ESTIMATES OF DIFFERENCE NORMS FOR FUNCTIONS IN ANISOTROPIC SOBOLEV SPACES

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Abstract. We investigate the spaces of functions on $\mathbb{R}^n$ for which the generalized partial derivatives $D_k^{r_k} f$ exist and belong to different Lorentz spaces L^{p_k,s_k}. For the functions in these spaces, the sharp estimates of the Besov type norms are found. The methods used in the paper are based on estimates of non-increasing rearrangements. These methods enable us to cover also the case when some of p_k's are equal to 1.

1. Introduction

In this paper we study the spaces of functions f on $\mathbb{R}^n$ which possess the generalized partial derivatives

(1) $$D_k^{r_k} f \equiv \frac{\partial^{r_k} f}{\partial x_k^{r_k}} \qquad (r_k \in \mathbb{N}).$$

Our main goal is to obtain sharp estimates for the norms of the differences

(2) $$\Delta_k^{r_k}(h) f(x) \equiv \sum_{j=0}^{r_k} (-1)^{r_k-j} \binom{r_k}{j} f(x + jhe_k) \qquad (h \in \mathbb{R})$$

(e_k is the unit coordinate vector). We will specify this problem below; here we only note that it was completely solved in the case when all derivatives (1) belong to the same space $L^p(\mathbb{R}^n)$. Nevertheless, it is natural to suppose that the derivatives $D_k^{r_k}$ $(k = 1, \dots, n)$ belong to different spaces L^{p_k}. The corresponding classes of functions naturally appear in the embedding theory as well as in applications. The most extended theory of these classes is contained in the monography [2]. Furthermore, many authors have studied Sobolev and Nikol'skii-Besov spaces whose construction involves, instead of L^p-norms, norms in more general spaces (see [12]). In this paper we suppose that derivatives belong to different Lorentz spaces $L^{p_k,s_k}(\mathbb{R}^n)$ (where $1 \le p_k, s_k < \infty$ and $s_k = 1$, if $p_k = 1$). Note that very interesting comments and results concerning this type of Sobolev spaces can be found in [19]. There are many important problems in Analysis which lead to these spaces. It was proved by E.M.Stein [17] that the sharp condition for the differentiability a.e. for a function $f \in W_1^1$ is that $\bigtriangledown f$ belongs to the Lorentz space $L^{n,1}$. The use of Lorentz type limitations on the derivatives can be crucial in the estimates of Fourier transforms (as it can be deduced from [9, 11, 15]). That is, if we look for a sharp conditions on the derivatives to guarantee a given integrability property of the Fourier transform, then these conditions generally will be expressed in terms of Lorentz norms.

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Let us return to the our main problem - estimates for the norms of the differences (2). As it was mentioned above, estimates of this type are already known. In particular, they give a refinement of the classical Sobolev embedding theorem with the limiting exponent. The simplest version of this theorem asserts that for any function f in the Sobolev space $W_p^1(\mathbb{R}^n)$ $(1 \le p < n)$

$$\|f\|_{q^*} \le c \sum_{k=1}^n \left\| \frac{\partial f}{\partial x_k} \right\|_p , \quad q^* = \frac{np}{n-p} . \tag{3}$$

Sobolev proved this inequality in 1938 for $p > 1$; his method, based on integral representations, did not work in the case $p = 1$. Only at the end of fifties Gagliardo and Nirenberg gave simple proofs of the inequality (3) for all $1 \le p < n$.

The inequality (3) has been generalized and developed in various directions (see [2, 10, 12, 13, 20, 21] for details and references). It was proved that the left hand side in (3) can be replaced by the stronger Lorentz norm; that is, there holds the inequality

$$\|f\|_{q^*,p} \le c \sum_{k=1}^n \left\| \frac{\partial f}{\partial x_k} \right\|_p , \quad 1 \le p < n. \tag{4}$$

For $p > 1$ this result follows by interpolation (see [14, 18]). In the case $p = 1$ some geometric inequalities were used to prove (4) (see [3, 4, 7, 8, 16]).

An elementary approach to the study of Sobolev type inequalities, based on estimates of non-increasing rearrangements, has been worked out in [8]. In [8] there was proved an extension of the inequality (4) to the anisotropic Sobolev spaces $W_p^{r_1,\dots,r_n}(\mathbb{R}^n)$ $(p \ge 1, r_k \in \mathbb{N})$ defined by conditions $f, D_k^{r_k} f \in L^p(\mathbb{R}^n)$. Afterwards, it was shown in [10] that the same methods give an analogous result in the case when the derivatives $D_k^{r_k} f$ belong to different spaces L^{p_k}. Observe that this approach has been still further simplified in the work [11], where the iterative rearrangements were used.

The sharp estimates of the norms of differences for the functions in Sobolev spaces firstly have been proved by V.P.Il'in [2, vol.2, p.72]. For the space $W_p^1(\mathbb{R}^n)$ Il'in's result reads as follows: if $n \in \mathbb{N}$, $1 < p < q < \infty$ and $\alpha \equiv 1 - n(1/p - 1/q) > 0$, then

$$\sum_{k=1}^n \left(\int_0^\infty \left[h^{-\alpha} \|\Delta_k^1(h) f\|_q \right]^p \frac{dh}{h} \right)^{1/p} \le c \sum_{k=1}^n \left\| \frac{\partial f}{\partial x_k} \right\|_p . \tag{5}$$

Actually, this means that there holds the continuous embedding to the Besov space

$$W_p^1(\mathbb{R}^n) \hookrightarrow B_{q,p}^\alpha(\mathbb{R}^n) .$$

It is easy to see that the inequality (5) fails to hold for $p = n = 1$. Nevertheless, it was proved in [6] that (5) is true in the case $p = 1$, $n \ge 2$.

The inequality (5) for $p = 1$, $n \ge 2$ was used to prove some estimates of Fourier transforms of functions in Sobolev spaces (see [15], [9]). In particular, using these results, we can compare the inequalities (3) and (5). Let us consider the case $p = 1$, $n = 2$. The inequality (3) means that for any function $f \in W_1^1(\mathbb{R}^2)$ its Fourier transform $\widehat{f}$ belongs to $L^2(\mathbb{R}^2)$. At the same time, as it was shown in [9], the stronger result can be easily derived from (5); that is, if $f \in W_1^1(\mathbb{R}^2)$, then $\widehat{f} \in L^{2,1}(\mathbb{R}^2)$. Note that this assertion does not follow from (4).

The extension of the inequality (5) to the spaces $W_p^{r_1,\dots,r_n}$ was given in [8]. This is the following inequality

$$\sum_{k=1}^{n}\left(\int_0^\infty \left[h^{-\alpha_k}\|\Delta_k^{r_k}(h)f\|_{q,p}\right]^p \frac{dh}{h}\right)^{1/p} \le c\sum_{k=1}^{n}\|D_k^{r_k}f\|_p\,, \tag{6}$$

where $0 < 1/p - 1/q < r/n$, $r \equiv n\left(\sum_{i=1}^n r_i^{-1}\right)^{-1}$ and $\alpha_k = r_k\left[1-\frac{n}{r}\left(\frac{1}{p}-\frac{1}{q}\right)\right]$; the inequality is valid if $p > 1$, $n \ge 1$ or $p = 1$, $n \ge 2$. Using (6), we get the following continuous embedding

$$W_p^{r_1,\dots,r_n}(\mathbb{R}^n) \hookrightarrow B_{q,p}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)\,.$$

For $p > 1$ this embedding was proved by Il'in [2, Vol.2, p.72]. The main result in [8] is the proof of (6) for $p = 1$, $n \ge 2$. This result was applied in [9] to obtain Fourier transforms estimates for functions in $W_1^{r_1,\dots,r_n}$.

Now we can specify our main problem: find the sharp estimates of the type (6) for the case when the derivatives $D_k^{r_k}f$ belong to *different* Lorentz spaces L^{p_k,s_k}. The main result of the paper is the following inequality (see Theorem 1 below)

$$\left(\int_0^\infty \left[h^{-\alpha_j}\|\Delta_j^{r_j}(h)f\|_{q_j,1}\right]^{\theta_j} \frac{dh}{h}\right)^{1/\theta_j} \le c\sum_{k=1}^{n}\|D_k^{r_k}f\|_{p_k,s_k}\,. \tag{7}$$

We shall not specify here the conditions on the parameters. Technically, the most complicated case is one when some of p_k's are equal to 1 and some of them are greater than 1. The basic difficulty is to find the *sharp* values of the parameters θ_j; let us emphasize that it is exactly the main result of the work. In this connection observe that an inequality similar to (7) was proved by Il'in [2, Vol.2, p.72] in the case $p_k = s_k > 1$ $(k = 1,\dots,n)$, but with the value of the parameter $\theta = \max_{1\le k\le n} p_k$, which is not sharp when p_k are different.

The general base of our approach is contained in the Lemmas 2, 3 and 4 given below. These lemmas were proved earlier by the first named author. Lemmas 3 and 4 give estimates of non-increasing rearrangement of a function in terms of its derivatives. We use also the scheme of the proof of the inequality (6) developed in [8]. Observe that in our case some essential modifications of this scheme are requiered.

Note also that as in the articles [9], [11], [15], the results of this paper can be applied to the study of estimates of Fourier transforms in Sobolev spaces.

2. Auxiliary propositions

Let $S_0(\mathbb{R}^n)$ be the class of all measurable and almost everywhere finite functions f on $\mathbb{R}^n$ such that for each $y > 0$,

$$\lambda_f(y) \equiv |\{x \in \mathbb{R}^n : |f(x)| > y\}| < \infty.$$

A non-increasing rearrangement of a function $f \in S_0(\mathbb{R}^n)$ is a non-increasing function f^* on $\mathbb{R}_+ \equiv (0,+\infty)$ that is equimeasurable with $|f|$. The rearrangement f^* can be defined by the equality

$$f^*(t) = \sup_{|E|=t}\inf_{x\in E}|f(x)| \quad , \quad 0 < t < \infty\,.$$

The following relation holds [1, Ch.2]

$$\sup_{|E|=t}\int_E |f(x)|dx = \int_0^t f^*(u)du\,.$$

In what follows we set

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(u)du\,.$$

Assume that $0 < q, p < \infty$. A function $f \in S_0(\mathbb{R}^n)$ belongs to the Lorentz space $L^{q,p}(\mathbb{R}^n)$ if

$$\|f\|_{q,p} \equiv \left(\int_0^\infty \left(t^{1/q} f^*(t)\right)^p \frac{dt}{t}\right)^{1/p} < \infty\,.$$

We have the inequality [1, p.217]

$$\|f\|_{q,s} \le c\|f\|_{q,p} \quad (0 < p < s < \infty),$$

so that $L^{q,p} \subset L^{q,s}$ for $p < s$. In particular, for $0 < p \le q$

$$L^{q,p} \subset L^{q,q} \equiv L^q\,.$$

Lemma 1. *Let $\psi \in L^{p,s}(\mathbb{R}_+)$ $(1 \le p, s < \infty)$ be a non-negative non-increasing function on $\mathbb{R}_+$. Then for any $\delta > 0$ there exists a continuously differentiable function φ on $\mathbb{R}_+$ such that:*
(i) $\psi(t) \le \varphi(t), \quad t \in \mathbb{R}_+$;
(ii) $\varphi(t)\, t^{1/p-\delta}$ *decreases and* $\varphi(t)\, t^{1/p+\delta}$ *increases on* $\mathbb{R}_+$;
(iii) $\|\varphi\|_{p,s} \le c\|\psi\|_{p,s}$,
where c is a constant that only depends on p and δ.

Proof. We can suppose that $\delta < 1/p$. Set

$$\varphi_1(t) = 2\, t^{\delta-1/p}\int_{t/2}^\infty u^{1/p-\delta}\psi(u)\,\frac{du}{u}\,.$$

Then $\varphi_1(t)\, t^{1/p-\delta}$ decreases and

$$\varphi_1(t) \ge 2\, t^{\delta-1/p}\psi(t)\int_{t/2}^t u^{1/p-\delta-1}\, du \ge \psi(t)\,.$$

Furthermore, applying Hardy's inequality [1, p.124], we easily get that

$$\|\varphi_1(t)\|_{p,s} \le c\, \|\psi\|_{p,s}\,. \tag{8}$$

Set now

$$\varphi(t) = (\delta + 1/p)\, t^{-1/p\,-\delta}\int_0^t \varphi_1(u)u^{\delta+1/p}\,\frac{du}{u}\,. \tag{9}$$

Then $\varphi(t)t^{1/p\,+\delta}$ increases on $\mathbb{R}_+$ and

$$\varphi(t) \ge \varphi_1(t) \ge \psi(t) \quad t \in \mathbb{R}_+\,.$$

Furthermore, the change of variable $v = u^{2\delta}$ in the right hand side of (9) gives that

$$t^{1/p\,-\delta}\varphi(t) = c\, t^{-2\delta}\int_0^{t^{2\delta}} \eta(v^{1/(2\delta)})\, dv\,,$$

where $\eta(u) = \varphi_1(u)\, u^{1/p\,-\delta}$ is a decreasing function on $\mathbb{R}_+$. Thus, $t^{1/p\,-\delta}\varphi(t)$ decreases. Finally, using Hardy's inequality and (8), we get (iii). The lemma is proved. □

Let $r_k \in \mathbb{N}$ and $1 \le p_k < \infty$ for $k = 1, \dots, n \quad (n \ge 2)$.
Denote

$$r = n \left(\sum_{j=1}^{n} \frac{1}{r_j} \right)^{-1}, \qquad p = \frac{n}{r} \left(\sum_{j=1}^{n} \frac{1}{p_j r_j} \right)^{-1} \tag{10}$$

and

$$\gamma_k = 1 - \frac{1}{r_k} \left(\frac{r}{n} + \frac{1}{p_k} - \frac{1}{p} \right). \tag{11}$$

Then $\gamma_k > 0$ and

$$\sum_{k=1}^{n} \gamma_k = n - 1. \tag{12}$$

Indeed,

$$\left(\frac{r}{n} + \frac{1}{p_k} - \frac{1}{p} \right) \sum_{j=1}^{n} \frac{1}{r_j} = 1 + \sum_{j \ne k} \left(\frac{1}{p_k} - \frac{1}{p_j} \right) \frac{1}{r_j} < 1 + \sum_{j \ne k} \frac{1}{r_j} \le r_k \sum_{j=1}^{n} \frac{1}{r_j}.$$

Thus, $\gamma_k > 0$. The equality (12) follows immediately from (10).

To prove our main results we use estimates of the rearrangement of a given function in terms of its derivatives $D_k^{r_k} f$ $(k = 1, ..., n)$. Thus, we apply simultaneously n estimates in which upper bounds involve functions belonging to different Lorentz spaces. The following lemma enables us to find a sharp "intermediate" estimate.

We will use the notations (10) and (11).

Lemma 2. *Let $r_k \in \mathbb{N}$, $1 \le p_k, s_k < \infty$ for $k = 1, \dots, n \quad (n \ge 2)$ and $s_k = 1$ if $p_k = 1$. Set*

$$s = \frac{n}{r} \left(\sum_{j=1}^{n} \frac{1}{s_j r_j} \right)^{-1}.$$

Let

$$0 < \delta \le \frac{1}{4} \min_{\gamma_j < 1} \min(\gamma_j, 1 - \gamma_j). \tag{13}$$

Suppose that $\varphi_k \in L^{p_k, s_k}(\mathbb{R}_+)$ $(k = 1, \dots, n)$ are positive continuously differentiable functions with $\varphi_k'(t) < 0$ on $\mathbb{R}_+$ such that $\varphi_k(t) t^{1/p_k - \delta}$ decreases and $\varphi_k(t) t^{1/p_k + \delta}$ increases on $\mathbb{R}_+$. Set for $u, t > 0$

$$\eta_k(u, t) = \begin{cases} (t/u)^{r_k - 1} \varphi_k(u), & \text{if } p_k = 1, \\ (t/u)^{r_k} \varphi_k(t), & \text{if } p_k > 1, \end{cases}$$

and

$$\sigma(t) = \sup \left\{ \min_{1 \le k \le n} \eta_k(u_k, t) : \prod_{k=1}^{n} u_k = t^{n-1}, \, u_k > 0 \right\}. \tag{14}$$

Then:
(i) there holds the inequality

$$\left(\int_0^\infty t^{s(1/p - r/n) - 1} \sigma(t)^s dt \right)^{1/s} \le c' \prod_{k=1}^{n} \|\varphi_k\|_{p_k, s_k}^{r/(n r_k)}; \tag{15}$$

(ii) there exist positive continuously differentiable functions $u_k(t)$ *on* $\mathbb{R}_+$ *such that*

$$\prod_{k=1}^{n} u_k(t) = t^{n-1} \tag{16}$$

and

$$\sigma(t) = \eta_k(u_k(t), t) \quad (t \in \mathbb{R}_+\,,\ k = 1, \ldots, n); \tag{17}$$

(iii) for any k *such that*

$$\frac{1}{p_k} > \frac{1}{p} - \frac{r}{n} \tag{18}$$

the function $u_k(t)t^{\delta-1}$ *decreases on* $\mathbb{R}_+$*;*
(iv) if $p_k = 1$*, then*

$$\int_0^\infty \frac{u_k(t)}{t} \varphi_k(u_k(t))\, dt \le c\, \|\varphi_k\|_1. \tag{19}$$

Proof. Fix $t > 0$ and denote

$$\mu_t(u) = \min_{1 \le k \le n} \eta_k(u_k, t), \quad u = (u_1, \ldots, u_n) \in \mathbb{R}^n_+.$$

This is a continuous function in $\mathbb{R}^n_+$. Observe that every function $\eta_k(s,t)$ is strictly decreasing and continuous with respect to s in $\mathbb{R}_+$. Furthermore, $\eta_k(s,t) \to 0$ as $s \to +\infty$. Thus,

$$\mu_t(u) \to 0 \quad \text{as} \quad \max u_k \to +\infty.$$

This implies the existence of a point $u^* \in \mathbb{R}^n_+$ such that

$$\mu_t(u^*) = \sigma(t) \quad \text{and} \quad \prod_{k=1}^{n} u^*_k = t^{n-1}.$$

For any $k = 1, \ldots, n$ there exists a unique point $u_k(t) > 0$ such that $\eta_k(u_k(t), t) = \sigma(t)$. It is clear that $u^*_k \le u_k(t)$ for all k (otherwise we would have that $\mu_t(u^*) < \sigma(t)$). Suppose that $u^*_j < u_j(t)$ for some j. Take $u'_j \in (u^*_j, u_j(t))$ and choose $u'_k \in (0, u^*_k)$ $(k \ne j)$ such that $\prod_{k=1}^n u'_k = t^{n-1}$. Then we obtain that $\mu_t(u') > \sigma(t)$, in contradiction with the definition of $\sigma(t)$. Thus, $u^*_k = u_k(t)$ $(k = 1, \ldots, n)$, and we get that the functions $u_k(t)$ satisfy both equalities (16) and (17).

Further, for any $j = 1, \ldots, n$

$$\eta_j(u_j(t), t) = \eta_n(u_n(t), t). \tag{20}$$

It follows that there exist functions $\psi_j(s,t) \in C^1(\mathbb{R}^2_+)$ $\quad (j = 1, \ldots, n-1)$ such that

$$\frac{\partial \psi_j}{\partial s}(s,t) > 0, \quad (s,t) \in \mathbb{R}^2_+, \tag{21}$$

and

$$u_j(t) = \psi_j(u_n(t), t) \quad (j = 1, ..., n-1). \tag{22}$$

Indeed, if $p_j = 1$, then (20) implies that

$$\lambda_j(u_j(t)) = t^{1-r_j} \eta_n(u_n(t), t),$$

where $\lambda_j(s) \equiv s^{1-r_j} \varphi_j(s)$ is a continuously differentiable function with $\lambda'_j(s) < 0$ $(s > 0)$. Thus, (22) holds with

$$\psi_j(s,t) = \lambda_j^{-1}(t^{1-r_j} \eta_n(s,t));$$

clearly, $\psi_j \in C^1(\mathbb{R}^2_+)$ and satisfies (21). If $p_j > 1$, then (22) holds with the function

$$\psi_j(s,t) = t[\varphi_j(t)/\eta_n(s,t)]^{1/r_j},$$

which also belongs to $C^1(\mathbb{R}^2_+)$ and satisfies (21).

It follows from (16) and (22) that for any $t > 0$

$$\Phi(u_n(t),t) = t^{n-1},$$

where

$$\Phi(s,t) = s\prod_{j=1}^{n-1}\psi_j(s,t).$$

Since $\Phi'_s(s,t) > 0$, we get that $u_n \in C^1(\mathbb{R}_+)$ and therefore, by (22), $u_j \in C^1(\mathbb{R}_+)$ for any $j = 1,...,n$. The statement (ii) is proved. Note also that by (17) the function σ is continuously differentiable in $\mathbb{R}_+$.

Now we will prove that for all $t > 0$

$$\frac{r/n - 1/p - \delta}{t} \le \frac{\sigma'(t)}{\sigma(t)} \le \frac{r/n - 1/p + \delta}{t}. \tag{23}$$

Our conditions on φ_k imply that for any $k = 1,\dots,n$

$$\left(\frac{1}{p_k} - \delta\right)\frac{1}{t} \le -\frac{\varphi'_k(t)}{\varphi_k(t)} \le \left(\frac{1}{p_k} + \delta\right)\frac{1}{t}. \tag{24}$$

Further, if $p_k > 1$, then by (17)

$$\frac{\sigma'(t)}{\sigma(t)} = \frac{r_k}{t} - r_k\frac{u'_k(t)}{u_k(t)} + \frac{\varphi'_k(t)}{\varphi_k(t)} \tag{25}$$

and by (24)

$$\frac{r_k - 1/p_k - \delta}{t} - r_k\frac{u'_k(t)}{u_k(t)} \le \frac{\sigma'(t)}{\sigma(t)} \le \frac{r_k - 1/p_k + \delta}{t} - r_k\frac{u'_k(t)}{u_k(t)}. \tag{26}$$

If $p_k = 1$, then we have by (17)

$$\frac{\sigma'(t)}{\sigma(t)} = \frac{r_k - 1}{t} - (r_k + \alpha_k(t))\frac{u'_k(t)}{u_k(t)}, \tag{27}$$

where

$$\alpha_k(t) = -\left[1 + u_k(t)\frac{\varphi'_k(u_k(t))}{\varphi_k(u_k(t))}\right].$$

By (24) ($p_k = 1$),

$$-\delta \le \alpha_k(t) \le \delta. \tag{28}$$

Now, differentiating (16) and taking into account (12), we get that for any $t > 0$ there exists $m \equiv m(t)$ such that

$$\frac{u'_m(t)}{u_m(t)} \le \frac{\gamma_m}{t}.$$

If $p_m > 1$, then by the first of the inequalities (26),

$$\frac{\sigma'(t)}{\sigma(t)} \ge \frac{r_m - 1/p_m - r_m\gamma_m - \delta}{t} = \frac{r/n - 1/p - \delta}{t}.$$

If $p_m = 1$ (in this case $\gamma_m < 1$), then by (27) and (28)

$$\frac{\sigma'(t)}{\sigma(t)} \ge \frac{r_m - 1 - \gamma_m(r_m + \delta)}{t} \ge \frac{r/n - 1/p - \delta}{t}.$$

Thus, we have the first inequality in (23). To prove the second inequality observe that by (12) and (16) for any $t>0$ there exists $l \equiv l(t)$ such that

$$\frac{u_l'(t)}{u_l(t)} \ge \frac{\gamma_l}{t}.$$

It remains to apply the right hand side inequality of (26) in the case $p_l > 1$ or (27) and (28) in the case $p_l = 1$.

To prove (iii) assume that k satisfies the condition (18) (that is, $\gamma_k < 1$). Let $p_k > 1$. By (25), (24) and (23),

$$r_k \frac{u_k'(t)}{u_k(t)} = \frac{r_k}{t} - \frac{\sigma'(t)}{\sigma(t)} + \frac{\varphi_k'(t)}{\varphi_k(t)} \le$$

$$\le \frac{r_k + 1/p - r/n - 1/p_k + 2\delta}{t} = \frac{r_k\gamma_k + 2\delta}{t}.$$

Thus, by (13),

$$\frac{u_k'(t)}{u_k(t)} \le \frac{1-\delta}{t},$$

which implies (iii) (in the case $p_k > 1$). If $p_k = 1$, then by (27) and (23)

$$(r_k + \alpha_k(t))\frac{u_k'(t)}{u_k(t)} \le \frac{r_k - 1 + 1/p - r/n + \delta}{t} =$$

$$= \frac{r_k\gamma_k + \delta}{t}.$$

From here (see (13)),

$$\frac{u_k'(t)}{u_k(t)} \le \frac{r_k\gamma_k + \delta}{(r_k + \alpha_k(t))t} \le \frac{r_k\gamma_k + \delta}{(r_k - \delta)t} \le \frac{1-\delta}{t}.$$

This implies (iii).

To prove (iv) assume that $p_k = 1$. By (27) and (23)

$$(r_k + \alpha_k(t))\frac{u_k'(t)}{u_k(t)} \ge \frac{r_k - 1 + 1/p - r/n - \delta}{t} =$$

$$= \frac{r_k\gamma_k - \delta}{t}.$$

From here (see (13)),

$$\frac{u_k'(t)}{u_k(t)} \ge \frac{r_k\gamma_k - \delta}{(r_k + \alpha_k(t))t} \ge \frac{r_k\gamma_k - \delta}{(r_k + \delta)t} > \frac{\delta}{t}.$$

It follows that

$$\int_0^\infty \frac{u_k(t)}{t}\varphi_k(u_k(t))dt \le$$

$$\le \frac{1}{\delta}\int_0^\infty u_k'(t)\varphi_k(u_k(t))dt = \frac{1}{\delta}\,\|\varphi_k\|_1.$$

Thus, we obtain (19).

It remains to prove the inequality (15). By (17), we have

$$\sigma(t)^{r/(nr_k)} = \left(\frac{t}{u_k(t)}\right)^{r/n}\left[\varphi_k(u_k(t))\frac{u_k(t)}{t}\right]^{r/(nr_k)}, \text{ if } \; p_k = 1,$$

and

$$\sigma(t)^{r/(nr_k)} = \left(\frac{t}{u_k(t)}\right)^{r/n} \varphi_k(t)^{r/(nr_k)}, \text{ if } p_k > 1.$$

Multiplying these equalities and using (16), we get

$$\sigma(t) = t^{r/n} \prod_{p_k=1} \left[\frac{u_k(t)}{t}\varphi_k(u_k(t))\right]^{r/(nr_k)} \prod_{p_k>1} (\varphi_k(t))^{r/(nr_k)}. \tag{29}$$

Denote

$$q_k = \frac{nr_k s_k}{rs}.$$

Then

$$\sum_{k=1}^{n} \frac{1}{q_k} = 1 \quad \text{and} \quad \sum_{k=1}^{n} \frac{s_k}{p_k q_k} = \frac{s}{p}.$$

Therefore, applying Hölder's inequality with the exponents q_k and using (19), we get from (29)

$$\int_0^\infty t^{s(1/p - r/n)-1} \sigma(t)^s dt \leq$$
$$\leq c \prod_{p_k=1} \|\varphi_k\|_1^{rs/(nr_k)} \prod_{p_k>1} \|\varphi_k\|_{p_k,s_k}^{rs/(nr_k)}.$$

The proof is now complete. □

The Lebesgue measure of a measurable set $A \subset \mathbb{R}^k$ will be denoted by $\operatorname{mes}_k A$.

For any F_σ-set $E \subset \mathbb{R}^n$ denote by E^j the orthogonal projection of E onto the coordinate hyperplane $x_j = 0$. By the Loomis-Whitney inequality [5, 4.4.2],

$$(\operatorname{mes}_n E)^{n-1} \leq \prod_{j=1}^{n} \operatorname{mes}_{n-1} E^j . \tag{30}$$

As usual, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we denote by $\hat{x}_k$ the $(n-1)$-dimensional vector obtained from x by removal of its kth coordinate.

Let $f \in S_0(\mathbb{R}^n)$, $t > 0$ and let E_t be a set of type F_σ and measure t such that

$$|f(x)| \geq f^*(t) \quad \text{for all } x \in E_t .$$

Denote by $\lambda_j(t)$ the $(n-1)$-dimensional measure of the projection E_t^j $(j = 1, \ldots, n)$. By (30), we have that

$$\prod_{j=1}^{n} \lambda_j(t) \geq t^{n-1}. \tag{31}$$

The following lemma was proved in [8] (see also [10]).

Lemma 3. *Let $n \geq 2$, $r_k \in \mathbb{N}$ $(k = 1, \ldots, n)$. Assume that a locally integrable function $f \in S_0(\mathbb{R}^n)$ has weak derivatives $D_k^{r_k} f \in L_{loc}(\mathbb{R}^n)$ $(k = 1, \ldots, n)$. Then for all $0 < t < \tau < \infty$ and $k = 1, \ldots, n$ we have*

$$f^*(t) \leq K \left[f^*(\tau) + \left(\frac{\tau}{\lambda_k(t)}\right)^{r_k} (D_k^{r_k} f)^{**}(\tau)\right] \tag{32}$$

and

$$f^*(t) \leq K \left[f^*(\tau) + \left(\frac{\tau}{\lambda_k(t)}\right)^{r_k - 1} \psi_k^* \left(\frac{\lambda_k(t)}{2}\right)\right], \tag{33}$$

where K is a constant depending only on $r_1,\dots,r_n$ and

$$\psi_k(\hat{x}_k) = \int_{\mathbb{R}} |D_k^{r_k} f(x)|\,dx_k\,, \quad \hat{x}_k \in \mathbb{R}^{n-1}. \tag{34}$$

Lemma 4. *Let $n \ge 2$, $r_k \in \mathbb{N}$, $1 \le p_k, s_k < \infty$ for $k = 1,\dots,n$ and $s_k = 1$ if $p_k = 1$. Set*

$$r = n\left(\sum_{k=1}^n \frac{1}{r_k}\right)^{-1}, \qquad p = \frac{n}{r}\left(\sum_{k=1}^n \frac{1}{p_k r_k}\right)^{-1}. \tag{35}$$

and

$$s = \frac{n}{r}\left(\sum_{k=1}^n \frac{1}{s_k r_k}\right)^{-1}. \tag{36}$$

Assume that a locally integrable function $f \in S_0(\mathbb{R}^n)$ has weak derivatives $D_k^{r_k} f \in L^{p_k,s_k}(\mathbb{R}^n)$ $(k = 1,\dots,n)$. Then for any $\xi > 1$

$$f^*(t) \le K\,\left[f^*(\xi t) + \xi^{\bar{r}}\sigma(t)\right]\,, \tag{37}$$

where $\bar{r} = \max r_k$, the constant K depends only on $r_1,\dots,r_n$ and

$$\left(\int_0^\infty t^{s(1/p\,-r/n)-1}\sigma(t)^s\,dt\right)^{1/s} \le c\prod_{k=1}^n \|D_k^{r_k} f\|_{p_k,s_k}^{r/(nr_k)}. \tag{38}$$

Proof. For every fixed $k = 1,\dots,n$ we take (see (34))

$$\psi(t) \equiv \psi_k(t) = \begin{cases} \psi_k^*(t/2), & \text{if } p_k = 1, \\ (D_k^{r_k} f)^{**}(t), & \text{if } p_k > 1. \end{cases}$$

Then $\|\psi_k\|_1 = 2\|D_k^{r_k}\|_1$, if $p_1 = 1$, and by Hardy's inequality [1, p.124]

$$||\psi_k||_{p_k,s_k} \le c\|D_k^{r_k} f\|_{p_k,s_k},$$

if $p_k > 1$. Next we apply Lemma 1 with δ defined as in Lemma 2. This way we obtain the functions which we denote by $\varphi_k(t)$ $(k = 1,\dots,n)$. Further, with these functions φ_k we define the function $\sigma(t)$ by (14). By Lemma 2, we have the inequality (38). Using Lemma 3 with $\tau = \xi t$, we obtain

$$f^*(t) \le K\left[f^*(\xi t) + \xi^{\bar{r}}\left(\frac{t}{\lambda_k(t)}\right)^{r_k}\varphi_k(t)\right],$$

if $p_k > 1$, and

$$f^*(t) \le K\left[f^*(\xi t) + \xi^{\bar{r}}\left(\frac{t}{\lambda_k(t)}\right)^{r_k-1}\varphi_k(\lambda_k(t))\right],$$

if $p_k = 1$. Taking into account (14) and (31), we immediately get (37). □

Note that in the case $p_1 = \ldots = p_n$, $s_1 = \ldots = s_n$ Lemma 4 actually is contained in [10] (see Lemmas 7 and 8 in [8]).

Corollary 1. *Assume that a function f satisfies the conditions of Lemma 4 and $f \in L^1(\mathbb{R}^n) + L^{p_0}(\mathbb{R}^n)$ for some $p_0 > 0$ such that*

$$\frac{1}{p_0} > \frac{1}{p} - \frac{r}{n}.$$

Let $\max(1,p_0)<q<\infty$ *and*

$$\frac{1}{q}>\frac{1}{p}-\frac{r}{n}\,. \tag{39}$$

Then for any $\theta>0$ $f\in L^{q,\theta}(\mathbb{R}^n)$ *and*

$$\|f\|_{q,\theta}\le c\left[\|f\|_{L^1+L^{p_0}}+\prod_{k=1}^n\|D_k^{r_k}f\|_{p_k,s_k}^{r/(nr_k)}\right]. \tag{40}$$

Proof. We can assume that $\theta<\min(1,p_0,s)$. Let $f=g+h$, with $g\in L^1(\mathbb{R}^n)$ and $h\in L^{p_0}(\mathbb{R}^n)$. Applying Hölder inequality, we obtain

$$J_1\equiv\int_1^\infty\left[t^{1/q}f^*(t)\right]^\theta\frac{dt}{t}\le$$

$$\le 2^\theta\left[\int_1^\infty\left[t^{1/q}g^*(t/2)\right]^\theta\frac{dt}{t}+\int_1^\infty\left[t^{1/q}h^*(t/2)\right]^\theta\frac{dt}{t}\right]\le$$

$$\le c\left[\left(\int_0^\infty g^*(t)dt\right)^\theta+\left(\int_0^\infty h^*(t)^{p_0}dt\right)^{\theta/p_0}\right].$$

It follows that

$$J_1\le c'\,\|f\|_{L^1+L^{p_0}}^\theta. \tag{41}$$

Let $0<\delta<1$. Using (37) with $\xi=(2^{1/\theta}K)^q$, we get by Hölder inequality and (39):

$$J_\delta\equiv\int_\delta^\infty\left[t^{1/q}f^*(t)\right]^\theta\frac{dt}{t}\le J_1+K^\theta\int_\delta^1\left[t^{1/q}f^*(\xi t)\right]^\theta\frac{dt}{t}+$$

$$+c\int_0^1 t^{\theta/q-1}\sigma(t)^\theta dt\le J_1+\frac{1}{2}J_\delta+$$

$$+c'\left(\int_0^1 t^{s(1/p-r/n)-1}\sigma(t)^s dt\right)^{\theta/s}.$$

By (41), $J_\delta<\infty$. The inequality (40) follows now from (38) and (41). $\square$

Remark 1. *Let* $r_k\in\mathbb{N}$, $1\le p_k,s_k<\infty$ *for* $k=1,\dots,n$ $(n\ge 2)$ *and* $s_k=1$, *if* $p_k=1$. *Let* r,p *and* s *be the numbers defined by (35) and (36). Assume that* $p<n/r$ *and set* $q^*=np/(n-rp)$. *Then for any function* $f\in C^\infty(\mathbb{R}^n)$ *with the compact support we have*

$$\|f\|_{q^*,s}\le c\prod_{k=1}^n\|D_k^{r_k}f\|_{p_k,s_k}^{r/(nr_k)}. \tag{42}$$

This statement follows immediately from the Lemma 4. The inequality (42) gives a generalization of the classical Sobolev's inequality with limiting exponent. A slightly different scheme of the proof of (42) was given in [10, Theorem 13.1]. *In the case* $p_k=s_k>1$ $(k=1,\dots,n)$ *the inequality (42) contains in* [2, Ch.4]. *For* $r_1=\dots=r_n=1$ *the proof of (42) was given in* [19]. *One can find a detailed description of the preceding results in* [10] *(see also* [19]*).*

3. The main theorem

Theorem 1. *Let $n \ge 2$, $r_k \in \mathbb{N}$, $1 \le p_k, s_k < \infty$ for $k = 1, \dots, n$ and $s_k = 1$ if $p_k = 1$. Let r, p and s be the numbers defined by (35) and (36). For every p_j $(1 \le j \le n)$ satisfying the condition*

$$\rho_j \equiv \frac{r}{n} + \frac{1}{p_j} - \frac{1}{p} > 0\,,$$

take arbitrary $q_j > p_j$ such that

$$\frac{1}{q_j} > \frac{1}{p} - \frac{r}{n}$$

and denote

$$\varkappa_j = 1 - \frac{1}{\rho_j}\left(\frac{1}{p_j} - \frac{1}{q_j}\right),$$

$$\alpha_j = \varkappa_j r_j \quad , \qquad \frac{1}{\theta_j} = \frac{1-\varkappa_j}{s} + \frac{\varkappa_j}{s_j}\,.$$

Then for any function $f \in S_0(\mathbb{R}^n)$ which has the weak derivatives $D_k^{r_k} f \in L^{p_k,s_k}(\mathbb{R}^n)$ $(k = 1, \dots, n)$ there holds the inequality

$$\left(\int_0^\infty \left[h^{-\alpha_j}\|\Delta_j^{r_j}(h)f\|_{q_j,1}\right]^{\theta_j}\frac{dh}{h}\right)^{1/\theta_j} \le c\sum_{k=1}^n \|D_k^{r_k}f\|_{p_k,s_k}\,, \tag{43}$$

where c is a constant that does not depend on f.

Proof. First observe that by our conditions $0 < \varkappa_j < 1$. Denote

$$g_k(x) = |D_k^{r_k} f(x)|.$$

Further, assume that $j = 1$ and set for $h > 0$

$$f_h(x) = |\Delta_1^{r_1}(h)f(x)|\,.$$

For almost all $x \in \mathbb{R}^n$ we have (see [2, Vol.1, p.101])

$$f_h(x) \le \int_0^h \cdots \int_0^h g_1(x + (u_1 + \cdots + u_{r_1})e_1)\,du_1 \cdots du_{r_1}\,. \tag{44}$$

From here,

$$f_h^*(t) \le h^{r_1} g_1^{**}(t). \tag{45}$$

Indeed, for any subset $A \subset \mathbb{R}^n$ with $|A| = t$

$$\int_A f_h(x)\,dx \le h^{r_1} \sup_{B\subset\mathbb{R}^n, |B|=t} \int_B g_1(y)\,dy = h^{r_1} t g_1^{**}(t).$$

From this, it follows (45).

If $p_1 = 1$ (in this case $s_1 = 1$), then it follows from (44) that $f_h \in L^1(\mathbb{R}^n)$. If $p_1 > 1$, then (45) implies that $f_h \in L^{p_1,s_1}(\mathbb{R}^n)$. Thus, by Corollary 1 we have that $f_h \in L^{q_1,1}(\mathbb{R}^n)$.

Denote for $h > 0$

$$J(h) \equiv \|f_h\|_{q_1,1} = \int_0^\infty t^{1/q_1 - 1} f_h^*(t)dt\,.$$

Set $\xi_0 = (4K)^{q_1}$ and

$$Q(h) = \{t > 0 \,:\; f_h^*(t) \ge 2K f_h^*(\xi_0 t)\}\,, \tag{46}$$

where K is the constant in Lemma 3. Then

$$\int_{\mathbb{R}_+\setminus Q(h)} t^{1/q_1-1} f_h^*(t)dt \le 2K\int_0^\infty t^{1/q_1-1} f_h^*(\xi_0 t)dt =$$
$$= 2K\xi_0^{-1/q_1}\int_0^\infty t^{1/q_1-1} f_h^*(t)dt = \frac{1}{2}J(h).$$

Therefore,

$$J(h) \le 2\int_{Q(h)} t^{1/q_1-1} f_h^*(t)dt \equiv 2J'(h)\,. \tag{47}$$

Denote

$$\psi_k(\hat{x}_k) = \int_{\mathbb{R}} g_k(x)dx_k\,, \quad \text{if } p_k = 1.$$

Let $\varepsilon = (1-\varkappa_1)/2$ and

$$0 < \delta < \varepsilon \min\left(\left(\frac{r_1 n}{r} - 1\right)^{-1}, \frac{1}{2}\min_{\gamma_j<1}\min(\gamma_j, 1-\gamma_j)\right). \tag{48}$$

Now for every $k = 1,\dots,n$ we apply Lemma 1 with $\psi(t) = \psi_k^*(t/2)$ in the case $p_k = 1$ and $\psi(t) = g_k^{**}(t)$ in the case $p_k > 1$. We obtain that there exist functions $\varphi_k(t)$ $(k = 1,\dots,n)$ on $\mathbb{R}_+$ such that

$$\varphi_k(t)\, t^{1/p_k-\delta} \downarrow, \quad \varphi_k(t)\, t^{1/p_k+\delta} \uparrow, \tag{49}$$

$$\psi_k^*(t/2) \le \varphi_k(t)\,, \text{ if } p_k = 1\,, \tag{50}$$

$$g_k^{**}(t) \le \varphi_k(t)\,, \text{ if } p_k > 1\,, \tag{51}$$

and

$$\|\varphi_k\|_{p_k,s_k} \le c\,\|D_k^{r_k} f\|_{p_k,s_k}\,. \tag{52}$$

We shall estimate $f_h^*(t)$ for fixed $h > 0$ and $t \in Q(h)$. Let $E(t,h)$ be a set of type F_σ and measure t such that

$$f_h(x) \ge f_h^*(t) \quad \text{for all } x \in E(t,h). \tag{53}$$

Denote by $\lambda_k(t,h)$ the $(n-1)$-dimensional measure of the orthogonal projection of $E(t,h)$ onto the coordinate hyperplane $x_k = 0$. By Lemma 3, (50) and (51), we have that for each $t \in Q(h)$

$$f_h^*(t) \le c\left(\frac{t}{\lambda_k(t,h)}\right)^{r_k-1} \varphi_k(\lambda_k(t,h))\,, \quad \text{if } p_k = 1\,, \tag{54}$$

and

$$f_h^*(t) \le c\left(\frac{t}{\lambda_k(t,h)}\right)^{r_k} \varphi_k(t)\,, \quad \text{if } p_k > 1\,. \tag{55}$$

Applying inequality (30) and Lemma 2, we obtain that there exist a non-negative function $\sigma(t)$ and positive continuously differentiable functions $u_k(t)$ $(k = 1,\dots,n)$ on $\mathbb{R}_+$ satisfying the following conditions:

$$f_h^*(t) \le c\,\sigma(t)\,, \quad t \in Q(h)\,, \tag{56}$$

$$\left(\int_0^\infty t^{s(1/p-r/n)-1}\sigma(t)^s dt\right)^{1/s} \le c\prod_{k=1}^n \|D_k^{r_k} f\|_{p_k,s_k}^{r/(nr_k)}\,, \tag{57}$$

$$\sigma(t)=\begin{cases}(t/u_k(t))^{r_k-1}\varphi_k(u_k(t))\,, & \text{if } p_k=1,\\ (t/u_k(t))^{r_k}\varphi_k(t)\,, & \text{if } p_k>1,\end{cases} \tag{58}$$

$$\prod_{k=1}^n u_k(t)=t^{n-1}, \tag{59}$$

$$u_1(t)t^{\delta-1} \text{ decreases}, \tag{60}$$

$$\int_0^\infty \frac{u_k(t)}{t}\varphi_k(u_k(t))\,dt\le c\,\|D_k^{r_k}f\|_1\,,\ \text{if } p_k=1. \tag{61}$$

The estimate (56) can be used for t "sufficiently small". For "large" t we need different estimates, involving h.

First, we have the estimate (45). Nevertheless, this estimate does not work in the case $p_1=1$ (the operator $g\to g^{**}$ is unbounded in L^1).

We shall prove an estimate which can be applied for all values of $p_1\ge 1$. Denote

$$\beta(t)=t/u_1(t)\,. \tag{62}$$

We shall prove that for any $h>0$ and any $t\in Q(h)$

$$f_h^*(t)\le c\,h^{r_1-\varepsilon}\beta(t)^{\varepsilon}\chi(t), \tag{63}$$

where $\varepsilon=(1-\varkappa_1)/2$ and

$$\chi(t)\equiv\sigma(t)\beta(t)^{-r_1}=\begin{cases}u_1(t)\varphi_1(u_1(t))/t\,, & \text{if } p_1=1\\ \varphi_1(t)\,, & \text{if } p_1>1\end{cases} \tag{64}$$

(see (58)). By (52) and (61),

$$\|\chi\|_{p_1,s_1}\le c\,\|D_1^{r_1}f\|_{p_1,s_1}\,. \tag{65}$$

For $h\ge\beta(t)$ ($t\in Q(h)$) the inequality (63) follows directly from (56) and (64). Assume that $0<h\le\beta(t)$, $t\in Q(h)$. If $p_1>1$, then (63) is the immediate consequence of (45), (51) and (64).

Let $p_1=1$. First suppose that there exists $1\le j\le n$ such that

$$\lambda_j(t,h)\ge\frac12 u_j(t)\left(\frac{\beta(t)}{h}\right)^{r_1/r_j}.$$

If $p_j>1$, then by (55) and (58)

$$f_h^*(t)\le c\left(\frac{t}{u_j(t)}\right)^{r_j}\varphi_j(t)\left(\frac{h}{\beta(t)}\right)^{r_1}=c\,\sigma(t)\left(\frac{h}{\beta(t)}\right)^{r_1}.$$

If $p_j=1$, then we apply (54). Notice that

$$\lambda_j(t,h)\ge\frac12 u_j(t).$$

Taking into account that for $\delta_j=\varepsilon r_j/r_1$ the function $\varphi_j(u)\,u^{1-\delta_j}$ decreases and the function $\varphi_j(u)\,u^{1+\delta_j}$ increases (see (48) and (49)), we get that

$$(\lambda_j(t,h))^{1-\delta_j}\varphi_j(\lambda_j(t,h))\le\left(\frac12 u_j(t)\right)^{1-\delta_j}\varphi_j(\frac{u_j(t)}{2})\le$$

$$\le c(u_j(t))^{1-\delta_j}\varphi_j(u_j(t)).$$

Thus, by (54)

$$f_h^*(t) \le c\, h^{r_1-\varepsilon}\beta(t)^{\varepsilon-r_1}\left(\frac{t}{u_j(t)}\right)^{r_j-1}\varphi_j(u_j(t)).$$

From these estimates and (58) it follows the inequality (63), where $\chi(t)$ is defined by (64).

Now assume that for each $j = 1, \dots, n$.

$$\lambda_j(t,h) < \frac{1}{2}u_j(t)\left(\beta(t)/h\right)^{r_1/r_j}. \tag{66}$$

First of all, it follows that

$$\lambda_1(t,h) < \frac{t}{2h}. \tag{67}$$

Further, for any F_σ-set $A \subset E \equiv E(t,h)$ denote by A_j the orthogonal projection of A onto the hyperplane $x_j = 0$. If

$$\operatorname{mes}_{n-1} A_1 \le \frac{1}{2}u_1(t)\left(\frac{h}{\beta(t)}\right)^{\frac{r_1 n}{r}-1} \equiv \frac{1}{2}\gamma(t,h), \tag{68}$$

then

$$\operatorname{mes}_n A \le \frac{t}{2}.$$

Indeed, otherwise we would have by (68) and (30)

$$\prod_{j=2}^{n}\operatorname{mes}_{n-1} A_j \ge \frac{t^{n-1}}{2^{n-2}\gamma(t,h)} =$$

$$= \frac{1}{2^{n-2}}\left(\frac{\beta(t)}{h}\right)^{r_1\sum_{j=2}^{n} r_j^{-1}}\prod_{j=2}^{n} u_j(t),$$

contrary to the assumption (66).

Using Lemma 3 of [8], we decompose the projection $E_1(t,h)$ into measurable disjoint subsets P and S such that

$$\operatorname{mes}_{n-1} S = \frac{1}{2}\gamma(t,h)$$

and

$$\int_P \psi_1(\hat{x}_1)d\hat{x}_1 \le \int_{\gamma(t,h)/2}^{t/(2h)} \psi_1^*(u)du. \tag{69}$$

It follows from the observation given above that the measure of the set

$$E' = \{x \in E(t,h) : \quad \hat{x}_1 \in P\}$$

is at least $t/2$. For $\hat{x}_1 \in E_1(t,h)$ we denote by $T(\hat{x}_1)$ the section of the set $E(t,h)$ by the line that passes through $\hat{x}_1$ and is perpendicular to the hyperplane $x_1 = 0$ (note that $T(\hat{x}_1)$ is a set of type F_σ). For almost all $\hat{x}_1 \in E_1(t,h)$ we have (see (44))

$$f_h^*(t)\ \operatorname{mes}_1 T(\hat{x}_1) \le \int_{T(\hat{x}_1)} f_h(x)dx_1 \le$$

$$\le h^{r_1}\int_{\mathbb{R}} |D_1^{r_1} f(x)|dx_1 = h^{r_1}\psi_1(\hat{x}_1).$$

Integrating this inequality with respect to $\hat{x}_1$ over P and taking into account (69) and the inequality

$$\int_P \operatorname{mes}_1 T(\hat{x}_1)d\hat{x}_1 = |E'| \ge \frac{t}{2}\,,$$

we get (see also (50))

$$f_h^*(t) \le \frac{h^{r_1}}{t}\int_{\gamma(t,h)}^{t/h}\varphi_1(u)du\,. \tag{70}$$

For $0 < h \le \beta(t)$ we have

$$\gamma(t,h) \le u_1(t) \le t/h\,.$$

Furthermore, let $\eta = \varepsilon/\left(\frac{r_1 n}{r} - 1\right)$. By (49), $\varphi_1(u)\,u^{1+\eta}$ increases and $\varphi_1(u)\,u^{1-\varepsilon}$ decreases on $(0,\infty)$. Thus, we have

$$\int_{\gamma(t,h)}^{t/h}\varphi_1(u)du = \int_{\gamma(t,h)}^{u_1(t)}\varphi_1(u)du + \int_{u_1(t)}^{t/h}\varphi_1(u)du \le$$

$$\le \varphi_1(u_1(t))u_1(t)^{1+\eta}\gamma(t,h)^{-\eta}/\eta \;+\; \varphi_1(u_1(t))u_1(t)^{1-\varepsilon}(t/h)^{\varepsilon}/\varepsilon =$$

$$= c\,h^{-\varepsilon}\beta(t)^{\varepsilon}u_1(t)\varphi_1(u_1(t)).$$

From here and (70) it follows (63).

Finally, taking into account (56) and (63), we obtain that for any $h > 0$ and any $t \in Q(h)$

$$f_h^*(t) \le c\,\Phi(t,h), \tag{71}$$

where

$$\Phi(t,h) = \min(\sigma(t), h^{r_1-\varepsilon}\beta(t)^{\varepsilon}\chi(t)) \tag{72}$$

and $\chi(t)$ is defined by (64).

Further, we have (see (47))

$$J'(h) \le c\int_0^\infty t^{1/q_1\,-1}\Phi(t,h)dt$$

and

$$J \equiv \int_0^\infty h^{-\alpha_1\theta_1-1}J(h)^{\theta_1}dh \le$$

$$\le c\int_0^\infty h^{-\alpha_1\theta_1-1}dh\left(\int_0^\infty t^{1/q_1\,-1}\Phi(t,h)dt\right)^{\theta_1}.$$

By (60), the function $\beta(t)t^{-\delta}$ increases on $\mathbb{R}_+$. It easily follows that the inverse function β^{-1} exists on $\mathbb{R}_+$ and satisfies the condition

$$\beta^{-1}(2z) \le 2^{1/\delta}\beta^{-1}(z). \tag{73}$$

Furthermore, we have

$$J \le c\left[\int_0^\infty h^{-\alpha_1\theta_1-1}dh\left(\int_0^{\beta^{-1}(h)} t^{1/q_1\,-1}\Phi(t,h)dt\right)^{\theta_1} +\right.$$

$$\left. + \int_0^\infty h^{-\alpha_1\theta_1-1}dh\left(\int_{\beta^{-1}(h)}^\infty t^{1/q_1\,-1}\Phi(t,h)dt\right)^{\theta_1}\right] \equiv c\,(J_1+J_2)\,.$$

Applying Minkowsi's inequality, we obtain

$$J_1^{1/\theta_1} = \left(\int_0^\infty h^{-\alpha_1\theta_1-1}dh\left(\sum_{k=0}^\infty\int_{\beta^{-1}(2^{-k-1}h)}^{\beta^{-1}(2^{-k}h)} t^{1/q_1-1}\sigma(t)dt\right)^{\theta_1}\right)^{1/\theta_1} \le$$

$$\le \sum_{k=0}^\infty \left(\int_0^\infty h^{-\alpha_1\theta_1-1}dh\left(\int_{\beta^{-1}(2^{-k-1}h)}^{\beta^{-1}(2^{-k}h)} t^{1/q_1-1}\sigma(t)dt\right)^{\theta_1}\right)^{1/\theta_1} =$$

$$= \sum_{k=0}^\infty 2^{-k\alpha_1}\left(\int_0^\infty z^{-\alpha_1\theta_1-1}dz\left(\int_{\beta^{-1}(z/2)}^{\beta^{-1}(z)} t^{1/q_1-1}\sigma(t)dt\right)^{\theta_1}\right)^{1/\theta_1}.$$

Further, using the Hölder inequality and (73), we get

$$\int_{\beta^{-1}(z/2)}^{\beta^{-1}(z)} t^{1/q_1-1}\sigma(t)dt \le c\left(\int_0^{\beta^{-1}(z)} t^{\theta_1/q_1-1}\sigma(t)^{\theta_1}dt\right)^{1/\theta_1}$$

Thus, by Fubini's theorem and (64)

$$J_1 \le c\int_0^\infty z^{-\alpha_1\theta_1-1}dz\int_0^{\beta^{-1}(z)} t^{\theta_1/q_1-1}\sigma(t)^{\theta_1}dt =$$

$$= c\int_0^\infty t^{\theta_1/q_1-1}\sigma(t)^{\theta_1}dt\int_{\beta(t)}^\infty z^{-\alpha_1\theta_1-1}dz =$$

$$= c'\int_0^\infty t^{\theta_1/q_1-1}\sigma(t)^{\theta_1}\beta(t)^{-\alpha_1\theta_1}dt =$$

$$= c'\int_0^\infty t^{\theta_1/q_1-1}\chi(t)^{\varkappa_1\theta_1}\sigma(t)^{(1-\varkappa_1)\theta_1}dt. \tag{74}$$

The same reasonings give that

$$J_2 \le c\int_0^\infty z^{[r_1(1-\varkappa_1)-\varepsilon]\theta_1}\frac{dz}{z}\int_{\beta^{-1}(z)}^\infty t^{\theta_1/q_1-1}\beta(t)^{\theta_1\varepsilon}\chi(t)^{\theta_1}dt =$$

$$= c\int_0^\infty t^{\theta_1/q_1-1}\beta(t)^{\theta_1\varepsilon}\chi(t)^{\theta_1}dt\int_0^{\beta(t)} z^{[r_1(1-\varkappa_1)-\varepsilon]\theta_1-1}dz =$$

$$= c'\int_0^\infty t^{\theta_1/q_1-1}\chi(t)^{\theta_1}\beta(t)^{r_1(1-\varkappa_1)\theta_1}dt\,.$$

By (64) the last integral is the same as one in the right hand side of (74). Therefore, we have that

$$J \le c\int_0^\infty t^{\theta_1/q_1-1}\chi(t)^{\varkappa_1\theta_1}\sigma(t)^{(1-\varkappa_1)\theta_1}dt$$

Now we apply Hölder inequality with the exponents $u = s_1/(\varkappa_1\theta_1)$ and $u' = s_1/(s_1-\varkappa_1\theta_1)$. Observe that

$$(1-\varkappa_1)\theta_1 u' = s\,, \quad \left(\frac{\theta_1}{q_1}-\frac{s_1}{p_1u}\right)u' = s\left(\frac{1}{p}-\frac{r}{n}\right).$$

Thus, we obtain, using (57) and (65):

$$J^{1/\theta_1} \le c\left(\int_0^\infty t^{s(1/p-r/n)-1}\sigma(t)^s dt\right)^{(1-\varkappa_1)/s}\|D_1^{r_1}f\|_{p_1,s_1}^{\varkappa_1}\,.$$

$$\le c\left(\prod_{j=1}^{n}\|D_j^{r_j}f\|_{p_j,s_j}^{r/(nr_j)}\right)^{1-\varkappa_1}\|D_1^{r_1}f\|_{p_1,s_1}^{\varkappa_1}.$$

Since

$$\sum_{j=1}^{n}\frac{r}{nr_j}=1,$$

we obtain the inequality (43). The theorem is proved. □

Remark 2. *First we recall the definition of the Besov space in the direction of the coordinate axis x_j (see* [13, Ch.4]*).*

Let $\alpha > 0$, $1 \le p,\theta < \infty$ and $1 \le j \le n$. Define the space $B^{\alpha}_{p,\theta;j}(\mathbb{R}^n)$ as the class of all functions $f \in L^p(\mathbb{R}^n)$ for which

$$\|f\|_{B^{\alpha}_{p,\theta;j}} \equiv \|f\|_p + \left(\int_0^\infty \left[h^{-\alpha}\|\Delta_j^r(h)f\|_p\right]^\theta \frac{dh}{h}\right)^{1/\theta} < \infty \tag{75}$$

for any integer $r > \alpha$. Of course, the right hand side in (75) depends on r, but every choice of the integer $r > \alpha$ leads to equivalent norms [13, Ch.4].

Now observe that the conditions of Theorem 1 do not imply the belongness of the function f to some $L^\nu(\mathbb{R}^n)$. However, if we assume in addition that $f \in L^{p_0}(\mathbb{R}^n)$ for some $p_0 \ge 1$ and that $q_j > p_0$, then by Corollary 1 we get $f \in L^{q_j,1}(\mathbb{R}^n)$. Thus, with these additional conditions Theorem 1 implies that $f \in B^{\alpha_j}_{q_j,\theta_j;j}(\mathbb{R}^n)$ and

$$\|f\|_{B^{\alpha_j}_{q_j,\theta_j;j}} \le c\left[\|f\|_{p_0} + \sum_{k=1}^{n}\|D_k^{r_k}f\|_{p_k,s_k}\right].$$

Remark 3. *It is important to emphasize that the values of parameters θ_k found in the Theorem 1 are sharp. To verify this statement we shall consider the following simple example.*

Assume that $n=2$, $r_1=r_2=1$, $1 \le p_1,p_2 < \infty$ and $s_1=p_1$, $s_2=p_2$. Furthermore, suppose that

$$p \equiv 2\left(\frac{1}{p_1}+\frac{1}{p_2}\right)^{-1} < 2. \quad \text{Then } \frac{1}{p_i} > \frac{1}{p}-\frac{1}{2} \quad (i=1,2).$$

Let $q_1 > p_1$ be such that

$$\frac{1}{q_1} > \frac{1}{p}-\frac{1}{2}.$$

As in Theorem 1, set

$$\varkappa_1 = 1-\frac{1/p_1-1/q_1}{1/p_1-1/p+1/2},$$

$$\alpha_1=\varkappa_1 \quad , \quad \frac{1}{\theta_1}=\frac{1-\varkappa_1}{p}+\frac{\varkappa_1}{p_1}.$$

Let $0<\varepsilon<\theta_1$; define the following numbers

$$\alpha=\frac{2/p-1}{1+2(1/p_1-1/p)} \quad , \quad \beta=\frac{2/p-1}{1+2(1/p_2-1/p)},$$

$$\delta=\frac{1}{p_1[1+2(1/p_1-1/p)]}\frac{\theta_1}{\theta_1-\varepsilon} \quad , \quad \gamma=\frac{1}{p_2[1+2(1/p_2-1/p)]}\frac{\theta_1}{\theta_1-\varepsilon}.$$

Further, denote for $(x,y) \in [-1,1]^2$

$$\varphi_0(x,y) = |x|^\alpha \left(\log \frac{e}{|x|}\right)^\delta + |y|^\beta \left(\log \frac{e}{|y|}\right)^\gamma .$$

Set

$$D = \left\{(x,y) \in [-1,1]^2 : \varphi_0(x,y) \le 1\right\}$$

and

$$f(x,y) \equiv f_\varepsilon(x,y) = \begin{cases} [\varphi_0(x,y)]^{-1} - 1 & , \text{ if } (x,y) \in D, \\ 0 & , \text{ if } (x,y) \notin D. \end{cases}$$

Carrying out a routine calculations, one can show that the function f *has the following properties:*

(i) *for any* $1 \le \nu \le 2p/(2-p) \quad f \in L^\nu(\mathbb{R}^2)$;

(ii) $\dfrac{\partial f}{\partial x} \in L^{p_1}(\mathbb{R}^2), \qquad \dfrac{\partial f}{\partial y} \in L^{p_2}(\mathbb{R}^2)$;

(iii) $\displaystyle\int_0^\infty \left[h^{-\alpha_1} \|\Delta_1^1(h) f\|_{q_1}\right]^{\theta_1 - \varepsilon} \frac{dh}{h} = +\infty .$

This implies that the values of θ_k *in Theorem 1 can not be reduced.*

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