Computing Spectral Sequences with Kenzo

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- Other spectral sequences: Bockstein, Grothendieck, Hurewicz, Kunneth, Quillen, van Kampen, ...

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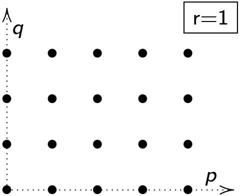

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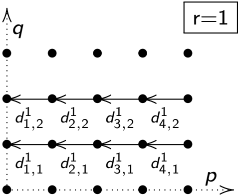

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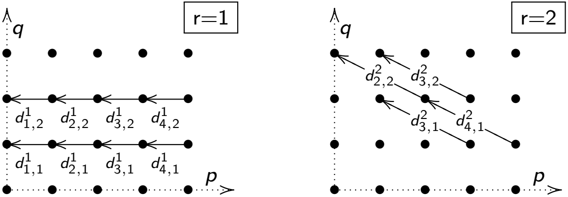

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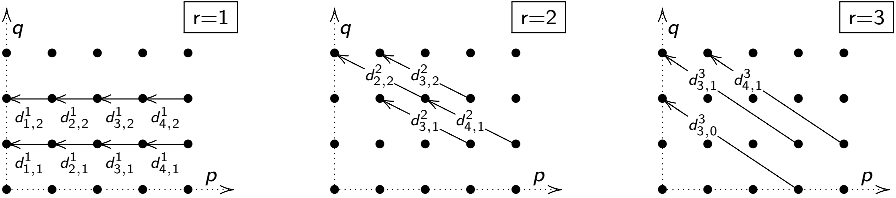

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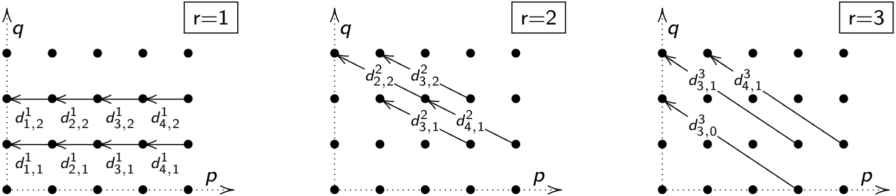

Since $E^{r+1}_{p,q} \subseteq E^r_{p,q}$ for each $r \geq 1$, one can define the *final groups* of the spectral sequence as $E^\infty_{p,q} = \bigcap_{r \geq 1} E^r_{p,q}$.

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Theorem (Serre spectral sequence)

Let $G \hookrightarrow E \to B$ *be a fibration with a simply connected base space* B*. Then a first quadrant spectral sequence* $E = (E^r, d^r)_{r \geq 2}$ *can be defined with* $E^2_{p,q} = H_p(B, H_q(G))$ *and* $E^2 \Rightarrow H_*(E)$*.*

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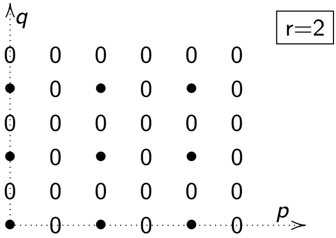

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●	0	●	0	●	0
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●	0	●	0	●	0

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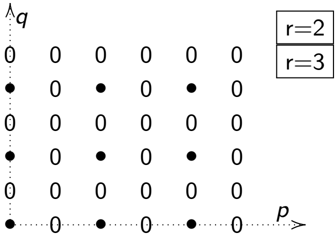

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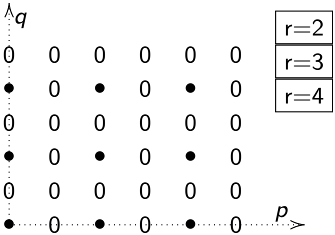

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0	0	0	0	0	0
•	0	•	0	•	0

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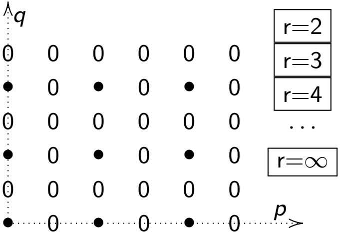

0	0	0	0	0	0
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0	0	0	0	0	0
●	0	●	0	●	0
0	0	0	0	0	0
●	0	●	0	●	0

Algorithmic problems of spectral sequences

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If we think of a spectral sequence as a black box, then the input is a differential bigraded module, usually $E^1_{*,*}$*, and, with each turn of the handle, the machine computes a successive homology according to a sequence of differentials. If some differential is unknown, then some other (any other) principle is needed to proceed. [...] In the nontrivial cases, it is often a deep geometric idea that is caught up in the knowledge of a differential.*

John McCleary, User's guide to spectral sequences
(Publish or Perish, 1985)

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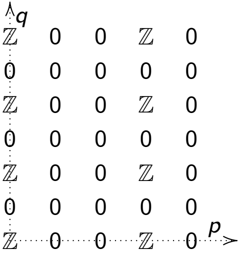

$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
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0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0

q p

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$\boxed{r = 3}$

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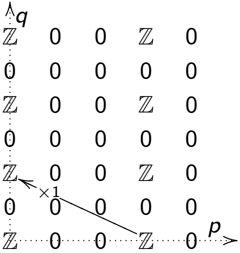

$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
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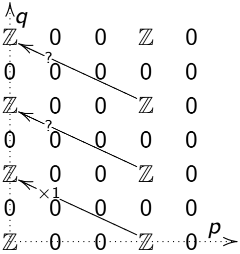

$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
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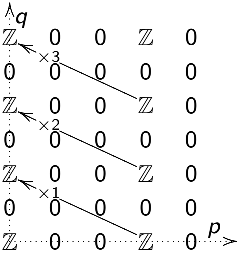

$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
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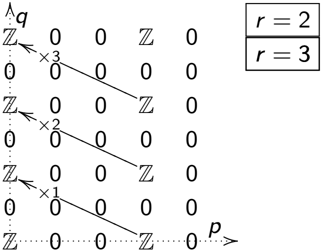
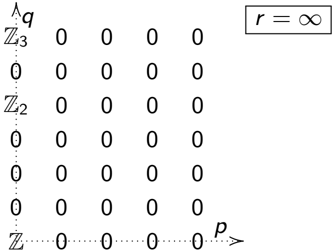


$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0



$\mathbb{Z}_3$	0	0	0	0
0	0	0	0	0
$\mathbb{Z}_2$	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
$\mathbb{Z}$	0	0	0	0

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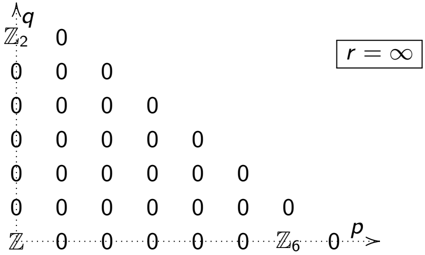

$\mathbb{Z}_2$	0						
0	0	0					
0	0	0	0				
0	0	0	0	0			
0	0	0	0	0	0		
0	0	0	0	0	0	0	
$\mathbb{Z}$	0	0	0	0	0	$\mathbb{Z}_6$	0

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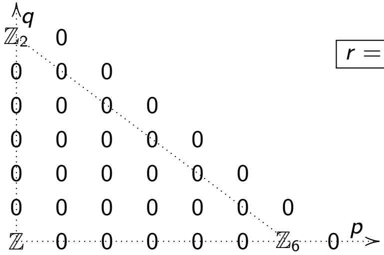

$\mathbb{Z}_2$	0						
0	0	0					
0	0	0	0				
0	0	0	0	0			
0	0	0	0	0	0		
0	0	0	0	0	0	0	
$\mathbb{Z}$	0	0	0	0	0	$\mathbb{Z}_6$	0

$\boxed{r = \infty}$

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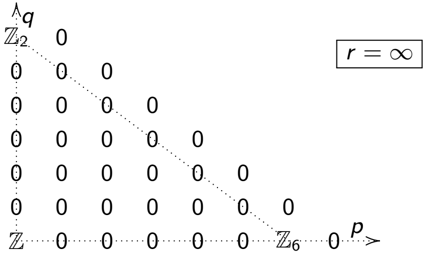

$\mathbb{Z}_2$	0						
0	0	0					
0	0	0	0				
0	0	0	0	0			
0	0	0	0	0	0		
0	0	0	0	0	0	0	
$\mathbb{Z}$	0	0	0	0	0	$\mathbb{Z}_6$	0

We obtain a short exact sequence:

$$0 \leftarrow \mathbb{Z}_6 \leftarrow H_6(E) \leftarrow \mathbb{Z}_2 \leftarrow 0$$

but now there are two possible extensions: the trivial one $\mathbb{Z}_2 \oplus \mathbb{Z}_6$ and the twisted one $\mathbb{Z}_{12}$.

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\end{tikzcd}$$

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A *reduction* ρ between two chain complexes C_* and D_* (denoted by $\rho : C_* \Rrightarrow D_*$) is a triple $\rho = (f, g, h)$

$$\begin{tikzcd} C_* \arrow[loop left, "h"] \arrow[r, bend left, "f"] & D_* \arrow[l, bend left, "g"] \end{tikzcd}$$

satisfying the following relations:

1) $fg = \mathrm{Id}_{D_*}$;
2) $d_C h + h d_C = \mathrm{Id}_{C_*} - gf$;
3) $fh = 0; \quad hg = 0; \quad hh = 0.$

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$$\begin{tikzcd} C_* \arrow[loop left, "h"] \arrow[r, bend left, "f"] & D_* \arrow[l, bend left, "g"] \end{tikzcd}$$

satisfying the following relations:

1) $fg = \mathrm{Id}_{D_*}$;
2) $d_C h + h d_C = \mathrm{Id}_{C_*} - gf$;
3) $fh = 0; \quad hg = 0; \quad hh = 0.$

If $C_* \Rightarrow\!\!\!\!\Rightarrow D_*$, then $C_* \cong D_* \oplus A_*$, with A_* acyclic, which implies that $H_n(C_*) \cong H_n(D_*)$ for all n.

Effective homology

Effective homology

Definition

A *(strong chain) equivalence* ε between C_* and D_*, $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$, is a triple $\varepsilon = (B_*, \rho, \rho')$ where B_* is a chain complex, $\rho : B_* \Rrightarrow C_*$ and $\rho' : B_* \Rrightarrow D_*$.

$$\begin{tikzcd} & B_* \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ C_* & & D_* \end{tikzcd} \qquad \begin{tikzcd} & \frac{42}{30} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \frac{14}{10} & & \frac{21}{15} \end{tikzcd}$$

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The Kenzo system

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The Kenzo system uses the notion of *object with effective homology* to compute homology groups of some complicated spaces.

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- Otherwise, the program uses the effective homology.

Example:

$$X = \Omega(\Omega(\Omega(P^{\infty}\mathbb{R}/P^{3}\mathbb{R}) \cup_4 D^4) \cup_2 D^2)$$

$$H_5(X) = \mathbb{Z}_2^{23} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{16}$$

$$H_6(X) = \mathbb{Z}_2^{52} \oplus \mathbb{Z}_4^{3} \oplus \mathbb{Z}^{3}$$

$$H_7(X) = \mathbb{Z}_2^{113} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8^{3} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{32} \oplus \mathbb{Z}$$

Spectral sequences of filtered complexes

Spectral sequences of filtered complexes

Theorem

Let F be a filtration of a chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$. There exists a spectral sequence $E = E(C_*, F) = (E^r, d^r)_{r\geq 1}$, defined by

$$E^r_{p,q} = \frac{Z^r_{p,q} \cup F_{p-1}C_{p+q}}{d_{p+q+1}(Z^{r-1}_{p+r-1,q-r+2}) \cup F_{p-1}C_{p+q}}$$

where $Z^r_{p,q} = \{a \in F_pC_{p+q} | \; d_{p+q}(a) \in F_{p-r}C_{p+q-1}\} \subseteq F_pC_{p+q}$, and $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is the morphism induced on these subquotients by the differential map $d_{p+q} : C_{p+q} \to C_{p+q-1}$.

If F is bounded, then $E^1 \Rightarrow H_*(C_*)$.

Saunders MacLane, Homology (Springer, 1963)

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Spectral sequences of filtered complexes

With the formal expresion for the groups $E^r_{p,q}$ and the differential maps $d^r_{p,q}$, they can only be determined in very simple situations.

However, the effective homology method allows us to determine, as a by-product, this kind of spectral sequences, obtaining a real algorithm.

This algorithm has been implemented as a set of programs (about 2500 lines) enhancing the Kenzo system, that determine the groups $E^r_{p,q}$ and also the differential maps $d^r_{p,q}$ for every level r.

Algorithm computing spectral seq. of filtered complexes

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Algorithm computing spectral seq. of filtered complexes

These new methods work in a way that is similar to the mechanism of Kenzo for computing homology groups:

- Given an effective chain complex C_* with a filtration F_C, the different components of the associated spectral sequence can be computed by means of elementary methods with integer matrices.
- If the filtered chain complex (C_*, F_C) is not effective, but with effective homology $C_* \twoheadleftarrow D_* \twoheadrightarrow HC_*$, then appropriate filtrations of D_* and HC_* can also produce the spectral sequence.

Example of computation

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Example: $K(\mathbb{Z},1) \hookrightarrow K(\mathbb{Z},1) \times_\tau S^2 \to S^2$ (Hopf fibration)

$$\tau : S^2 \to K(\mathbb{Z},1) \text{ given by } \tau(\mathtt{s2}) = [1]$$

$$X = K(\mathbb{Z},1) \times_\tau S^2 = S^3$$

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$\boxed{r = 2}$

$$\begin{tikzcd}
q & & & & & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
\mathbb{Z} & 0 & \mathbb{Z} & 0 & 0 & \\
\mathbb{Z} & 0 & \mathbb{Z} \arrow[llu, "\times 1"'] & 0 & 0 & p
\end{tikzcd}$$

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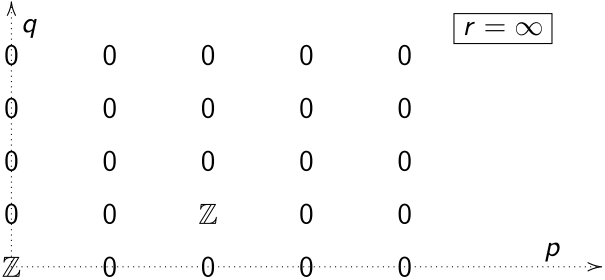

$\boxed{r = \infty}$

0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	$\mathbb{Z}$	0	0
$\mathbb{Z}$	0	0	0	0

Example of computation

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First of all, we construct the object $X = K(\mathbb{Z}, 1) \times_\tau S^2$

```
>(setf s2 (sphere 2))
[K208 Simplicial-Set]
>(setf kz1 (k-z 1))
[K1 Abelian-Simplicial-Group]
> (setf tau (build-smmr
:sorc s2
:trgt kz1
:degr -1
:sintr #'(lambda (dmns gmsm) (absm 0 '(1)))
:orgn '(kz1-tw-s2)))
[K213 Fibration K208 -> K1]
> (setf X (fibration-total tau))
[K219 Simplicial-Set]
```

Example of computation

Example of computation

Then the space X and its effective equivalent object are filtered

```
> (change-chcm-to-flcc X crpr-flin '(crpr-flin))
[K219 Filtered-Simplicial-Set]
> (change-chcm-to-flcc (rbcc (efhm X)) tnpr-flin '(tnpr-flin))
[K279 Filtered-Chain-Complex]
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Computation of some groups:

```
> (spsq-group X 2 2 0)
Spectral sequence E^2_{2,0}
Component Z
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Component Z
```

The differential maps can also be obtained

```
> (spsq-dffr X 2 2 0 '(1))
(1)
```

Example of computation

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For $n = 1$ the convergence level is 3.

```
>(spsq-cnvg X 1)
3
```

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```
>(spsq-cnvg X 1)
3
```

And finally, we can determine the filtration of the homology groups

```
> (hmlg-fltr X 3 1)
Filtration F_1 H_3
nil
> (hmlg-fltr X 3 2)
Filtration F_2 H_3
Component Z
```

About our programs

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The new Kenzo module for spectral sequences allows the computation of spectral sequences associated with filtered complexes, including bicomplexes and the classical Serre and Eilenberg-Moore spectral sequences.

It has made it possible to determine some examples of spectral sequences (Serre, Eilenberg-Moore) of infinite spaces which had not been determined before.

Applications: homotopy of suspended classifying spaces

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The homotopy groups of suspended classifying spaces $\Sigma K(G, 1)$ can be computed by means of the Serre spectral sequence associated with some fibrations involved in the Postnikov tower of these spaces, or directly using the effective homology of these fibrations.

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Mikhailov and Wu say that:

> *Theorem 5.4*: *Let A_4 be the 4-th alternating group.*
> *Then* $\pi_4(\Sigma K(A_4,1)) = \mathbb{Z}_4$

but we have obtained $\pi_4(\Sigma K(A_4,1)) = \mathbb{Z}_{12}$

Applications: persistent homology

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This has allowed us to detect an error in the book

Computational Topology: An Introduction
Herbert Edelsbrunner and John Harer
American Mathematical Society

Bousfield-Kan spectral sequence

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We have tried to generalize our work to other spectral sequences which are not associated with filtered complexes, and we have considered first the Boufield-Kan spectral sequence, related with the computation of the homotopy groups of a simplicial set.

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- We have already developed algorithms computing the first two levels of the spectral sequence, based on the computation of the effective homology of the free simplicial Abelian group of a simplicial set X, denoted RX.

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- We have already developed algorithms computing the first two levels of the spectral sequence, based on the computation of the effective homology of the free simplicial Abelian group of a simplicial set X, denoted RX.
- The computation of the higher levels is more difficult. For this task we have used discrete vector fields and we have introduced the notion of *effective homotopy*, but the algorithm is not yet completed.