Effective Homology of Free Simplicial Abelian Groups: the Acyclic Case

A. Romero

Universidad de La Rioja (Spain)

Sevilla, September 2006

Computation of Homotopy Groups

Computation of Homotopy Groups

X a simplicial set, $\pi_i(X)$???

Computation of Homotopy Groups

X a simplicial set, $\pi_i(X)$???

- Theoretical algorithm by Edgar Brown

Computation of Homotopy Groups

X a simplicial set, $\pi_i(X)$???

- Theoretical algorithm by Edgar Brown
- Postnikov tower

Computation of Homotopy Groups

X a simplicial set, $\pi_i(X)$???

- Theoretical algorithm by Edgar Brown
- Postnikov tower
- Bousfield-Kan spectral sequence

Computation of Homotopy Groups

X a simplicial set, $\pi_i(X)$???

- Theoretical algorithm by Edgar Brown
- Postnikov tower
- Bousfield-Kan spectral sequence

Spectral sequences

Spectral sequences

Definition. A *Spectral Sequence* $E = \{E^r, d^r\}$ is a family of $\mathbb{Z}$-bigraded modules E^1, E^2,…, each provided with a differential $d^r = \{d^r_{p,q}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$, $r = 1, 2, \ldots$

Spectral sequences

Definition. A *Spectral Sequence* $E = \{E^r, d^r\}$ is a family of $\mathbb{Z}$-bigraded modules E^1, E^2,..., each provided with a differential $d^r = \{d^r_{p,q}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$, $r = 1, 2, \ldots$

They are not algorithms!

Spectral sequences

Definition. A *Spectral Sequence* $E = \{E^r, d^r\}$ is a family of $\mathbb{Z}$-bigraded modules E^1, E^2,..., each provided with a differential $d^r = \{d^r_{p,q}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$, $r = 1, 2, \ldots$

They are not algorithms!

- Spectral sequences associated with filtered chain complexes: using the *effective homology* method, an algorithm computing the whole set of their components has been developed.

Spectral sequences

Definition. A *Spectral Sequence* $E = \{E^r, d^r\}$ is a family of $\mathbb{Z}$-bigraded modules E^1, E^2,…, each provided with a differential $d^r = \{d^r_{p,q}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$, $r = 1, 2, \ldots$

They are not algorithms!

- Spectral sequences associated with filtered chain complexes: using the *effective homology* method, an algorithm computing the whole set of their components has been developed.

- Generalization: spectral sequences which are not associated with filtered complexes.

Spectral sequences

Definition. A *Spectral Sequence* $E = \{E^r, d^r\}$ is a family of $\mathbb{Z}$-bigraded modules E^1, E^2,..., each provided with a differential $d^r = \{d^r_{p,q}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$, $r = 1, 2, \ldots$

They are not algorithms!

- Spectral sequences associated with filtered chain complexes: using the *effective homology* method, an algorithm computing the whole set of their components has been developed.

- Generalization: spectral sequences which are not associated with filtered complexes. In particular, the Bousfield-Kan spectral sequence.

Bousfield-Kan spectral sequence

Bousfield-Kan spectral sequence

Associated with a simplicial set X with base point $* \in X_0$

Bousfield-Kan spectral sequence

Associated with a simplicial set X with base point $* \in X_0$

Definition. Let X be a pointed simplicial set and R a ring. Then RX is defined as the free simplicial R-module generated by X.

Bousfield-Kan spectral sequence

Associated with a simplicial set X with base point $* \in X_0$

Definition. Let X be a pointed simplicial set and R a ring. Then RX is defined as the free simplicial R-module generated by X.

For the computation of the Bousfield-Kan spectral sequence associated with X, the first step is the determination of the homology groups

$$H_i(RX)$$

Bousfield-Kan spectral sequence

Associated with a simplicial set X with base point $* \in X_0$

Definition. Let X be a pointed simplicial set and R a ring. Then RX is defined as the free simplicial R-module generated by X.

For the computation of the Bousfield-Kan spectral sequence associated with X, the first step is the determination of the homology groups

$$H_i(RX)$$

Effective homology

Effective homology

A method that provides algorithms for the computation of homology groups of complicated spaces: twisted cartesian products, classifying spaces, loop spaces...

Effective homology

A method that provides algorithms for the computation of homology groups of complicated spaces: twisted cartesian products, classifying spaces, loop spaces...

$$
\begin{tikzcd}
X_1, X_2, \ldots, X_n \arrow[d, "\varphi"] \\
X
\end{tikzcd}
$$

Effective homology

A method that provides algorithms for the computation of homology groups of complicated spaces: twisted cartesian products, classifying spaces, loop spaces...

$$\begin{array}{ccc} X_1, X_2, \ldots, X_n & \qquad & X_1^{EH}, X_2^{EH}, \ldots, X_n^{EH} \\ \Big\downarrow \varphi & & \\ X & & \end{array}$$

Effective homology

A method that provides algorithms for the computation of homology groups of complicated spaces: twisted cartesian products, classifying spaces, loop spaces...

$$
\begin{array}{ccc}
X_1, X_2, \ldots, X_n & \qquad & X_1^{EH}, X_2^{EH}, \ldots, X_n^{EH} \\
\Big\downarrow \varphi & & \Big\downarrow \varphi^{EH} \\
X & & X^{EH}
\end{array}
$$

Effective homology

Effective homology

Definition. A *reduction* ρ between two chain complexes A and B (denoted by $\rho : A \Rrightarrow B$) is a triple $\rho = (f, g, h)$
$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$
satisfying the following relations:

$$fg = \mathrm{id}_B;\ gf + d_A h + h d_A = \mathrm{id}_A;$$

$$fh = 0;\ hg = 0;\ hh = 0.$$

Effective homology

Definition. A *reduction* ρ between two chain complexes A and B (denoted by $\rho : A \Rrightarrow B$) is a triple $\rho = (f, g, h)$

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

satisfying the following relations:

$$fg = \mathrm{id}_B;\ gf + d_A h + h d_A = \mathrm{id}_A;$$

$$fh = 0;\ hg = 0;\ hh = 0.$$

Remark. If $A \Rrightarrow B$, then $A = B \oplus C$, with C acyclic, which implies that $H_*(A) \cong H_*(B)$.

Effective homology

Definition. A *reduction* ρ between two chain complexes A and B (denoted by $\rho : A \Rrightarrow B$) is a triple $\rho = (f, g, h)$

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

satisfying the following relations:

$$fg = \mathrm{id}_B;\ gf + d_A h + h d_A = \mathrm{id}_A;$$

$$fh = 0;\ hg = 0;\ hh = 0.$$

Remark. If $A \Rrightarrow B$, then $A = B \oplus C$, with C acyclic, which implies that $H_*(A) \cong H_*(B)$.

Definition. Roughly speaking, an *object with effective homology* is a triple (X, EC, ρ) where EC is an effective chain complex and ρ is a reduction $\rho : C(X) \Rrightarrow EC$.

Effective homology: the acyclic case

Effective homology: the acyclic case

X an acyclic simplicial set

$H_i(X) = 0$ for all $i > 0$

Effective homology: the acyclic case

X an acyclic simplicial set

$$H_i(X) = 0 \text{ for all } i > 0$$

In this case, the effective homology of X is given by a reduction

$$C(X) \Rrightarrow \mathbb{Z}$$

Effective homology: the acyclic case

X an acyclic simplicial set

$H_i(X) = 0$ for all $i > 0$

In this case, the effective homology of X is given by a reduction

$$C(X) \Rrightarrow \mathbb{Z}$$

defined by a contraction homotopy

$$h : C_*(X) \to C_{*+1}(X)$$

such that $dh + hd = \mathrm{id}$

Effective homology of RX: the acyclic case

Effective homology of RX: the acyclic case

X an acyclic pointed simplicial set ($\Rightarrow H_i(X) = 0$ for all $i > 0$) with effective homology, $h : C_*(X) \longrightarrow C_{*+1}(X)$ such that $dh + hd = \mathrm{id}$.

Effective homology of RX: the acyclic case

X an acyclic pointed simplicial set ($\Rightarrow H_i(X) = 0$ for all $i > 0$) with effective homology, $h : C_*(X) \to C_{*+1}(X)$ such that $dh + hd = \mathrm{id}$.

The Hurewicz Theorem and the relation $\pi_i(RX) \cong H_i(X)$ imply RX is also acyclic ($\Rightarrow H_i(RX) = 0$ for all $i > 0$).

Effective homology of RX: the acyclic case

X an acyclic pointed simplicial set ($\Rightarrow H_i(X) = 0$ for all $i > 0$) with effective homology, $h : C_*(X) \to C_{*+1}(X)$ such that $dh + hd = \mathrm{id}$.

The Hurewicz Theorem and the relation $\pi_i(RX) \cong H_i(X)$ imply RX is also acyclic ($\Rightarrow H_i(RX) = 0$ for all $i > 0$).

We want to find the effective homology of RX, that is, a contraction homotopy

$$\bar{h} : C_*(RX) \to C_{*+1}(RX)$$

$$\text{such that } \bar{d}\bar{h} + \bar{h}\bar{d} = \mathrm{id}$$

Effective homology of RX: the acyclic case

An algorithm determining the contraction homotopy

$$\bar{h} : C_*(RX) \to C_{*+1}(RX)$$

Effective homology of RX: the acyclic case

An algorithm determining the contraction homotopy

$$\bar{h} : C_*(RX) \rightarrow C_{*+1}(RX)$$

- First, a simplicial homotopy $h_i : RX_q \rightarrow RX_{q+1}, 0 \leq i \leq q$, is built.

Effective homology of RX: the acyclic case

An algorithm determining the contraction homotopy

$$\bar{h} : C_*(RX) \to C_{*+1}(RX)$$

- First, a simplicial homotopy $h_i : RX_q \to RX_{q+1}, 0 \leq i \leq q$, is built. It is defined recursively using the following ideas:

  $- \pi_i(RX) \cong H_i(X) = 0 \; \forall i > 0$

Effective homology of RX: the acyclic case

An algorithm determining the contraction homotopy

$$\bar{h} : C_*(RX) \to C_{*+1}(RX)$$

- First, a simplicial homotopy $h_i : RX_q \longrightarrow RX_{q+1}, 0 \leq i \leq q$, is built.

  It is defined recursively using the following ideas:

  – $\pi_i(RX) \cong H_i(X) = 0 \ \forall i > 0$

  – the Kan property holds since RX is a simplicial group

Effective homology of RX: the acyclic case

An algorithm determining the contraction homotopy

$$\bar{h} : C_*(RX) \to C_{*+1}(RX)$$

- First, a simplicial homotopy $h_i : RX_q \to RX_{q+1}, 0 \leq i \leq q$, is built.

  It is defined recursively using the following ideas:

  – $\pi_i(RX) \cong H_i(X) = 0 \ \forall i > 0$

  – the Kan property holds since RX is a simplicial group

- Then, the chain homotopy $\bar{h} : C_n(RX) \to C_{n+1}(RX)$ is given by

$$\bar{h}(x) = \sum_{i=0}^{n} (-1)^i h_i(x) \text{ if } x \in X_n$$

Effective homology of RX: the general case

Effective homology of RX: the general case

X a simplicial set with effective homology such that $C(X) = E \oplus A$, where E is an effective chain complex and A is acyclic.

Effective homology of RX: the general case

X a simplicial set with effective homology such that $C(X) = E \oplus A$, where E is an effective chain complex and A is acyclic.

A general algorithm computing the effective homology of RX has been developed.

Effective homology of RX: the general case

X a simplicial set with effective homology such that $C(X) = E \oplus A$, where E is an effective chain complex and A is acyclic.

A general algorithm computing the effective homology of RX has been developed. Several ideas are used:

- the Dold-Kan correspondence between the categories of chain complexes and simplicial Abelian groups
- Eilenberg-MacLane spaces
- the Eilenberg-Zilber theorem
- the Basic Perturbation Lemma...

Further work

Further work

- Implementation in Common Lisp (similar to the set of programs computing spectral sequences associated with filtered complexes).

Further work

- Implementation in Common Lisp (similar to the set of programs computing spectral sequences associated with filtered complexes).

- Algorithm computing the Bousfield-Kan spectral sequence associated with a simplicial set X. The homology groups $H_i(RX)$ are only one ingredient!

Further work

- Implementation in Common Lisp (similar to the set of programs computing spectral sequences associated with filtered complexes).

- Algorithm computing the Bousfield-Kan spectral sequence associated with a simplicial set X. The homology groups $H_i(RX)$ are only one ingredient! Other elements:
  - cosimplicial spaces
  - additive relations...