Effective Homology and Spectral Sequences

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Computation of Homotopy Groups

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Spectral sequences

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Definition. A *spectral sequence* $E = (E^r, d^r)_{r \geq 1}$ is a family of bigraded $\mathbb{Z}$-modules $E^r = \{E^r_{p,q}\}$, each provided with a differential $d^r = \{d^r_{p,q}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$ for every $r \geq 1$.

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> "*If we think of a spectral sequence as a black box, then the input is a differential bigraded module, usually* $E^1_{*,*}$*, and, with each turn of the handle, the machine computes a successive homology according to a sequence of differentials. If some differential is unknown, then some other (any other) principle is needed to proceed. In the nontrivial cases, it is often a deep geometric idea that is caught up in the knowledge of a differential.*"

John McCleary, User's guide to spectral sequences (Publish or Perish, 1985)

Spectral sequences of filtered complexes

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It has been implemented as a new module for the Kenzo system.

This algorithm can be applied to calculate two classical examples of spectral sequences: Serre and Eilenberg-Moore.

Bousfield-Kan spectral sequence

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Definition. Let X be a simplicial set with a base point $\star \in X_0$ and R a ring. Then RX is defined as the free simplicial R-module generated by X, where the base point and its degeneracies are put equal to zero.

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$$\pi_*(RX, \star) \cong \widetilde{H}_*(X; R)$$

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$$R^k X \equiv R(R^{k-1}X) \text{ for all } k \in \mathbb{N}$$

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There exist canonical maps $\Phi : X \rightarrow RX$ and $\Psi : R^2X \rightarrow RX$ given by $\Phi(x) = 1 * x$ for all $x \in X$ and $\Psi(1 * y) = y$ for all $y \in RX$.

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For every pair (p, j) such that $0 \leq j \leq p$, *coface* and *codegeneracy* operators are defined as

$$\begin{aligned} \partial^j : \quad R^p X &\longrightarrow R^{p+1} X, \quad \partial^j = R^j \Phi R^{p-j} \\ \eta^j : R^{p+2} X &\longrightarrow R^{p+1} X, \quad \eta^j = R^j \Psi R^{p-j} \end{aligned}$$

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Theorem (Bousfield-Kan spectral sequence). *Let X be a simplicial set with base point $\star \in X_0$, and R a ring. There exists a second quadrant spectral sequence $E = (E^r, d^r)_{r \geq 1}$, whose E^1 term is given by*

$$E^1_{p,q} = \pi'_q(R^{p+1} X) = \pi_q(R^{p+1} X) \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{p-1}$$

which in the case $R = \mathbb{Z}$ and under suitable hypotheses (for instance, if X is 1-reduced) converges to the homotopy groups $\pi_*(X; \star)$.

A. K. Bousfield and D. M. Kan. The homotopy spectral sequence of a space with coefficients in a ring. Topology, 11, pp. 79–106, 1972.

Bousfield-Kan spectral sequence

Bousfield-Kan spectral sequence

For the computation of the Bousfield-Kan spectral sequence associated with a simplicial set X, the first step is the determination of groups

$$\pi_q(R^{p+1}X) \cong \widetilde{H}_q(R^p X)$$

Effective homology

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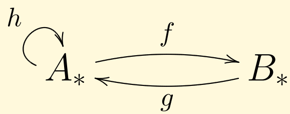
Definition. A *reduction* ρ between two chain complexes A_* and B_* (denoted by $\rho : A_* \Rrightarrow B_*$) is a triple $\rho = (f, g, h)$



satisfying the following relations:

$fg = \mathrm{id}_B$; $gf + d_A h + h d_A = \mathrm{id}_A$;

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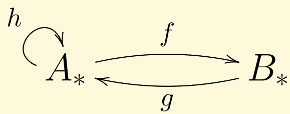

satisfying the following relations:

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Remark. If $A_* \Rrightarrow B_*$, then $A_* = B_* \oplus C_*$, with C_* acyclic, which implies that $H_n(A_*) \cong H_n(B_*)$ for all n.

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Definition. A *(strong chain) equivalence* between the complexes A_* and B_* (denoted $A_* \lll\!\!\Longleftrightarrow\!\!\ggg B_*$) is a triple (D_*, ρ, ρ') where D_* is a chain complex, $\rho : D_* \Rrightarrow A_*$ and $\rho' : D_* \Rrightarrow B_*$.

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Remark. This implies that $H_n(X) \cong H_n(HX_*)$ for all n.

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Our goal: an algorithm computing the effective homology of RX

$$C_*(RX) \Leftarrow\!\Leftarrow DR_* \Rightarrow\!\Rightarrow HR_*$$

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- Dold-Kan correspondence between the category $\mathcal{A}$ of simplicial Abelian groups and the category $\mathcal{C}$ of (positive) chain complexes:

  functors $N_* : \mathcal{A} \rightarrow \mathcal{C}$ and $\Gamma : \mathcal{C} \rightarrow \mathcal{A}$

  which satisfy $\Gamma \circ N_* = \mathrm{Id}_{\mathcal{A}}$ and $N_* \circ \Gamma = \mathrm{Id}_{\mathcal{C}}$.

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- Eilenberg-MacLane spaces: $K = K(\pi, n)$ such that $\pi_n(K) = \pi$ and $\pi_i(K) = 0$ if $i \neq n$. They can be defined as $K = \Gamma(C_*(\pi, n))$ and are objects with effective homology.

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Proposition 2. *Let A_* and B_* be chain complexes and $\rho : A_* \Rrightarrow B_*$ a reduction between them. Then one can construct a new reduction*

$$\Gamma(\rho) : C_*(\Gamma(A_*)) \Rrightarrow C_*(\Gamma(B_*))$$

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Algorithm 1.

Input:

- a 1-reduced pointed simplicial set X,
- an equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$, where HX_* is an effective complex.

Output: an equivalence $\mu_L : C_*(RX) \Lleftarrow C_*(\Gamma(\widetilde{DX}_*)) \Rrightarrow C_*(\Gamma(\widetilde{HX}_*))$, where $\widetilde{DX}_*$ and $\widetilde{HX}_*$ are chain complexes obtained respectively from DX_* and HX_*, $\widetilde{HX}_*$ is effective and $\widetilde{HX}_0 = \widetilde{HX}_1 = 0$.

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Our problem can be solved computing the effective homology of $\Gamma(C_*)$, for C_* an elementary chain complex.

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  Proposition 3. *Let C_* be an elementary chain complex with $C_m \cong C_{m+1} \cong \mathbb{Z}$ and $C_n = 0$ for $n \neq m, m+1$. Then*

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which implies $\Gamma(C_*)$ is also an object with effective homology.

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Algorithm 2.

Input: an effective chain complex E_* such that $E_0 = E_1 = 0$.

Output: an equivalence $C_*(\Gamma(E_*)) \Lleftarrow D\Gamma E_* \Rrightarrow H\Gamma E_*$, where $H\Gamma E_*$ is an effective chain complex.

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Output: an equivalence $C_*(\Gamma(E_*)) \Lleftarrow D\Gamma E_* \Rrightarrow H\Gamma E_*$, where $H\Gamma E_*$ is an effective chain complex.

For the chain complex $\widetilde{HX}_*$, we obtain an equivalence

$$\mu_R : C_*(\Gamma(\widetilde{HX}_*)) \Lleftarrow \widetilde{DR}_* \Rrightarrow HR_*$$

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Algorithm 4.

Input:

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- an equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$, where HX_* is an effective complex.

Output: the groups $E^1_{p,q} = \pi'_q(R^{p+1}X)$ for each $p, q \in \mathbb{Z}$.

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Algorithm 5.

Input:

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- an equivalence $C_*(X) \twoheadleftarrow\!\!\twoheadleftarrow DX_* \twoheadrightarrow\!\!\twoheadrightarrow HX_*$, where the chain complex HX_* is effective.

Output: the groups $E^2_{p,q}$ for each pair $p, q \in \mathbb{Z}$.

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  - *effective homotopy* theory