A new algorithm for computing the twisting operator of the Whitehead tower method in the not 1-reduced case

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The Whitehead tower method [Whi52] for computing homotopy groups is defined theoretically for simply connected simplicial sets. In that case, the homotopy groups of X can be determined by means of a tower of fibrations

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
G_{n+3} \equiv K(H_{n+2}(X_{n+2}), n+1) & \longrightarrow & X_{n+3} \\
 & & \downarrow \\
G_{n+2} \equiv K(H_{n+1}(X_{n+1}), n) & \longrightarrow & X_{n+2} \\
 & & \downarrow \\
G_{n+1} \equiv K(H_{n}(X), n-1) & \longrightarrow & X_{n+1} \\
 & & \downarrow \\
 & & X
\end{array}
$$

such that each X_k in the tower satisfies $\pi_k(X_k) \cong H_k(X_k) \cong \pi_k(X)$, for $k \geq n+1$.

However, the implementation in the Kenzo system [DRSS99] is only valid for simplicial sets X which are 1-reduced, that is, which have only one vertex (0-simplex) and no non-degenerate 1-simplex. Each fibration $K(H_k(X_k), k-1) \longrightarrow X_{k+1} \longrightarrow X_k$ in the tower is defined in that case by means of a simplicial morphism of degree -1, called *twisting operator* [May67], $\tau_k : X_K \rightarrow K(H_k(X_k), k-1)$. This simplicial morphism is described by a well-known existing isomorphism between $H_k(X_k)$ and $K(H_k(X_k), k-1)_{k-1}$ (the set of the $(k-1)$-simplices of $K(H_k(X_k), k-1)$). In order to construct this twisting operator, Kenzo uses the *Smith Normal Form* (SNF) [New72] of the differential matrix of degree $k+1$ of the simplicial set X_k, denoted D^k_{k+1}, and the fact that the space X is 1-reduced.

Now, we have also considered the situation where the initial simplicial set X is not 1-reduced but only simply connected (that is, such that $H_1(X) = 0$ but there exist non-degenerate simplices of dimension 1). In that situation, the twisting operator $\tau_k : X_K \to K(H_k(X_k), k-1)$ corresponding to the existing isomorphism between $H_k(X_k)$ and $K(H_k(X_k), k-1)_{k-1}$ cannot be as easily determined as in the 1-reduced case. However, Kenzo's algorithm can be modified by considering the two differential matrices of X_k of degrees k and $k+1$, denoted D^k_k and D^k_{k+1} respectively, and following the same steps used for computing homology groups.

These steps lead to the following new Algorithm 1:

Algorithm 1: Whitehead twisting operator

Input: An integer $n \geq 2$ and an $(n-1)$-connected simplicial set X.
Output: A twisting operator $\tau : X \to K(H_n(X), n-1)$ defining the Whitehead fibration.

Compute the SNF D of the matrix D^k_k and the matrices P, P^{-1}, Q and Q^{-1} corresponding to the changes of bases.
Multiply the matrix D^k_{k+1} by Q^{-1}, denoted D'^k_{k+1}.
Select a submatrix of D'^k_{k+1} representing the image of D^k_{k+1}, denoted I^k_{k+1}.
Compute the SNF D' of I^k_{k+1} and the matrices P', P'^{-1}, Q' and Q'^{-1} of the change of bases. The columns of P'^{-1} are the generators of $H_n(X)$.
For each n-simplex x in X, define $\tau(x)$ by computing the coordinates of x with respect to the columns of P'^{-1}.
For each i-simplex x in X with $i \neq n$, define $\tau(x)$ as the degeneration of the base point.
return τ

Our implementation in Kenzo of Algorithm 1 is slightly modified taking into account that, when $H_n(X)$ is a sum of $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$, the fibration $G_{k+1} \equiv K(H_k(X_k), k-1) \longrightarrow X_{k+1} \longrightarrow X_k$ in the tower is simulated by several fibrations. Combining the implementation of Algorithm 1 with some of the functions for computing the Whitehead fibrations of 1-reduced simplicial sets which were already available in Kenzo and the effective homology technique, we have developed and implemented the following new Algorithm 2.

Both algorithms do terminate in a finite number of steps and are correct since are implementations of the well-known Whitehead tower method [Whi52].

References

[DRSS99] X. Dousson, J. Rubio, F. Sergeraert, and Y. Siret. The Kenzo program. Institut Fourier, Grenoble, 1999. `http://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo/`.

[May67] J. P. May. *Simplicial objects in Algebraic Topology.* Van Nostrand Mathematical Studies. University of Chicago Press, 1967.

Algorithm 2: Homotopy group

Input: An integer $n \geq 2$ and a simply connected simplicial set X.
Output: The homotopy group $\pi_n(X)$.
Compute the first non null homology group $h \equiv H_i(X)$. **if** $n < i$ **then**
  return *Trivial group*
else if $n = i$ **then**
  return h
else
  Compute the Whitehead twisting operator τ from X and i using Algorithm 1.
  Compute X_{i+1} as the total space of the fibration given by τ.
  return $\pi_n(X_{i+1})$

[New72] Morris Newman. *Integral matrices.* Academic Press, 1972.

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