Free presentation for central extensions

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Let $0 \to A \to E \to C \to 1$ be a central extension of two finitely generated Abelian groups A and C. It is well-known there exists a set-theoretic map $\gamma : C \times C \to A$ which satisfies:

1. $\gamma(g, 0) = 0 = \gamma(0, g)$
2. $\gamma(g + h, k) = \gamma(h, k) - \gamma(g, h) + \gamma(g, h + k)$

In addition, the initial extension is equivalent to another extension

$$0 \to A \to A \times_\gamma C \to C \to 1$$

where the elements of $A \times_\gamma C$ are pairs (a, c) with $a \in A$ and $c \in C$, and the group law is defined by

$$(a_1, c_1)(a_2, c_2) \equiv (a_1 + a_2 + \gamma(c_1, c_2), c_1 + c_2).$$

The set-theoretic map γ is called the 2-*cocycle* of the extension, since it corresponds to a map $\gamma : K(C, 1)_2 \to A$ in $H^2(C, A)$.

Let us suppose that the groups A and C are represented respectively by means of the matrices $\alpha : A_1 \to A_0$ and $\beta : C_1 \to C_0$ (which are supposed to be in canonical form). This gives us the resolutions (exact sequences) $A_1 \stackrel{\alpha}{\to} A_0 \stackrel{\alpha'}{\to} A$ and $C_1 \stackrel{\beta}{\to} C_0 \stackrel{\beta'}{\to} C$. Since the elements in $A \times_\gamma C$ are pairs (a, c) with $a \in A$ and $c \in C$, we consider the groups $A_1 + C_1$ and $A_0 + C_0$ and we are going to determine an exact sequence $A_1 + C_1 \stackrel{\varepsilon}{\to} A_0 + C_0 \stackrel{\varepsilon'}{\to} A \times_\gamma C$. The map $A_1 + C_1 \stackrel{\varepsilon}{\to} A_0 + C_0$ will be given by a block matrix

$$\varepsilon = \left[\begin{array}{c|c} \alpha & F \\ \hline 0 & \beta \end{array} \right] \tag{1}$$

We have the following diagram:

$$
\begin{tikzcd}
A_1 \arrow[r, shift left, "i"] \arrow[d, "\alpha"] & A_1 + C_1 \arrow[l, shift left, "\rho"] \arrow[r, shift left, "j"] \arrow[d, shift right=2, "\alpha"] \arrow[d, shift left=2, "\beta"] \arrow[d, dotted, "F?"'] & C_1 \arrow[l, shift left, "\sigma"] \arrow[d, "\beta"] \\
A_0 \arrow[r, shift left, "i"] \arrow[d, "\alpha'"] & A_0 + C_0 \arrow[l, shift left, "\rho"] \arrow[r, shift left, "j"] \arrow[d, dotted, "\varepsilon'?"] & C_0 \arrow[l, shift left, "\sigma"] \arrow[d, "\beta'"] \\
A \arrow[r, shift left, "i"] & A \times_\gamma C \arrow[l, shift left, "\rho"] \arrow[r, shift left, "j"] & C \arrow[l, shift left, "\sigma"]
\end{tikzcd}
$$

where the maps i and j correspond to the canonical inclusion and projection respectively and σ and ρ and the natural section and retraction (which in each row satisfy that $\rho i = \mathrm{Id}$, $i\rho + \sigma j = \mathrm{Id}$ and $j\sigma = \mathrm{Id}$). It is now necessary to define ε' and F.

The map $\varepsilon' : A_0 + C_0 \to A \times_\gamma C$ is defined over the generators of $A_0 + C_0$ as follows. Given a_i a generator of A_0, we define $\varepsilon'(a_i, 0) = i\alpha'(a_i) = (\alpha'(a_i), 0)$. Let us remark that since the matrix α is in canonical form then $\alpha'(a_i) = a_i$ and then $\varepsilon'(a_i, 0) = (a_i, 0)$. Similarly, ε' is defined over the generators of C_0 as $\varepsilon'(0, c_i) = \sigma\beta'(c_i) = (0, \beta'(c_i)) = (0, c_i)$.

Then, given any element $(a, c) \in A_0 + C_0$, we express it as a sum of generators of $A_0 + C_0$ and we compute $\varepsilon'(a, c)$ by applying the group operation in $E \cong A \times_\gamma C$ (denoted $+^E$ or $\sum_E$) over the images of the corresponding generators. Let us observe that given $(a, 0) \in A_0 + C_0$ with $a = \sum_i a_i$ and a_i generators of A_0, then $\varepsilon'(a, 0) = \varepsilon'(\sum_i a_i, 0) = \sum_i^E \varepsilon'(a_i, 0) = \sum_i^E (a_i, 0) = (\sum_i a_i, 0) = (a, 0)$. However, given $(0, c) \in A_0 + C_0$ with $c = \sum_i c_i$ and c_i generators of C_0, then $\varepsilon'(0, c) = \varepsilon'(0, \sum_i c_i) = \varepsilon'(\sum_i (0, c_i)) = \sum_i^E \varepsilon'(0, c_i) \neq (0, \sum_i c_i) = (0, c)$ because in the previous sum the cocycle must be considered. One has $\varepsilon'(0, c) = \sum_i^E \varepsilon'(0, c_i) = (a, c)$ for some $a \in A$.

For the definition of F, we do the following steps. Given c'_i a generator of C_1, we consider $\varepsilon'(0, \beta(c'_i))$ which satisfies $j\varepsilon'(0, \beta(c'_i)) = \beta'\beta(c'_i) = 0$, that is, $\varepsilon'(0, \beta(c'_i)) \in \operatorname{Ker} j = \operatorname{Im} i$. We take now $\rho\varepsilon'(0, \beta(c'_i)) \in A$ which can be seen as an element $a \in A_0$. We define $F(c'_i) = -a$.

We can observe that both morphisms ε and ε' are well constructed because they have been defined over the generators. Moreover, the four squares in the previous diagram are commutatives. Since each column in the diagram is a chain complex and each row is a short exact sequence, we have a short exact sequence of chain complexes. Finally, since α and β are injective, the homology groups of the left and right chain complexes are null and then the homology groups of the chain complex $A_1+C_1 \xrightarrow{\varepsilon} A_0+C_0 \xrightarrow{\varepsilon'} A\times_\gamma C$ are also null. In this way, we obtain a resolution for the group $E \cong A\times_\gamma C$ which could be used to represent the group E by means of the matrix ϵ.

However, let us observe now that the maps ε' and F cannot be directly implemented since we do not know the groups A, C and $E \cong A\times_\gamma C$ (and then the bottom maps i, j, σ and ρ can not be implemented). This problem can be solved because the calculation of F can be done without using the those maps. Let c'_i be a generator of C_1, $c'_i = (0,\dots,0,1,0,\dots,0)$. Since β is a matrix in canonical form, then $\beta(c'_i) = (0,\dots,0,d_i,0,\dots,0) \in C_0$ for some $d_i > 1$. Let us remark here that $(0,\dots,0,d_i,0,\dots,0)$ is not a generator of C_0 and therefore its image by ε' is not directly computed and we must express it as $(0,\dots,0,d_i,0,\dots,0) = (0,\dots,0,1,0,\dots,0) + \overset{d_i}{\cdots} + (0,\dots,0,1,0,\dots,0)$. Therefore,

$$
\begin{aligned}
\varepsilon'(0,\beta(c'_i)) =& \varepsilon'(0,(0,\dots,0,d_i,0,\dots,0)) = \varepsilon'(0,(0,\dots,0,1,0,\dots,0)) +^E \cdots \\
& +^E \varepsilon'(0,(0,\dots,0,1,0,\dots,0)) = (\gamma((0,\dots,0,1,0,\dots,0), \\
& (0,\dots,0,1,0,\dots,0)) + \gamma((0,\dots,0,2,0,\dots,0),(0,\dots,0,1,0,\dots,0)) \\
& + \gamma((0,\dots,0,d_i-1,0,\dots,0),(0,\dots,0,1,0,\dots,0)),(0,\dots,0,d_i,0,\dots,0))
\end{aligned}
$$

and then $F(c'_i)$ is defined directly as

$$
\begin{aligned}
F(c'_i) =& -\rho\varepsilon'(0,(0,\dots,0,d_i,0,\dots,0)) = -(\gamma((0,\dots,0,1,0,\dots,0), \\
& (0,\dots,0,1,0,\dots,0)) + \gamma((0,\dots,0,2,0,\dots,0),(0,\dots,0,1,0,\dots,0)) + \\
& \gamma((0,\dots,0,d_i-1,0,\dots,0),(0,\dots,0,1,0,\dots,0))) \\
& = -(\gamma(c'_i,c'_i) + \gamma(2*c'_i,c'_i) + \cdots + \gamma((d_i-1)*c'_i,c'_i))
\end{aligned}
$$

A similar proof of the formula of $F(c'_i)$ can be deduced directly from Lemma 4.23 in the book Advanced Algebra by A.W. Knapp (Birkäuser, 2008). Considering $M_A = A_1$, $P_A = A_0$, $M_C = C_1$ and $P_C = C_0$ in the diagram by Knapp, the map ε_B corresponds to ε' in our diagram and is defined in the same way. In order to compute $M_B = \operatorname{Ker}\varepsilon_B$ in Knapp's proof, we must take into account that the group B (E in our case) is not defined, but we know it is isomorphic to $A\times_\gamma C$. Moreover, it happens and element $x \in A_0 + C_0$ is in $\operatorname{Ker} \varepsilon'$ if and only if $x \in \operatorname{Ker}[\rho\varepsilon' : A_0 + C_0 \to A]$ and $x \in \operatorname{Ker}[j\varepsilon' : A_0 + C_0 \to C]$. Then, it is easy to observe that $\operatorname{Ker}[j\varepsilon'] = A_0 + C_1$ and $\operatorname{Ker}[\rho\varepsilon' : A_0 + C_1 \to A] = A_1 + C_1$ with inclusion $A_1 + C_1 \hookrightarrow A_0 + C_0$ given by the matrix M_E defined before.

Following Knapp's idea as explained in the previous paragraph, the matrix M'_E can be determined directly from the representation of the groups A and C and the group structure (product operation) on the elements of E without knowing the cocycle γ. This can be useful when working with central extensions coming from the computation of the effective homotopy of a fibration.

Finally let us observe that the block matrix ε is not necessarily in canonical form and therefore it could not be used directly for our implementation of groups. It is necessary to compute the canonical form of this matrix, producing a new matrix $\varepsilon_2 : E_1 \to E_0$ which is valid for the definition.