} z_i^{-a} \frac{dz_i}{z_i}.$$

Now,

$$I_i = \lambda_i^{\frac{b\alpha}{\alpha_i}} \int_0^{(\frac{\lambda}{\lambda_i})^{1/\alpha_i}} z_i^{b\alpha-a} \frac{dz_i}{z_i} + \lambda^{\frac{b\alpha}{\alpha_i}} \int_{(\frac{\lambda}{\lambda_i})^{1/\alpha_i}}^\infty z_i^{-a} \frac{dz_i}{z_i} = c\left[\lambda^{b\alpha-a}\lambda_i^a\right]^{1/\alpha_i}. \qquad (4.3) \quad \boxed{\texttt{lp3}}$$

By (4.2) and (4.3), we obtain immediately (4.1). □

From now on, let $n \in \mathbb{N}, 0 < r_i < \infty, 1 \leq \theta_i \leq \infty$. Assume that $\phi_i \in \mathcal{L}^{\theta_i}$ are positive functions ($i = 1, \ldots, n$). Define

$$\rho(t) = \min\{t_1^{r_1}\phi_1(t_{l_1}), t_2^{r_2}\phi_2(t_{l_2}), \ldots, t_n^{r_n}\phi_n(t_{l_n})\}, \qquad (4.4) \quad \boxed{\texttt{defro}}$$

where $l_1, \ldots, l_n \in \{1, \ldots, n\}$.

Remark 2. Note that the function $\rho_\sigma(t)$ defined in (3.20) is a particular case of (4.4) ($\theta_i = p$ if $r_i \in \mathbb{N}$, $\theta_i = \infty$, $\phi_i \equiv \Omega_i$ if $r_i \notin \mathbb{N}$).

The lemma below gives us the integrability for functions of the type (4.4).

$\boxed{\texttt{mainl}}$ **Lemma 5 (The main Lemma).** *Let $n \in \mathbb{N}, 0 < r_i < \infty, 1 \leq \theta_i \leq \infty$ ($i = 1, \ldots, n$). Let $\rho(t) = \min_{1\leq i\leq n}\{t_i^{r_i}\phi_i(t_{l_i})\}$, $\phi_i \in \mathcal{L}^{\theta_i}$, $l_i \in \{1, \ldots, n\}$. Set*

$$r = n\left(\sum_{i=1}^{n} \frac{1}{r_i}\right)^{-1}, \quad s = \frac{n}{r}\left(\sum_{i=1}^{n} \frac{1}{r_i\theta_i}\right)^{-1}. \qquad (4.5) \quad \boxed{\texttt{rys}}$$

Then

$$\left(\int_{\mathbb{R}_+^n} \rho(t)^s \pi(t)^{-rs/n} \frac{dt}{\pi(t)}\right)^{\frac{1}{s}} \leq c \sum_{i=1}^{n} \|\phi_i\|_{\mathcal{L}^{\theta_i}}, \tag{4.6}$$

ml2

where c is a finite constant that only depends on n, r_i, θ_i.

Proof. We can assume that

$$\sum_{i=1}^{n} \|\phi_i\|_{\mathcal{L}^{\theta_i}} = 1. \tag{4.7}$$

ml3

Besides, we can suppose that not all θ_i's are equal to infinity[1].

Denote $\phi(u) = \sum_{i=1,\ldots,n;\theta_i \neq \infty} \phi_i(u)^{\theta_i}$ $(u > 0)$. By (4.7), $\|\phi\|_{\mathcal{L}^1} \leq 1$. Set

$$B_k = \{t \in \mathbb{R}_+^n : \max_{i=1,\ldots,n;\theta_i \neq \infty} \phi(t_{l_i}) \leq \phi(t_k)\}.$$

It is clear that $\bigcup_{k=1}^n B_k = \mathbb{R}_+^n$.

Without loss of generality we consider the integral of the left part of (4.6) only over B_1. We get for almost all $t \in B_1$

$$\rho(t) \leq \rho_{B_1}(t) \equiv \min\{t_1^{r_1}\phi(t_1)^{1/\theta_1}, t_2^{r_2}\phi(t_1)^{1/\theta_2}, \ldots, t_n^{r_n}\phi(t_1)^{1/\theta_n}\}. \tag{4.8}$$

ml4

From here

$$\int_{B_1} \rho(t)^s \pi(t)^{-rs/n} \frac{dt}{\pi(t)} \leq \int_{\mathbb{R}_+^n} \rho_{B_1}(t)^s \pi(t)^{-rs/n} \frac{dt}{\pi(t)} =$$

$$= \int_0^\infty t_1^{-rs/n} \frac{dt_1}{t_1} \int_{\mathbb{R}_+^{n-1}} \rho_{B_1}(t_1, \hat{t}_1)^s \pi(\hat{t}_1)^{-rs/n} \frac{d\hat{t}_1}{\pi(\hat{t}_1)}. \tag{4.9}$$

paso1

For each fixed $t_1 \in \mathbb{R}_+$, applying Lemma 4 and (4.5), we get

$$\int_{\mathbb{R}_+^{n-1}} \rho_{B_1}(t_1, \hat{t}_1)^s \pi(\hat{t}_1)^{-rs/n} d\hat{t}_1 \leq$$

$$\leq c[t_1^{r_1}\phi(t_1)^{1/\theta_1}]^{s - \sum_{i=2}^n \frac{rs}{nr_i}} \phi(t_1)^{\sum_{i=2}^n \frac{rs}{n\theta_i r_i}} = c\phi(t_1) t_1^{rs/n}. \tag{4.10}$$

paso2

Since $\|\phi\|_{\mathcal{L}^1} \leq 1$, (4.9) and (4.10) yield that

$$\int_{B_1} \rho(t)^s \pi(t)^{-rs/n} \frac{dt}{\pi(t)} \leq c.$$

□

[1] otherwise $s = \infty$ and the result is trivial.

We will obtain a generalization of Lemma 5. For this purpose, we need the following Hardy type inequality.

auxl **Lemma 6.** *Let φ a measurable non-negative function on $\mathbb{R}_+$. Let δ, $\alpha > 0$ and let $1 \le \gamma < \infty$. Assume that β is a measurable and positive function on $\mathbb{R}_+$ such that $\beta(u)u^{-\delta}$ increases. Then*

$$\int_0^\infty h^{-\alpha-1}dh \left(\int_{\{h \ge \beta(u)\}} \varphi(u) \frac{du}{u} \right)^\gamma \le c \int_0^\infty \beta(u)^{-\alpha} \varphi(u)^\gamma \frac{du}{u} \tag{4.11}$$ al1

and

$$\int_0^\infty h^{\alpha-1}dh \left(\int_{\{h \le \beta(u)\}} \varphi(u) \frac{du}{u} \right)^\gamma \le c \int_0^\infty \beta(u)^{\alpha} \varphi(u)^\gamma \frac{du}{u}. \tag{4.12}$$ al2

where c is a constant that only depends on α, δ and γ.

Proof. As $\beta(u)u^{-\delta} \uparrow$, this implies that the inverse function β^{-1} exists on $\mathbb{R}_+$ and satisfies the condition

$$\beta^{-1}(2u) \le 2^{1/\delta} \beta^{-1}(u) \tag{4.13}$$ al3

Denote by I the left hand side of (4.11). We have

$$I \equiv \int_0^\infty h^{-\alpha-1}dh \left(\int_0^{\beta^{-1}(h)} \varphi(u) \frac{du}{u} \right)^\gamma =$$

$$= \int_0^\infty h^{-\alpha-1}dh \left(\sum_{k=0}^\infty \int_{\beta^{-1}(2^{-k-1}h)}^{\beta^{-1}(2^{-k}h)} \varphi(u) \frac{du}{u} \right)^\gamma .$$

Next, by Minkowski's inequality

$$I^{1/\gamma} \le \sum_{k=0}^\infty \left(\int_0^\infty h^{-\alpha-1}dh \left(\int_{\beta^{-1}(2^{-k-1}h)}^{\beta^{-1}(2^{-k}h)} \varphi(u) \frac{du}{u} \right)^\gamma \right)^{1/\gamma} =$$

$$= \sum_{k=0}^\infty 2^{-k\alpha/\gamma} \left(\int_0^\infty z^{-\alpha-1}dz \left(\int_{\beta^{-1}(z/2)}^{\beta^{-1}(z)} \varphi(u) \frac{du}{u} \right)^\gamma \right)^{1/\gamma} .$$

Further, using the Hölder's inequality and (4.13)

$$\int_{\beta^{-1}(z/2)}^{\beta^{-1}(z)} \varphi(u) \frac{du}{u} \le c \left(\int_0^{\beta^{-1}(z)} \varphi(u)^\gamma \frac{du}{u} \right)^{1/\gamma} .$$

Thus, by Fubini's theorem

$$I \leq c\int_0^\infty z^{-\alpha-1}dz\int_0^{\beta^{-1}(z)}\varphi(u)^\gamma\frac{du}{u} = c\int_0^\infty \beta(u)^{-\alpha}\varphi(u)^\gamma\frac{du}{u}.$$

The same reasonings prove (4.12). □

secl **Lemma 7.** *Assume that the conditions of Lemma 5 hold and suppose that there exists* $0<\delta\leq\frac{1}{2}\min_{1\leq i,k\leq n,\theta_k\neq\infty}\{\frac{r_i}{\theta_k}\}$ *such that*

$$\phi_i(u)u^\delta\uparrow \quad \textit{and} \quad \phi_i(u)u^{-\delta}\downarrow \tag{4.14}$$ crecdecrec

for every i such that $\theta_i<\infty$. Then, for any $0<d\leq\infty$ and $j\in\{1,\dots,n\}$

$$\left(\int_0^\infty \left\|\rho(t)^s\pi(t)^{-rs/n}\right\|_{L^d(\mathbb{R}_+^{n-1},\frac{d\hat{t}_j}{\pi(\hat{t}_j)})}\frac{dt_j}{t_j}\right)^{\frac{1}{s}} \leq c\sum_{i=1}^n\|\phi_i\|_{\mathcal{L}^{\theta_i}} \tag{4.15}$$ sl1

where c is a constant that depends on $n, r_i, \theta_i, d, \delta$.

Note that the greater is d, the weaker is (4.15). Indeed, by (4.14), $\rho(t)\pi(t)^\delta$ is increasing at each one of its variables. So, it is easy to see that

$$\sup_{\hat{t}_j\in\mathbb{R}_+^{n-1}}\rho(t)^s\pi(t)^{-rs/n}\leq c\|\rho(t)^s\pi(t)^{-rs/n}\|_{L^d(\mathbb{R}_+^{n-1},\frac{d\hat{t}_j}{\pi(\hat{t}_j)})} \quad \text{for any } 0<d<\infty.$$

From here, it follows that if $q>d>0$, then

$$\begin{aligned}&\|\rho(t)^s\pi(t)^{-rs/n}\|_{L^q(\mathbb{R}_+^{n-1},\frac{d\hat{t}_j}{\pi(\hat{t}_j)})}\leq\\ &\leq[\|\rho(t)^s\pi(t)^{-rs/n}\|_{L^d(\mathbb{R}_+^{n-1},\frac{d\hat{t}_j}{\pi(\hat{t}_j)})}]^{\frac{q-d}{q}}\left(\int_{\mathbb{R}_+^{n-1}}[\rho(t)^s\pi(t)^{-rs/n}]^d\frac{d\hat{t}_j}{\pi(\hat{t}_j)}\right)^{1/q}\leq\\ &\qquad\leq c\|\rho(t)^s\pi(t)^{-rs/n}\|_{L^d(\mathbb{R}_+^{n-1},\frac{d\hat{t}_j}{\pi(\hat{t}_j)})}.\end{aligned}$$

Note also that for $d=1$ we get the same conclusion as in Lemma 5. So, for the proof, we can suppose that $0<d<1$.

Proof. As above, we can suppose that the condition (4.7) holds. Let ϕ and B_k $(k=1,\dots,n)$ be defined as in Lemma 5. Then the left hand side of (4.15) does not exceed the sum $\sum_{k=1}^n I_k$, where

$$I_k=\left(\int_0^\infty\left\|\rho(t)^s\pi(t)^{-rs/n}\chi_{B_k}(t)\right\|_{L^d(\mathbb{R}_+^{n-1},\frac{d\hat{t}_j}{\pi(\hat{t}_j)})}\frac{dt_j}{t_j}\right)^{\frac{1}{s}}.$$

We consider I_1. For almost all $t \in B_1$ we have the inequality (4.8). Thus,

$$I_1^s \le \int_0^\infty \|\rho_{B_1}(t)^s \pi(t)^{-rs/n}\|_{L^d(\mathbb{R}_+^{n-1}, \frac{d\hat{t}_j}{\pi(\hat{t}_j)})} \frac{dt_j}{t_j}.$$

Case 1. If $j = 1$, then

$$I_1^s \le c \int_0^\infty t_1^{-rs/n} G(t_1) \frac{dt_1}{t_1},$$

where

$$G(t_1)^d \equiv \int_{\mathbb{R}_+^{n-1}} [\rho_{B_1}(t)^s \pi(\hat{t}_1)^{-rs/n}]^d \frac{d\hat{t}_1}{\pi(\hat{t}_1)}.$$

Applying Lemma 4 to the variables $t_2, \dots, t_n$, we easily get

$$G(t_1)^d \le c[t_1^{r_1} \phi(t_1)^{1/\theta_1}]^{\frac{rsd}{nr_1}} \phi(t_1)^{d \sum_{i=2}^n \frac{rs}{nr_i\theta_i}} = [t_1^{rs/n} \phi(t_1)]^d.$$

This implies (4.15).

Case 2. Let $j \ne 1$. For $t \in \mathbb{R}_+^n$, denote by $\hat{t}_{1,j}$ the $(n-2)$-dimensional vector obtained from t by removal of t_1, t_j. Then

$$I_1^s \le c \int_0^\infty t_j^{-rs/n} \|R(t_1, t_j)\|_{\mathcal{L}^d, (t_1)} \frac{dt_j}{t_j}, \tag{4.16}$$

sl3

where

$$R(t_1, t_j)^d = t_1^{-rsd/n} \int_{\mathbb{R}_+^{n-2}} [\pi(\hat{t}_{1,j})^{-rs/n} \rho_{B_1}(t)^s]^d \frac{d\hat{t}_{1,j}}{\pi(\hat{t}_{1,j})}.$$

Fix t_1, t_j and apply Lemma 4 to coordinates of the vector $\hat{t}_{1,j}$. We obtain

$$R(t_1, t_j) \le c t_1^{-rs/n} \min\{t_1^{r_1} \phi(t_1)^{1/\theta_1}, t_j^{r_j} \phi(t_1)^{1/\theta_j}\}^{\frac{rs}{n}(\frac{1}{r_1} + \frac{1}{r_j})} \phi(t_1)^{\sum_{i \ne 1,j} \frac{rs}{nr_i\theta_i}}.$$

Now, we define $\beta(t_1) = [t_1^{r_1} \phi(t_1)^{1/\theta_1 - 1/\theta_j}]^{1/r_j}$. Note that by (4.14)

$$\beta(t_1) t_1^{-\delta/r_j} \uparrow . \tag{4.17}$$

betacrec

Besides ($b \equiv \frac{rs}{nr_j\theta_1} + \sum_{i \ne j} \frac{rs}{nr_i\theta_i}$, $b' \equiv \frac{rs}{nr_1\theta_j} + \sum_{i \ne 1} \frac{rs}{nr_i\theta_i}$),

$$R(t_1, t_j) \le c \begin{cases} t_1^{\frac{rs}{n}\frac{r_1}{r_j}} \phi(t_1)^b \equiv R_1(t_1), & \text{if } \beta(t_1) \le t_j, \\ t_j^{\frac{rs}{n}(1 + \frac{r_j}{r_1})} t_1^{-\frac{rs}{n}} \phi(t_1)^{b'} \equiv t_j^{\frac{rs}{n}(1 + \frac{r_j}{r_1})} R_2(t_1), & \text{if } \beta(t_1) \ge t_j. \end{cases} \tag{4.18}$$

sl4

Joining (4.16) and (4.18), we get

$$I_1^s \le c \int_0^\infty t_j^{-\frac{rs}{n}} \frac{dt_j}{t_j} \left(\int_{\{t_j \ge \beta(t_1)\}} R_1(t_1)^d \frac{dt_1}{t_1} \right)^{1/d} +$$

$$+ c \int_0^\infty t_j^{\frac{rs}{n}\frac{r_j}{r_1}} \frac{dt_j}{t_j} \left(\int_{\{t_j \le \beta(t_1)\}} R_2(t_1)^d \frac{dt_1}{t_1} \right)^{1/d}.$$

Taking into account (4.17), we apply Lemma 6 with $\gamma = 1/d$ ($\gamma > 1$). Using the definitions of β, R_1, and R_2, and (4.5), we obtain

$$I_1^s \le c \int_0^\infty \beta(t_1)^{-\frac{rs}{n}} R_1(t_1) \frac{dt_1}{t_1} + c \int_0^\infty \beta(t_1)^{\frac{rs}{n}\frac{r_j}{r_1}} R_2(t_1) \frac{dt_1}{t_1} = c' \int_0^\infty \phi(t_1) \frac{dt_1}{t_1}.$$

□

We will use also the following generalization of Lemma 7.

desequilibrio **Lemma 8.** *Let* $m \in \mathbb{N}$, $0 < r_i < \infty$, $1 \le \theta_i \le \infty$ ($i = 1, \ldots, m$). *Define the function* $\rho(z) = \min_{1 \le i \le m}\{z_i^{r_i} \phi_i(z_{l_i})\}$, $\phi_i \in \mathcal{L}^{\theta_i}$, $l_i \in \{1, \ldots, m\}$. *Suppose also that there exists* $0 < \delta \le \frac{1}{2} \min_{1 \le i,k \le m, \theta_k \ne \infty}\{\frac{r_i}{\theta_k}\}$ *such that*

$$\phi_i(u) u^\delta \uparrow \quad \text{and} \quad \phi_i(u) u^{-\delta} \downarrow$$

for every i *such that* $\theta_i < \infty$. *Let* $0 < a_i < \infty$ *be numbers verifying*

$$\sum_{i=1}^m \frac{a_i}{r_i \theta_i} = 1. \tag{4.19}$$ conda

Set $a \equiv \sum_{i=1}^m \frac{a_i}{r_i}$. *Then, for any* $0 < d \le \infty$ *and* $j \in \{1, \ldots, m\}$

$$\left(\int_0^\infty \| \rho(z)^a \prod_{i=1}^m z_i^{-a_i} \|_{L^d(\mathbb{R}_+^{m-1}, \frac{d\hat{z}_j}{\pi(\hat{z}_j)})} \frac{dz_j}{z_j} \right)^{\frac{1}{a}} \le c \sum_{i=1}^m \|\phi_i\|_{\mathcal{L}^{\theta_i}}, \tag{4.20}$$ cor1

where c *is a constant that depends on* $m, r_i, a_i, \theta_i, d, \delta$.

Note that Lemma 7 is the particular case $a_1 = \ldots = a_m$.

Proof. Let J be the left hand side of (4.20). The change of variable $z_i^{a_i/b} = u_i$ ($b = \min_{1\le k\le m} a_k$) gives us

$$J = c\left(\int_0^\infty \|\rho(u)^a \pi(u)^{-b}\|_{L^d(\mathbb{R}_+^{m-1}, \frac{d\hat{u}_j}{\pi(\hat{u}_j)})} \frac{du_j}{u_j}\right)^{\frac{1}{a}}$$

where $\rho(u) = \min_{1\le i\le m}\{u_i^{r_i'} F_i(u_{l_i})\}$, $r_i' = r_i b/a_i$, and $F_i(v) = \phi_i(v^{b/a_{l_i}})$ belongs to $\mathcal{L}^{\theta_i}$. Note now that $F_i(v)v^{\delta} \uparrow$ and $F_i(v)v^{-\delta} \downarrow$. So, it remains to apply Lemma 7 to the last integral ($r' = m(\sum 1/r_i')^{-1} = a^{-1}bm$ and $r's'/m = (\sum 1/r_i'\theta_i)^{-1} = b$ by (4.19)) and we get (4.20). $\square$

5 Embeddings of Lipschitz spaces

imbed

tl

Theorem 1. *Let $2 \le n \in \mathbb{N}$, $1 \le p < \infty$, $0 < r_i < \infty$ ($i = 1, \ldots, n$). Set*

$$r = n\left(\sum_{i=1}^n \frac{1}{r_i}\right)^{-1}, \quad r' = n\left(\sum_{i:r_i\in\mathbb{N}} \frac{1}{r_i}\right)^{-1}, \ s = \frac{r'p}{r}, \quad q^* = \frac{np}{n-rp}. \tag{5.1}$$

tl0

Then, if $p < n/r$

$$\|f\|_{q^*,s;\mathcal{R}} \le c\|f\|_{\lambda_p^{r_1,\ldots,r_n}} \qquad \text{for all } f \in \Lambda_p^{r_1,\ldots,r_n}(\mathbb{R}^n) \tag{5.2}$$

tl0bis

where c is a constant that doesn't depend on f.

Proof. First suppose that $f \in C_0^\infty(\mathbb{R}^n)$. Let $S = \|f\|_{q^*,s;\mathcal{R}}$. So, $S < \infty$.

It is well known that if $1 < p \le \infty$,

$$\|f\|_{\lambda_p^{r_1,\ldots,r_n}} \sim \sum_{j:r_j\in\mathbb{N}} \|D_j^{r_j} f\|_p + \sum_{j:r_j\notin\mathbb{N}} \sup_{u>0} u^{-r_j}\omega_j^{\bar{r}_j}(f;u)_p, \tag{5.3}$$

LWH

and it is still true for $p = 1$ restricted to functions in $C^\infty(\mathbb{R}^n)$ [10].

Now, taking into account (5.3) and Remark 1, we integrate inequality (3.19) and get for any $\sigma \in \mathcal{P}_n$

$$\left(\int_{\mathbb{R}_+^n} \pi(t)^{s/q^*-1} \mathcal{R}_\sigma f(t)^s dt\right)^{1/s} \le 2^{\bar{r}'+n} K^{-1/q^*} S + c(K)\left(\int_{\mathbb{R}_+^n} \pi(t)^{-\frac{rs}{n}-1} \rho_\sigma(t)^s dt\right)^{1/s}$$

with $\rho_\sigma(t)$ defined in (3.20). Consequently

$$S = \sum_{\sigma\in\mathcal{P}_n} \|f\|_{q^*,s;\mathcal{R}_\sigma} \le n!2^{\bar{r}'+n}K^{-1/q^*}S+$$

$$+c'(K)\sum_{\sigma\in\mathcal{P}_n}\left(\int_{\mathbb{R}^n_+}\pi(t)^{-\frac{rs}{n}-1}\rho_\sigma(t)^s dt\right)^{1/s}. \tag{5.4}$$ tl4

Now, we apply Lemma 5 with $\theta_i = p$ if $r_i \in \mathbb{N}$; $\theta_i = \infty$ and $\phi_i = \Omega_i$ if $r_i \notin \mathbb{N}$ (observe that the values of s in (5.1) and (4.5) coincide) and get

$$\left(\int_{\mathbb{R}^n_+}\pi(t)^{-\frac{rs}{n}-1}\rho_\sigma(t)^s dt\right)^{1/s} \le c\left[\sum_{r_i\in\mathbb{N}}\|\phi_i\|_{\mathcal{L}^p} + \sum_{r_i\notin\mathbb{N}}\Omega_i\right]. \tag{5.5}$$ tl5

Therefore, setting $K = (2^{\bar{r}'+n+1}n!)^{q^*}$, and using (5.4), (5.5), the definition of Ω_i, and (3.4) we obtain the inequality (5.2).

For $f \in \Lambda_p^{r_1,\dots,r_n}(\mathbb{R}^n)$, there exists a sequence of functions $f_k \in C_0^\infty(\mathbb{R}^n)$ such that $\lim_{k\to\infty}\|f_k\|_{\lambda_p^{r_1,\dots,r_n}} \le \|f\|_{\lambda_p^{r_1,\dots,r_n}}$ and $f_k \to f$ in L^p (use ϵ-regularizations and cut-off). So, applying Lemma 2 of [11] and Fatou's Lemma we obtain (5.2) in the general case. □

Remark 3. If all the r_i's are integers, then $s = p$ and we get the embedding of anisotropic Sobolev spaces into Lorentz spaces proved earlier in [9, 11]. In the general case, assume that $s \le q^*$. Then Theorem 1 yields an alternative proof of the results concerning embeddings into L^{q^*} [5] and $L^{q^*,s}$ [14, 10].

tb **Theorem 2.** *Let $2 \le n \in \mathbb{N}$, $1 \le p < q < \infty$. Let $0 < r_i < \infty$ $(i = 1,\dots,n)$. Define r, s as in (5.1). Suppose that*

$$\varkappa = 1 - \frac{n}{r}\left(\frac{1}{p} - \frac{1}{q}\right) > 0$$

and define

$$\alpha_i = \varkappa r_i, \quad \frac{1}{\gamma_i} = \begin{cases} \frac{1-\varkappa}{s} + \frac{\varkappa}{p}, & \text{if } r_i \in \mathbb{N}, \\ \frac{1-\varkappa}{s}, & \text{if } r_i \notin \mathbb{N}. \end{cases}$$

Then, for any $f \in \Lambda_p^{r_1,\dots,r_n}(\mathbb{R}^n)$,

$$\sum_{i=1}^n\left(\int_0^\infty\left[h^{-\alpha_i}\|\Delta_i^{\bar{r}_i}(h)f\|_{q,1;\mathcal{R}}\right]^{\gamma_i}\frac{dh}{h}\right)^{1/\gamma_i} \le c\|f\|_{\lambda_p^{r_1,\dots,r_n}(\mathbb{R}^n)}, \tag{5.6}$$ tb1

where c is a constant that does not depend on f.

Proof. Consider the first term in the left hand side of (5.6). Denote $f_{1,h}(x) = \Delta_1^{\bar{r}_1}(h)f(x)$. Estimate $J(h) = \|f_{1,h}\|_{q,1;\mathcal{R}} < \infty$. As in Theorem 1, we can suppose that $f \in C_0^\infty(\mathbb{R}^n)$.

Define now

$$\delta \equiv \frac{1}{2}\min\{\frac{r}{n}(1-\varkappa), \frac{1}{p}\min_j r_j\}.$$

Then, by (5.3), we proceed similarly to Remark 1. Applying Lemmas 2 and 3 to $f_{1,h}$, we easily obtain that for any $K > 1$

$$\mathcal{R}_\sigma f_{1,h}(t) \le 2^{\bar{r}'}\sum_{j=1}^n \mathcal{R}_{\sigma'_j} f_{1,h}\left(Kt_{m_j}, \frac{\hat{t}_{m_j}}{2}\right) + c(K)\pi(t)^{-1/p}\rho_\sigma(t),$$

where $\rho_\sigma(t)$ is defined in (3.20). Moreover, from (3.7) and (3.15) it follows that if $h < t_{m_1}$,

$$\mathcal{R}_\sigma f_{1,h}(t) \le c\pi(t)^{-1/p}\mu(t,h),$$

where

$$\mu(t,h) = \begin{cases} h^{r_1-\delta}t_{m_1}^\delta\phi_1(t_{l_1}), & \text{if } r_1 \in \mathbb{N}, \\ h^{r_1}\Omega_1, & \text{if } r_1 \notin \mathbb{N}, \end{cases} \qquad (5.7) \quad \boxed{\texttt{mu}}$$

and $\phi_1(t_{l_1})$ is defined in Lemma 2. Furthermore, setting

$$\bar{\rho}_\sigma(t,h) = \min\{\rho_\sigma(t), \mu(t,h)\}, \qquad (5.8) \quad \boxed{\texttt{robarra}}$$

we get for any $K > 1$

$$\mathcal{R}_\sigma f_{1,h}(t) \le 2^{\bar{r}'}\sum_{j=1}^n \mathcal{R}_{\sigma'_j} f_{1,h}\left(Kt_{m_j}, \frac{\hat{t}_{m_j}}{2}\right) + c(K)\pi(t)^{-1/p}\bar{\rho}_\sigma(t,h).$$

Multiplying by $\pi(t)^{1/q-1}$ and integrating over $\mathbb{R}^n_+$, we obtain

$$\|f_{1,h}\|_{q,1;\mathcal{R}_\sigma} \le 2^{\bar{r}'+n}K^{-1/q}J(h) + c(K)J_1(h,\sigma), \qquad (5.9) \quad \boxed{\texttt{aster}}$$

where

$$J_1(h,\sigma) = \int_{\mathbb{R}^n_+} \pi(t)^{-\frac{r}{n}(1-\varkappa)}\bar{\rho}_\sigma(t,h)\frac{dt}{\pi(t)}. \qquad (5.10) \quad \boxed{\texttt{jota1}}$$

Summing up inequalities (5.9) over all $\sigma \in \mathcal{P}_n$ and choosing $K = (2^{\bar{r}'+n+1}n!)^q$, we get

$$J(h) \le c' \sum_{\sigma \in \mathcal{P}_n} J_1(h,\sigma). \qquad (5.11) \quad \boxed{\texttt{jotajota1}}$$

Furthermore, denoting by I the first term of the left hand side of (5.6), we have (let's suppose that $\gamma_1 < \infty$[2])

$$I \equiv \left(\int_0^\infty h^{-\alpha_1\gamma_1 - 1} J(h)^{\gamma_1} dh \right)^{1/\gamma_1} \leq c' \sum_{\sigma \in \mathcal{P}_n} \bar{I}(\sigma),$$

where (by (5.11))

$$\bar{I}(\sigma) = \left(\int_0^\infty h^{-\alpha_1\gamma_1} J_1(h,\sigma)^{\gamma_1} \frac{dh}{h} \right)^{1/\gamma_1}. \qquad (5.12) \quad \boxed{\texttt{i1sigma}}$$

But now, it is clear that (see (5.8), (3.20) and (5.7))

$$\bar{\rho}_\sigma(t,h) \leq [1 + (\frac{t_{m_1}}{h})^\delta] \tilde{\rho}_\sigma(t,h) \qquad (5.13) \quad \boxed{\texttt{suma}}$$

where

$$\tilde{\rho}_\sigma(t,h) \equiv \begin{cases} \min\{h^{r_1}\phi_1(t_{l_1}), \rho_\sigma(t)\}, & \text{if } r_1 \in \mathbb{N} \\ \min\{h^{r_1}\Omega_1, \rho_\sigma(t)\}, & \text{if } r_1 \notin \mathbb{N}. \end{cases} \qquad (5.14) \quad \boxed{\texttt{rotilde}}$$

So, due to (5.12), (5.10) and (5.13) we have

$$\bar{I}(\sigma) \leq c\tilde{I}_1(\sigma) + c\tilde{I}_2(\sigma),$$

where

$$\tilde{I}_1(\sigma) \equiv \left(\int_0^\infty \| h^{-\alpha_1\gamma_1} \pi(t)^{-\frac{r}{n}(1-\varkappa)\gamma_1} \tilde{\rho}_\sigma(t,h)^{\gamma_1} \|_{L^{1/\gamma_1}(\mathbb{R}^n_+, \frac{dt}{\pi(t)})} \frac{dh}{h} \right)^{1/\gamma_1}$$

and

$$\tilde{I}_2(\sigma) \equiv \left(\int_0^\infty \| h^{-\alpha_1\gamma_1} \pi(t)^{-\frac{r}{n}(1-\varkappa)\gamma_1} (\frac{t_{m_1}}{h})^{\delta\gamma_1} \tilde{\rho}_\sigma(t,h)^{\gamma_1} \|_{L^{1/\gamma_1}(\mathbb{R}^n_+, \frac{dt}{\pi(t)})} \frac{dh}{h} \right)^{1/\gamma_1}.$$

It remains to estimate the last two integrals.

Joining (5.14) and (3.20) we get

$$\tilde{\rho}_\sigma(t,h) = \begin{cases} \min\{h^{r_1}\phi_1(t_{l_1}), \min_{r_j \in \mathbb{N}}\{t_{m_j}^{r_j}\phi_j(t_{l_j})\}, \min_{r_j \notin \mathbb{N}}\{t_{m_j}^{r_j}\Omega_j\}\}, & \text{if } r_1 \in \mathbb{N} \\ \min\{h^{r_1}\Omega_1, \min_{r_j \in \mathbb{N}}\{t_{m_j}^{r_j}\phi_j(t_{l_j})\}, \min_{r_j \notin \mathbb{N}}\{t_{m_j}^{r_j}\Omega_j\}\}, & \text{if } r_1 \notin \mathbb{N}. \end{cases}$$

[2]Otherwise, none of the r_i's belongs to $\mathbb{N}$, and the analogous of (5.6) follows from (5.11), (5.10) and Lemma 4.

So, $\tilde{\rho}_\sigma(t,h)$ has the form of $\rho(z)$ in Lemma 8 ($\rho(z) = \min_{1\le i\le m}\{z_i^{r_i}\phi_i(z_{l_i})\}$, $\phi_i \in \mathcal{L}^{\theta_i}$). Indeed, $m = n+1$, $r_{n+1} = r_1$,

$$z_1 = t_{m_1}, \quad \cdots \quad, \; z_n = t_{m_n}, \quad z_{n+1} = h.$$

$$(1 \le i \le n+1) \begin{cases} \text{If } r_i \in \mathbb{N},\ \theta_i = p \text{ and } \phi_i(v)v^{\delta} \uparrow,\ \phi_i(v)v^{-\delta} \downarrow \text{ (see (3.5))}. \\ \text{If } r_i \notin \mathbb{N},\ \theta_i = \infty \text{ and } \phi_i = \Omega_i. \end{cases}$$

To estimate $\tilde{I}_1(\sigma)$, note that it has the form of the left hand side of (4.20) with

$$a_1 = \cdots = a_n = \frac{r}{n}(1-\varkappa)\gamma_1 > 0,\ a_{n+1} = \alpha_1\gamma_1 > 0,\ d = 1/\gamma_1,\ j = n+1.$$

Then $a = \gamma_1$ and (4.19) holds. Applying Lemma 8 we get

$$\tilde{I}_1(\sigma) \le c \left[\sum_{r_i \in \mathbb{N}} \|\phi_i\|_{\mathcal{L}^p} + \sum_{r_i \notin \mathbb{N}} \Omega_i \right].$$

And by Ω_i definition and (3.4)

$$\tilde{I}_1(\sigma) \le c\|f\|_{\lambda_p^{r_1,\dots,r_n}(\mathbb{R}^n)}.$$

Next, $\tilde{I}_2(\sigma)$ also is similar to the left hand side of (4.20). Indeed,

$$a_{n+1} = (\alpha_1 + \delta)\gamma_1 > 0,\ a_1 = [\frac{r}{n}(1-\varkappa) - \delta]\gamma_1 > 0,$$

$$a_i = \frac{r}{n}(1-\varkappa)\gamma_1 > 0 \ (i = 2,\dots,n),\ d = 1/\gamma_1,\ j = n+1.$$

We have that $a = \gamma_1$ and (4.19) holds. Applying Lemma 8 we obtain the same estimate for $\tilde{I}_2(\sigma)$ as for $\tilde{I}_1(\sigma)$. □

In addition to the Theorem 1 (embedding with limit exponent) we have also the following theorem.

tlnolimite **Theorem 3.** *Let* $2 \le n \in \mathbb{N}$, $1 \le p < q < \infty$, *and* $0 < r_i < \infty$ $(i = 1,\dots,n)$. *Let* r *be as in (5.1). Suppose that* $1 - n/r(1/p - 1/q) > 0$. *Then, for any* $0 < \xi < \infty$

$$\|f\|_{q,\xi;\mathcal{R}} \le c\|f\|_{\Lambda_p^{r_1,\dots,r_n}(\mathbb{R}^n)} \tag{5.15}$$ tln1

where c *is a constant that doesn't depend on* f.

Proof. First of all we can suppose that $f \in C_0^\infty(\mathbb{R}^n)$, so $S \equiv \|f\|_{q,\xi;\mathcal{R}} < \infty$. By (2.4) we can suppose that $0 < \xi < 1$.

By Remark 1 we get (3.19) and (3.20). We define for $\sigma = m^{-1} \in \mathcal{P}_n$ and $j = 1, \dots, n$.

$$A = \{t \in \mathbb{R}^n_+ : t_i \geq 1,\, i = 1, \dots, n\}, \quad A_{\sigma,j} = \{t \in \mathbb{R}^n_+ : \min_{1 \leq i \leq n} t_{m_i}^{r_i} = t_{m_j}^{r_j} < 1\}.$$

It's clear that

$$A \cup \left(\bigcup_{j=1}^n A_{\sigma,j} \right) = \mathbb{R}^n_+.$$

Then, using (3.19),

$$\int_{\mathbb{R}^n_+} \pi(t)^{\xi/q-1} \mathcal{R}_\sigma f(t)^\xi dt \leq (2^{\bar{r}'+n} K^{-1/q} S)^\xi + I_0 + c(K) \sum_{j=1}^n I_j,$$

where

$$I_0 = \int_A \pi(t)^{\xi/q-1} \mathcal{R}_\sigma f(t)^\xi dt \quad \text{and} \quad I_j = \int_{A_{\sigma,j}} \pi(t)^{-\frac{r}{n}(1-\varkappa)\xi - 1} \rho_\sigma(t)^\xi dt.$$

Using the same methods as in Theorem 1, we choose $K = (n! 2^{\bar{r}'+n+1+1/\xi})^q$ and it only remains to estimate I_0 and I_j. Applying Holder's inequality with exponents p/ξ and $(p/\xi)'$, we have (due to $(\xi/q - 1)(p/\xi)' < -1$)

$$I_0 \leq c\|f\|_p^\xi.$$

On the other hand, let ζ be such that

$$1 > \frac{1}{\zeta} = \begin{cases} \frac{1-\varkappa}{s} & \text{if } r_j \notin \mathbb{N}, \\ \frac{1-\varkappa}{s} + \frac{1+\varkappa}{p} & \text{if } r_j \in \mathbb{N}. \end{cases}$$

Then, by Holder's inequality

$$I_j \leq \left(\int_{A_{\sigma,j}} \pi(t)^{-\alpha\zeta'} t_{m_j}^{\beta\zeta'} \frac{dt}{\pi(t)} \right)^{1/\zeta'} \cdot \left(\int_{\mathbb{R}^n_+} \pi(t)^{-\alpha\zeta} t_{m_j}^{-\beta\zeta} \rho_\sigma(t)^{\zeta\xi} \frac{dt}{\pi(t)} \right)^{1/\zeta} \equiv J_1 . J_2.$$

where $\alpha = \frac{r}{2n}(1-\varkappa)\xi$ and $\beta = \frac{r_j}{2}(1+\varkappa)\xi$. The first factor, J_1, is a constant by $A_{\sigma,j}$ definition. Indeed,

$$J_1^{\zeta'} = \int_0^1 \frac{dt_{m_j}}{t_{m_j}} \int_{\{\hat{t}_{m_j} : t_{m_i} \geq t_{m_j}^{r_j/r_i} (i \neq j)\}} \pi(t)^{-\alpha\zeta'} t_{m_j}^{\beta\zeta'} \frac{d\hat{t}_{m_j}}{\pi(\hat{t}_{m_j})} = c.$$

For the second factor, J_2, we apply Lemma 8 with

$$m = n,\ d = 1,\ z_i = t_{m_i},$$

$$a_i = \alpha\zeta \text{ if } i \neq j \quad \text{and } a_j = \alpha\zeta + \beta\zeta$$

(as above $\theta_i = p$ if $r_i \in \mathbb{N}$; and $\theta_i = \infty$, $\phi_i = \Omega_i$ if $r_i \notin \mathbb{N}$). Lastly we use the definition of Ω_i and (3.4). So,

$$J_2 \leq c(\|f\|_{\lambda_p^{r_1,\dots,r_n}(\mathbb{R}^n)})^\xi$$

and (5.15) is proved. □

Remark 4. As q is not a limit exponent, the embedding

$$\Lambda_p^{r_1,\dots,r_n}(\mathbb{R}^n) \hookrightarrow L^{q,\xi}(\mathbb{R}^n)$$

follows easily from the embeddings without limit exponent for spaces of Besov [10] or anothers. The goal of Theorem 3 is to present a proof based on the $\|.\|_{q,\xi,\mathcal{R}}$ norm (which is stronger than the usual Lorentz norm if $\xi < q$).

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