Effective homotopy of the fiber of a Kan fibration

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1 Introduction

Inspired by the fundamental ideas of the effective homology method (see [5] and [4]), which makes it possible to determine homology groups of complicated spaces, in [3] we started to develop a new *effective homotopy* theory, a technique which could allow the computation of *homotopy* groups of spaces. The spaces we work with in our paper [3] must be encoded as *constructive Kan simplicial sets* (that is, simplicial sets which satisfy the Kan extension property [1] in a constructive way, see [3] for details). The most important notion introduced in [3] is that of *a solution for the homotopical problem* of a constructive Kan simplicial set.

Definition 1.1. [3] A *solution for the homotopical problem* (SHmtP) posed by a constructive Kan simplicial set K is a graded 4-tuple $(\pi_n, f_n, g_n, h_n)_{n\geq 1}$ where:

- The component π_n is a standard presentation of a finitely generated Abelian group (that is to say, each π_n is a direct sum of several copies of the infinite cyclic group $\mathbb{Z}$ and some finite cyclic groups $\mathbb{Z}_{p_i^n}$, $\pi_n = \mathbb{Z}^{\alpha_n} \oplus \mathbb{Z}_{p_1^n}^{\beta_1^n} \oplus \cdots \oplus \mathbb{Z}_{p_r^n}^{\beta_r^n}$. The component π_n is therefore a well-known group which is given and where computations can be done). This group will be isomorphic to the homotopy group $\pi_n(K) = S_n(K)/(\sim)$, which is in principle unkonwn.
- The component g_n is an algorithm $g_n : \pi_n \to S_n(K)$ giving for every "abstract" homotopy class $a \in \pi_n$ a sphere $x = g_n(a) \in S_n(K)$ representing this homotopy class.

- The component f_n is an algorithm $f_n : S_n(K) \to \pi_n$ computing for every sphere $x \in S_n(K)$ "its" homotopy class $a = f_n(x) \in \pi_n$. This algorithm f_n must satisfy the following properties. First of all, the composition $f_n g_n$ must be the identity of π_n. Moreover, given $z \in K_{n+1}$ such that $\partial_i z = \star$ for all $0 \leq i \leq n$, then $f_n(\partial_{n+1} z) = 0$; in other words, $f_n(x) = 0$ for all $x \in S_n(K)$ with $x \sim \star$. Furthermore, f must be a "group" morphism, in the following sense: given $x, y \in S_n(K)$ and $w =$"$x + y$" representant of the homotopy class $[x] + [y]$ (computed by using the Kan property of K as explained in [1]), then $f_n(w) = f_n(x) + f_n(y)$.

- The component h_n is an algorithm $h_n : \ker f_n \to K_{n+1}$ satisfying $\partial_i h_n = \star$ for all $0 \leq i \leq n$ and $\partial_{n+1} h_n = \mathrm{Id}_{\ker f_n}$. This algorithm produces a *certificate* for a sphere $x \in S_n(K)$ claimed having a null homotopy class by the algorithm f_n.

If K is a Kan simpicial set and a solution for its homotopical problem is given, we say the K is an *object with effective homotopy*. The interesting point of this definition is the fact that, if K has effective homotopy, one can easily construct an algorithm computing the homotopy groups $\pi_n(K) = S_n(K)/(\sim)$ (with the corresponding generators). In this way, the effective homotopy technique can make it possible to determine homotopy groups of complicated spaces.

The problem now is how one can determine the effective homotopy of a given Kan simplicial set K. Similarly as done in the effective homology framework [4], we consider first some spaces whose effective homotopy can be directly determined (for example, the standard simplex $K = \Delta$ or Eilenberg-MacLane spaces $K(\pi, n)$'s for finitely generated Abelian groups π and $n \geq 1$, see [3] for details). Then, different constructors of Algebraic Topology should produce new spaces with effective homotopy.

As a first work in this research, we presented in [3] the following result allowing one to compute the effective homotopy of the total space of a constructive Kan fibration if the base and the fiber spaces are objects with effective homotopy.

Theorem 1.2. *[3] An algorithm can be written down:*

- ***Input:***
  - *A constructive Kan fibration $p : E \to B$ where B is a constructive Kan complex (which implies F and E are also constructive Kan simplicial sets), and F or B are simply connected.*
  - *Respective $SHmtP_F$ and $SHmtP_B$ for the simplicial sets F and B.*
- ***Output:*** *A $SHmtP_E$ for the Kan simplicial set E.*

The proof of this theorem is a bit tedious; it follows from successive applications of the Kan properties of F, B, E and p and the solutions for the homotopical problems of B and F. A sketch is included in [3]; the complete proof can be found in [2].

Let us remark that given a Kan fibration $p : E \to B$, in general it is not possible to determine the homotopy groups of the total space, $\pi_*(E)$, from the groups $\pi_*(F)$ and $\pi_*(B)$. However, the effective homotopy theory makes it possible to compute $\pi_*(E)$ if the fibration and the simplicial sets involved are *constructive* and in addition both F and B are provided with a solution for the homotopical problem, thanks to our Theorem 1.2. In this way, starting with objects B and F with trivial effective homotopy we compute the effective homotopy of the total space E; then, we can use E as base of fiber space of a different fibration, and repeating the process we can compute the homotopy groups of total spaces of complicated fibrations.

Let us consider now a similar situation, where E and B have effective homotopy, and one wants to determine the homotopy groups of the fiber of the fibration, F.

2 Solution for the homotopical problem of the fiber in a Kan fibration

Let $p : E \to B$ be a Kan fibration, with fiber F. It is necessary to remark here that the homotopy groups $\pi_*(E)$ and $\pi_*(B)$ are not sufficient in general to deduce $\pi_*(F)$, but the effective homotopy method allows one to determine

$\pi_*(F)$ if E and B are objects with effective homotopy, as explained by the following new result.

Theorem 2.1. *An algorithm can be written down:*

- ***Input:***
  - *A constructive Kan fibration $p : E \to B$ where B is a constructive Kan complex (which implies F and E are also constructive Kan simplicial sets), and F or B are simply connected.*
  - *Respective $SHmtP_E$ and $SHmtP_B$ for the simplicial sets E and B.*
- ***Output:*** *A $SHmtP_F$ for the Kan simplicial set F.*

Proof. The Kan fibration $p : E \to B$ produces a long exact sequence of homotopy [1]:

$$\cdots \xrightarrow{\text{inc}_*} \pi_{n+1}(E) \xrightarrow{p_*} \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \xrightarrow{inc_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \cdots$$

From this one can deduce a short exact sequence

$$0 \longrightarrow \text{Coker}[\pi_{n+1}(E) \xrightarrow{p_*} \pi_{n+1}(B)] \xrightarrow{i} \pi_n(F) \xrightarrow{j} \text{Ker}[\pi_n(E) \xrightarrow{p_*} \pi_n(B)] \longrightarrow 0$$

which implies the looked-for group $\pi_n(F)$ can be expressed as $\pi_n(F) \cong \text{Coker} \times_\chi \text{Ker}$ for a cohomology class $\chi \in H^2(\text{Ker}, \text{Coker})$ classifying the extension. This cohomology class is in principle unknown, but can be determined if the short exact sequence is *constructive* [6]. The most important part of the proof consists in defining a section $\sigma : \text{Ker} \to \pi_n(F)$ and a retraction $\rho : \pi_n(F) \to \text{Coker}$ such that $\rho i = \text{Id}_{\text{Coker}}$, $i\rho + \sigma j = \text{Id}_{\pi_n(F)}$ and $j\sigma = \text{Id}_{\text{Ker}}$. From them, we will give a *constructive* definition of the cohomology class which will allow us to *compute* the homotopy group $\pi_n(F)$. Furthermore the algorithms g_n and f_n will be immediately deduced from i, j, σ and ρ. We will end the proof with the computation of h_n.

Let us consider the long exact sequence of homotopy [1] of the fibration:

$$\cdots \xrightarrow{\text{inc}_*} \pi_{n+1}(E) \xrightarrow{p_*} \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \xrightarrow{inc_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \cdots$$

where the maps p_* and inc_* are the morphisms between the corresponding homotopy groups induced respectively by the fibration $p : E \to B$ and the inclusion $\mathrm{inc} : F \hookrightarrow E$, and $\partial : \pi_*(B) \to \pi_{*-1}(F)$ is the *connection morphism* [1].

The previous long exact sequence produces a short exact sequence:

$$0 \longrightarrow \mathrm{Coker}[\pi_{n+1}(E) \xrightarrow{p_*} \pi_{n+1}(B)] \xrightarrow{i} \pi_n(F) \xrightarrow{j} \mathrm{Ker}[\pi_n(E) \xrightarrow{p_*} \pi_n(B)] \longrightarrow 0$$

Let us emphasize here that the groups $\pi_*(E)$ and $\pi_*(B)$ with the corresponding generators can be computed thanks to the SHmtP for both simplicial sets E and B. Since they are finite type groups and the morphism $p_* : \pi_*(E) \to \pi_*(B)$ is induced by $p : E \to B$ (so that one can easily determine the definition of p_* on the generators of $\pi_*(E)$), it is possible to determine by means of elementary operations the groups $\mathrm{Ker} \equiv \mathrm{Ker}[\pi_n(E) \xrightarrow{p_*} \pi_n(B)]$ and $\mathrm{Coker} \equiv \mathrm{Coker}[\pi_{n+1}(E) \xrightarrow{p_*} \pi_{n+1}(B)] = \pi_{n+1}(B)/\operatorname{Im} p_*$.

The group $\pi_n(F)$ is on the contrary unknown, its computation is in fact the goal of our work. In order to represent the elements of $\pi_n(F)$, instead of homotopy classes inside an unknown group, we will use n-spheres in $S_n(F)$ *representing* these homotopy classes.

The previous short exact sequence implies the looked-for $\pi_n(F)$ is an extension of Ker by Coker, and then $\pi_n(F) \cong \mathrm{Coker} \times_\chi \mathrm{Ker}$ for a cohomology class $\chi \in H^2(\mathrm{Ker}, \mathrm{Coker})$ classifying the extension. But let us remark that for the moment this cohomology class is not known!

In order to obtain the cohomology class χ which will allow us to compute $\pi_n(F)$, we are going to make our exact sequence *constructive* [6], defining a section $\sigma : \mathrm{Ker} \to \pi_n(F)$ and a retraction $\rho : \pi_n(F) \to \mathrm{Coker}$ such that $\rho i = \mathrm{Id}_{\mathrm{Coker}}$, $i\rho + \sigma j = \mathrm{Id}_{\pi_n(F)}$, and $j\sigma = \mathrm{Id}_{\mathrm{Ker}}$. Before this, we describe the way the maps i and j will be represented.

The map i of the diagram is obtained by following the diagram chasing path:

$$\mathrm{Coker} \longrightarrow \pi_{n+1}(B) \xrightarrow{g_n} S_{n+1}(B) \overset{\partial}{\hookrightarrow} S_n(F)$$

where the map g_n is produced by SHmtP_B, the map $\mathrm{Coker} \to \pi_{n+1}(B)$ consists in choosing a representant of an element of the cokernel (which can be elementary done since $\pi_{n+1}(B)$ is of finite type), and $\partial : S_{n+1}(B) \to S_n(F)$ is the connection homomorphism, in our case *constructively* defined as follows: given $b \in S_{n+1}(B)$, we consider $x_i = \star \in E_n$ for $i = 1, \ldots, n+1$; this is a set of $(n+1)$ n-simplices such that $\partial_i x_j = \partial_{j-1} x_i = \star$ and such that $\partial_i b = \star = p(x_i)$ for all i. Since p is a constructive Kan fibration, one has an algorithm σ_p which provides us an element $x \in E_{n+1}$ with $\partial_i x = x_i = \star$, $1 \leq i \leq n+1$, and $p(x) = b$. Then $\partial_0 x$ satisfies $p(\partial_0 x) = \partial_0 p(x) = \partial_0 b = \star$, which implies $\partial_0 x \in F_n$, and $\partial_i \partial_0 x = \partial_0 \partial_{i+1} x = \star$ for all $0 \leq i \leq n$; therefore $\partial_0 x \in S_n(F)$. The connection morphism $\partial : S_{n+1}(B) \to S_n(F)$ is defined then by $\partial(b) = \partial_0 x$ (the definition of ∂ over the corresponding homotopy groups, $\partial : \pi_{n+1}(B) \to \pi_n(F)$, is given by $\partial(\beta) = [\partial_0 x] \in \pi_{n-1}(F)$ where b is any generator of a class $\beta \in \pi_{n+1}(B)$; as explained in [1], a different choice for the representant b of the class β would lead to the same homotopy class in $\pi_n(F)$). The morphism $i : \mathrm{Coker} \to \pi_n(F)$ is therefore implemented as a map $i : \mathrm{Coker} \to S_n(F)$.

The map j of the diagram is implemented as a map $j : S_n(F) \to \mathrm{Ker}$, obtained from the path:

$$S_n(F) \xrightarrow{\text{inc}} S_n(E) \xrightarrow{f_n} \mathrm{Ker} \subseteq \pi_n(E)$$

The map f_n (the third component in the effective homotopy of E) is in fact defined as $f_n : S_n(E) \to \pi_n(E)$, but given an element $x \in S_n(E)$ which is in $S_n(F)$, then $p(x) = 0$ and then its homotopy class is necessarily in Ker.

The desired section $\sigma : \mathrm{Ker} \to \pi_n(F)$, which will be implemented as a map $\sigma : \mathrm{Ker} \to S_n(F)$, is constructed as follows: let $\alpha \in \mathrm{Ker} \subseteq \pi_n(E)$, we choose a representant $e \in \alpha$ produced by g_n in SHmtP of E. Since $\alpha \in \mathrm{Ker}\, p_*$, one has $f_n(p(e)) = 0 \in \pi_n(B)$ and then the algorithm h_n of SHmtP for B returns $a \in B_{n+1}$ such that $\partial_i a = \star$ for $0 \leq i \leq n$ and $\partial_{n+1} a = p(e)$. The Kan property algorithm for B allows us then to obtain $b \in B_{n+1}$ such that $\partial_1 b = p(e)$ and $\partial_i b = \star$ for $i \neq 1$ (it suffices to take the $n+2$ $(n+1)$-simplices of B $\star, a, \star, \ldots, \star, -$). We consider then the $n+1$ n-simplices $-, e, \star, \ldots, \star$ of E which are compatible, and $b \in B_{n+1}$ such that $\partial_i b = p(x_i)$. The algorithm σ_p returns $x \in E_{n+1}$ such that $\partial_i x = \star$ for all $2 \leq i \leq n+1$, $\partial_1 x = e$, $p(x) = b$, and $\partial_0 x \in S_n(F)$ since $p(\partial_0 x) = \partial_0 p(x) = \partial_0 b = \star$ and $\partial_i \partial_0 x = \partial_0 \partial_{i+1} x = \star$ for all $0 \leq i \leq n$. We define $\sigma(\alpha) = \partial_0 x \in S_n(F)$.

The section $\sigma : \mathrm{Ker} \to \pi_n(F)$ is well-defined on Ker since the election of a representant for $\alpha \in \mathrm{Ker} \subseteq \pi_n(E)$ is uniquely done by g_n. Moreover, one can easily observe that $j\sigma = \mathrm{Id}_{\mathrm{Ker}}$: given $\alpha \in \mathrm{Ker} \subseteq \pi_n(E)$, let $e \in \alpha$ the representant produced by g_n, then $j\sigma(\alpha) = f_n(\partial_0 x)$ where x is obtained as explained in the previous paragraph. The element $x \in E_{n+1}$ satifies $\partial_1 x = e$

and $\partial_i x = \star$ for all $2 \leq i \leq n+1$ and therefore (thanks to Lemma 1.6 in [2]) $[e] = [\partial_0 x]$ in $\pi_n(E)$, so that $j\sigma(\alpha) = \alpha$.

It is now necessary to construct a retraction $\rho : \pi_n(F) \to \mathrm{Coker}$ that we will implement as a map $\rho : S_n(F) \to \mathrm{Coker}$. First of all, we define ρ over the elements $e \in S_n(F)$ such that $e \in \operatorname{Ker} j$, that is, $f_n(e) = 0 \in \pi_n(E)$. The algorithm h_n of SHmtP for E returns $y \in E_{n+1}$ such that $\partial_i y = \star$ for $0 \leq i \leq n$ and $\partial_{n+1} y = e$. Then, by Lemma 1.5 in [2], one obtains $x \in E_{n+1}$ such that $\partial_i x = \star$ for $1 \leq i \leq n+1$ and $\partial_0 x = e$. We consider now $b = p(x) \in B_{n+1}$, which satisfies $\partial_0 b = \partial_0 p(x) = p(\partial_0 x) = p(e) = \star$ (since $e \in S_n(F) \subseteq F_n$) and $\partial_i b = \partial_i p(x) = p(\partial_i x) = p(\star) = \star$ for $i \geq 1$, that is, $b \in S_{n+1}(B)$. We define $\rho(e) = [f_{n+1}(b)] = [f_{n+1}(p(x))] \in \mathrm{Coker} = \pi_{n+1}(B)/\operatorname{Im} p_*$.

First of all we should prove that ρ is well defined on $\operatorname{Ker} j$, that is, given $e, e' \in S_n(F)$ such that $f_n(e) = f_n(e') = 0$ (that is, $[e] = [e'] = 0$ in $\pi_n(E)$) and $e \sim e'$ in F, then $\rho(e)$ and $\rho(e')$ are the same element in the quotient group $\mathrm{Coker} = \pi_{n+1}(B)/\operatorname{Im} p_*$. Since $e \sim e'$ in F, there exists an $(n+1)$-simplex $z \in F_{n+1}$ such that $\partial_i z = \star$ for $0 \leq i \leq n-1$, $\partial_n z = e$ and $\partial_{n+1} z = e'$ (let us remark that, since F is not yet an object with effective homotopy, this element z can not be constructively determined; however, in this case it is sufficient to know that it exists, since we only want to prove that the map is well defined). On the other hand, since $[e] = [e'] = 0$ in E, one has $x, x' \in E_{n+1}$ such that $\partial_i x = \partial_i x' = \star$ for all $i \geq 1$, $\partial_0 x = e$ and $\partial_0 x' = e'$ (the algorithm h_n of E returns $y, y' \in E_{n+1}$ such that $\partial_i y = \partial_i y' = \star$ for all $0 \leq i \leq n$, $\partial_{n+1} y = e$ and $\partial_{n+1} y' = e'$, and then a simple *game* with the Kan property allows one to determine the desired x and x').

We consider now the $n+2$ $(n+1)$-simplices $z, -, \star, \ldots, \star, x, x'$ of E_{n+1} which satisfy the compatibility conditions:

$$\partial_i x_j = \begin{cases} \star = \partial_{j-1} x_i & \text{if } i > 0 \quad (i \neq 1) \text{ and then } j > 1 \\ \partial_0 x_j = \star = \partial_{j-1} x_0 & \text{if } i = 0, j \neq n+1, n+2 \\ \partial_0 x_{n+1} = \partial_0 x = e = \partial_n x_0 & \text{if } i = 0, j = n+1 \\ \partial_0 x_{n+2} = \partial_0 x' = e' = \partial_{n+1} x_0 & \text{if } i = 0, j = n+2 \end{cases}$$

The algorithm σ_E (which makes the simplicial set E a constructive Kan complex) returns an $(n+2)$-simplex $w \in E$ such that $\partial_0 w = z$, $\partial_i w = \star$ for $2 \leq i \leq n$, $\partial_{n+1} w = x$ and $\partial_{n+2} w = x'$. Then the $n+1$-simplex $\partial_1 w$ is an

$(n+1)$-sphere of E:

$$\partial_i \partial_1 w = \begin{cases} \partial_0 \partial_0 w = \partial_0 z = \star & \text{if } i = 0 \\ \partial_1 \partial_{i+1} w = \star & \text{for } 1 \leq i \leq n-1 \\ \partial_1 \partial_{n+1} w = \partial_1 x = \star & \text{if } i = n \\ \partial_1 \partial_{n+2} w = \partial_1 x' = \star & \text{if } i = n+1 \end{cases}$$

In particular, this implies that $[p(\partial_1 w)] \in \pi_{n+1}(B)$ is a class contained $\operatorname{Im} p_*$.

Moreover, $p(w)$ satisfies:

$$\partial_i p(w) = p(\partial_i w) = \begin{cases} p(z) = \star & \text{if } i = 0 \\ p(\partial_1 w) \in \operatorname{Im} p_* & \text{if } i = 1 \\ \star & \text{for } 2 \leq i \leq n \\ p(x) & \text{if } i = n+1 \\ p(x') & \text{if } i = n+2 \end{cases}$$

This proves the classes $[p(x)], p[x'] \in \pi_{n+1}(B)$ correspond to the same element in the quotient $\operatorname{Coker} = \pi_{n+1}(B)/\operatorname{Im} p_*$, and this implies the desired result $\rho[e] = \rho[e']$, so that ρ is well defined on $\operatorname{Ker} j$.

Furthermore, one can see that $\rho : \operatorname{Ker} j \to \operatorname{Coker}$ is a morphism of groups. Let $e, e' \in S_n(F) \cap \operatorname{Ker} j$ and e'' such that $e'' = e + e'$. Then we have to prove $\rho[e''] = \rho[e] + \rho[e']$. The proof is completely similar to the previous construction demonstrating that ρ is well-defined on $\operatorname{Ker} j$ (that one is in fact a particular case of this, where $e = \star$), so that the detailed descriptions of the necessary compatibility properties are skipped.

Let $e'' = e + e'$ in $S_n(F)$, that is, there exists an $(n+1)$-simplex $z \in F_{n+1}$ such that $\partial_i z = \star$ for $0 \leq i \leq n-2$, $\partial_{n-1} z = e, \partial_n z = e''$ and $\partial_{n+1} z = e'$. On the other hand, since $[e] = [e'] = [e''] = 0$ in E, one has $x, x', x'' \in E_{n+1}$ such that $\partial_i x = \partial_i x' = \partial_i x'' = \star$ for all $i \geq 1$, $\partial_0 x = e$, $\partial_0 x' = e'$ and $\partial_0 x'' = e''$.

We consider now the compatible $n+2$ $(n+1)$-simplices $z, -, \star, \ldots, \star, x, x'', x'$ of E_{n+1}. The algorithm σ_E returns an $(n+2)$-simplex $w \in E$ such that $\partial_0 w = z$, $\partial_i w = \star$ for $2 \leq i \leq n-1$, $\partial_n w = x$, $\partial_{n+1} w = x''$, and $\partial_{n+2} w = x'$. Then the $n+1$-simplex $\partial_1 w$ is an $(n+1)$-sphere of E (that is, $\partial_i(\partial_1 w) = \star$ for all $0 \leq i \leq n+1$), and $p(w) \in B_{n+2}$ satisfies:

$$\partial_i p(w) = p(\partial_i w) = \begin{cases} p(z) = \star & \text{if } i = 0 \\ p(\partial_1 w) \in \operatorname{Im} p_* & \text{if } i = 1 \\ \star & \text{for } 2 \leq i \leq n-1 \\ p(x) & \text{if } i = n \\ p(x'') & \text{if } i = n+1 \\ p(x') & \text{if } i = n+2 \end{cases}$$

which implies that $\rho[e''] = [p(x'')] = [p(x)] + [p(x')] = \rho[e] + \rho[e']$ in the quotient $\operatorname{Coker} = \pi_{n+1}(B)/\operatorname{Im} p_*$, so that we have proved ρ is a morphism of groups over $\operatorname{Ker} j$.

We have therefore defined ρ over the elements in $\operatorname{Ker} j$, which provides us a map $\rho' : S_n(F) \cap \operatorname{Ker} j \to \operatorname{Coker}$ which is a group morphism over the homotopy classes of $\pi_n(F)$. One can easily observe that $i\rho' = \operatorname{Id}_{\pi_n(F)\cap \operatorname{Ker} j}$: given $e \in S_n(F) \cap \operatorname{Ker} j$, we have defined $\rho'(e) = [p(x)] \in \operatorname{Coker}$ where $x \in E_{n+1}$ is an element (*constructively* determined) such that $\partial_0 x = e$ and $\partial_i x = \star$ for all $1 \leq i \leq n+1$. Then $i\rho'(e) = [\partial p(x)] \in \operatorname{Ker} \subseteq \pi_n(F)$ where ∂ is the connecting homomorphism. Since $\partial p(x) = \partial_0 x = e$ is one possible definition for ∂ and one knows that it is well defined, we have $i\rho'[e] = [e] \in \pi_n(F)$.

On the other hand, let $\alpha \in \operatorname{Coker} = \pi_{n+1}(B)/\operatorname{Im} p_*$, with a (uniquely determined) representant $b \in S_{n+1}(B)$. Then $i(\alpha) = \partial_0 x \in F$ where $x \in E_{n+1}$ satisfies $\partial_i x = \star$ for $i \geq 1$ and $p(x) = b$. Then $\rho' i(\alpha) = b$ since all the possible elections are unique.

We have defined therefore $\rho' : \pi_n(F) \cap \operatorname{Ker} j \to \operatorname{Coker}$ satisfying $i\rho' = \operatorname{Id}_{\pi_n(F)\cap \operatorname{Ker} j}$ and $\rho' i = \operatorname{Id}_{\operatorname{Coker}}$. The following step consists in defining ρ over all elements of $\pi_n(F)$. Let $e \in S_n(F)$ such that $e \notin \operatorname{Ker} j$. We consider $\sigma j(e) \in S_n(F)$ and the "difference" $z = e - \sigma j(e)$ which satisfies $[z] = [e] - [\sigma j(e)] \in \pi_n(F)$. Then

$$j(z) = j(e - \sigma j(e)) = j(e) - j\sigma j(e) = j(e) - j(e) = 0$$

Since $j(z) = 0$ then $z \in S_n(F) \cap \operatorname{Ker} j$ and one can determine $\rho'(z) \in \operatorname{Coker}$. We define then $\rho(e) \equiv \rho'(z)$. This new map $\rho : \pi_n(F) \to \operatorname{Coker}$ implemented as $\rho : S_n(F) \to \operatorname{Coker}$ satisfies $\rho i = \operatorname{Id}_{\operatorname{Coker}}$ and $i\rho + \sigma j = \operatorname{Id}_{\pi_n(F)}$. This last equation is directly deduced of the definition of ρ in terms of ρ' and the property $i\rho' = \operatorname{Id}_{\pi_n(F)\cap \operatorname{Ker} j}$:

$$i\rho([e]) + \sigma j([e]) = i\rho'([e] - \sigma j([e])) + \sigma j([e]) = [e] - \sigma j([e]) + \sigma j([e]) = [e]$$

From the previous identities, one can easily deduce that $\rho\sigma = 0$.

The section $\sigma : \mathrm{Ker} \to \pi_n(F)$ and the retraction $\rho : \pi_n(F) \to \mathrm{Coker}$ satisfying $\rho i = \mathrm{Id}_{\mathrm{Coker}}$, $i\rho + \sigma j = \mathrm{Id}_{\pi_n(F)}$ and $j\sigma = \mathrm{Id}_{\mathrm{Ker}}$ allow one to define the 2-cocycle $\chi \in H^2(\mathrm{Ker}, \mathrm{Coker})$ which classifies the extension. Given $\alpha, \beta \in \mathrm{Ker}$ two homotopy classes, $\chi(\alpha, \beta)$ is defined as:

$$\chi(\alpha, \beta) = \rho(\sigma(\alpha) + \sigma(\beta) - \sigma(\alpha + \beta)) = \rho'(\sigma(\alpha) + \sigma(\beta) - \sigma(\alpha + \beta))$$

It is not difficult to observe, using the facts that $\rho' : \mathrm{Ker}\, j \to \mathrm{Coker}$ is a group morphism and $\rho\sigma = 0$, that with this definition χ satisfies the necessary properties of a 2-cocycle:

$$\begin{array}{l} \chi(\alpha, 0) = 0 = \chi(0, \alpha) \\ \chi(\alpha + \beta, \gamma) = \chi(\beta, \gamma) - \chi(\alpha, \beta) + \chi(\alpha, \beta + \gamma) \end{array}$$

The standard extension theory proves then that $\pi_n(F) \cong \mathrm{Coker} \times_\chi \mathrm{Ker}$ and an elementary calculation can produce the unique possible isomorphism class $\pi_n \in \mathcal{M}$ of this group and also some explicit isomorphism $\pi_n \leftrightarrow \mathrm{Coker} \times_\chi \mathrm{Ker}$. In this way the first element π_n of the 4-tuple in dimension n of the solution for the homotopic problem of F has been reached.

We need also the three components g_n, f_n and h_n to achieve the construction of SHmtP_F. We define g_n and f_n firstly with respect to the model $\mathrm{Coker} \times_\chi \mathrm{Ker}$ of $\pi_n(F)$. It is easy to justify $g_n(\alpha, \beta) = i(\alpha) + \sigma(\beta) \in S_n(E)$ if $\alpha \in \mathrm{Coker}$ and $\beta \in \mathrm{Ker}$. In the same way, $f_n(x) = (\rho(x), j(x))$ is the unique possible definition of f_n.

In this way, one has the desired identity $f_n g_n = \mathrm{Id}_{\mathrm{Coker} \times_\chi \mathrm{Ker}}$:

$$\begin{aligned} f_n g_n(\alpha, \beta) &= f_n(i(\alpha) + \sigma(\beta)) = (\rho(i(\alpha) + \sigma(\beta)), j(i(\alpha) + \sigma(\beta))) \\ &= (\rho'(i(\alpha) + \sigma(\beta) - \sigma j(i(\alpha) + \sigma(\beta))), ji(\alpha) + j\sigma(\beta)) \\ &= (\rho'(i(\alpha) + \sigma(\beta) - \sigma(ji(\alpha) + j\sigma(\beta))), \beta) \\ &= (\rho'(i(\alpha) + \sigma(\beta) - \sigma j\sigma(\beta))), \beta) \\ &= (\rho'(i(\alpha) + \sigma(\beta) - \sigma(\beta))), \beta) = (\rho'(i(\alpha)), \beta) = (\alpha, \beta) \end{aligned}$$

Furthermore, f_n satisfies the two additional conditions that we have required. First of all, given $z \in F_{n+1}$ such that $\partial_i z = \star$ for all $0 \leq i \leq n$, one must verify that $f_n(\partial_{n+1} z) = 0$. The hypothesis $\partial_i z = \star$ for all $0 \leq i \leq n$

means that $\partial_{n+1} z \sim \star$ in F_n, so that $j([\partial_{n+1}]) = [\star] = 0 \in \text{Ker}$. On the other hand, since $\partial_{n+1} z \in \operatorname{Ker} j$, then $\rho(\partial_{n+1} z) = \rho'(\partial_{n+1} z)$, and taking into account that $\rho' : \operatorname{Ker} j \to \text{Coker}$ is a group morphism, one has $\rho'(\partial_{n+1} z) = [\star] \in \text{Coker}$. Therefore $f_n(\partial_{n+1} z) = (\rho(\partial_{n+1} z), j(\partial_{n+1} z)) = (0, 0) \equiv 0 \in \text{Coker} \times_\chi \text{Ker}$.

On the other hand, one can also prove that f_n is a "group morphism", that is: given $x, y \in S_n(F)$ and $z = x + y$ a representant of the class $[x] + [y] \in \pi_n(F)$, then $f_n(z) = f_n(x) + f_n(y)$. Let $x, y \in S_n(F)$, then $f_n(x) = (\rho(x), j(x))$ and $f_n(y) = (\rho(y), j(y))$. We consider the sum of both elements in the group $\text{Coker} \times_\chi \text{Ker}$:

$$f_n(x) + f_n(y) = (\rho(x) + \rho(y) + \chi(j(x), j(y)), j(x) + j(y))$$

The second factor is clearly equal to the desired $j(x + y)$, since j is a group morphism. For the first factor, we have

$$\begin{aligned}
\rho(x) + \rho(y) + \chi(j(x), j(y)) &= \rho'(x - \sigma j(x)) + \rho'(y - \sigma j(y)) \\
&\quad + \rho'(\sigma j(x) + \sigma j(y) - \sigma(j(x) + j(y))) \\
&= \rho'(x - \sigma j(x) + y - \sigma j(y) + \sigma j(x) + \sigma j(y) \\
&\quad - \sigma j(x + y)) = \rho'(x + y - \sigma j(x + y)) \\
&= \rho(x + y)
\end{aligned}$$

In this way we have proved $f_n(x) + f_n(y) = (\rho(x+y), j(x+y)) = f_n(x+y)$.

The definitions of $g_n : \text{Coker} \times_\chi \text{Ker} \to S_n(F)$ and $f_n : S_n(F) \to \text{Coker} \times_\chi \text{Ker}$ can then be converted into correspondances with π_n thanks to an arbitrary group isomophism $\text{Coker} \times_\chi \text{Ker} \cong \pi_n$, so that the required properties are still satisfied.

Constructing the map h_n is a little more complicated, a small *game* with the Kan extension properties is again necessary. Let $e \in S_n(F)$ such that $e \in \operatorname{Ker} f_n$, that is, $\rho(e) = 0 \in \text{Coker} = \pi_{n+1}(B)/\operatorname{Im} p_*$ and $j(e) = 0 \in \text{Ker} \subseteq \pi_n(E)$. The second property $j(e) = 0$ implies $e \sim \star$ in E. Following the definition of ρ, one has $x \in E_{n+1}$ such that $\partial_0 x = e$ and $\partial_i x = \star$ for all $i \geq 1$ and then we define $\rho(e) = [p(x)] \in \text{Coker}$. Since we have supposed $\rho(e) = 0$, this implies $p(x) \in \operatorname{Im} p_* \subseteq B_{n+1}$, so that $p(x) = p(y)$ where y is a sphere inE_{n+1}. Let us consider now the element $b = \eta_0 p(x) - \eta_1 p(x) + \eta_2 p(x)$ which satisfies $\partial_0 b = \partial_3 b = p(x)$ and $\partial_i b = \star$ for $i \neq 0, 3$, and then the $n + 1$

n-simplices of E $x, -, \star, y, \star, \ldots, \star$. The algorithm σ_p for the Kan extension property of the fibration returns $z \in E_{n+1}$ such that $p(z) = b$, $\partial_0 z = x$, $\partial_3 z = y$ and $\partial_i z = \star$ for $i \neq 0, 1, 3$. Then the unknown face $\partial_1 z$ satisfies

$$\partial_i \partial_1 z = \begin{cases} \partial_0 \partial_0 z = \partial_0 x = e & \text{if } i = 0 \\ \partial_1 \partial_2 z = \star & \text{if } i = 1 \\ \partial_1 \partial_3 z = \partial_1 y = \star & \text{if } i = 2 \\ \partial_1 \partial_{i+1} z = \star & \text{if } 3 \leq i \leq n+1 \end{cases}$$

Since that $p(\partial_1 z) = \partial_1 p(z) = \partial_1 b = \star$, one has that $\partial_1 z \in F_{n+1}$ and therefore we can define $h_n(e) = \partial_1 z$ which satisfies the desired properties.

□

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