# Constructive Spectral Sequences

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**Definition.** A *Spectral Sequence* $E = \{E^r, d^r\}$ is a family of $\mathbb{Z}$-bigraded modules $E^1$, $E^2$,..., each provided with a differential $d^r = \{d^r_{p,q}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$, $r = 1, 2, \ldots$

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*"If we think of a spectral sequence as a black box, then the input is a differential bigraded module, usually $E^1_{*,*}$, and, with each turn of the handle, the machine computes a successive homology according to a sequence of differentials. If some differential is unknown, then some other (any other) principle is needed to proceed. [...] In the nontrivial cases, it is often a deep geometric idea that is caught up in the knowledge of a differential."*

John McCleary, User's guide to spectral sequences (Publish or Perish, 1985)

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A fibration $G \hookrightarrow E \longrightarrow B$

$G$ is the fiber space, $B$ the base space, and $E = B \times_{\tau} G$ the total space.

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**Theorem (Serre spectral sequence).** *Let $G \hookrightarrow E \to B$ be a fibration with a base space $B$ simply connected. Then a first quadrant spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}_{r\geq 2}$ is defined with $E^2_{p,q} = H_p(B, H_q(G))$ and $E^r_{p,q} \Rightarrow H_{p+q}(E)$.*

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Using this spectral sequence, Serre computed many sphere homotopy groups.

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$r = 2$

| $q$ | | | | |
|---|---|---|---|---|
| $\mathbb{Z}$ | $0$ | $0$ | $\mathbb{Z}$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ |
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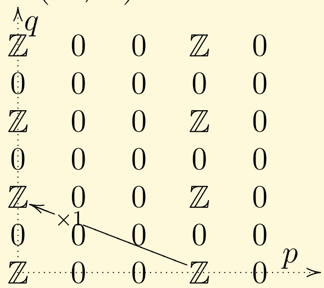


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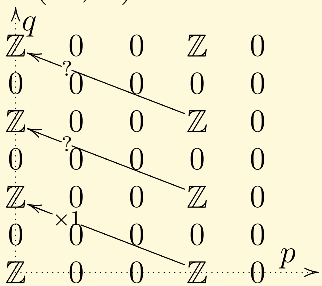


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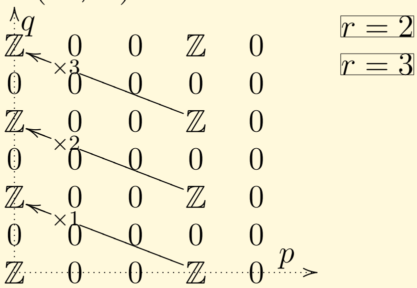

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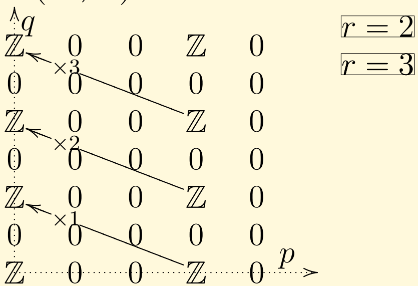


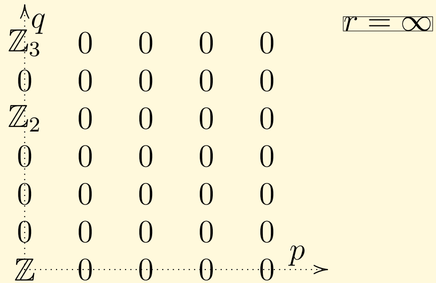

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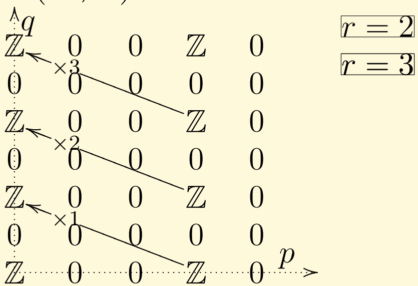


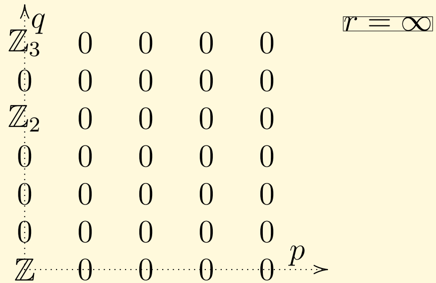


The Hurewicz theorem and the long exact sequence of homotopy imply that $\pi_4(S^3) = \pi_4(X_4) = H_4(X_4) = \mathbb{Z}_2$.

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- Then, a new fibration $F_3 \hookrightarrow X_5 \to X_4$ is considered to determine $\pi_5(S^3)$, where $F_3 = K(\mathbb{Z}_2, 3)$ is chosen because $\pi_4(X_4) = \mathbb{Z}_2$.

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  Using his spectral sequence, he proved $\pi_6(S^3)$ has 12 elements, but he was unable to choose between the two possible options $\mathbb{Z}_{12}$ and $\mathbb{Z}_2 + \mathbb{Z}_6$.

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$X$ $\qquad$ $H_*(X)???$

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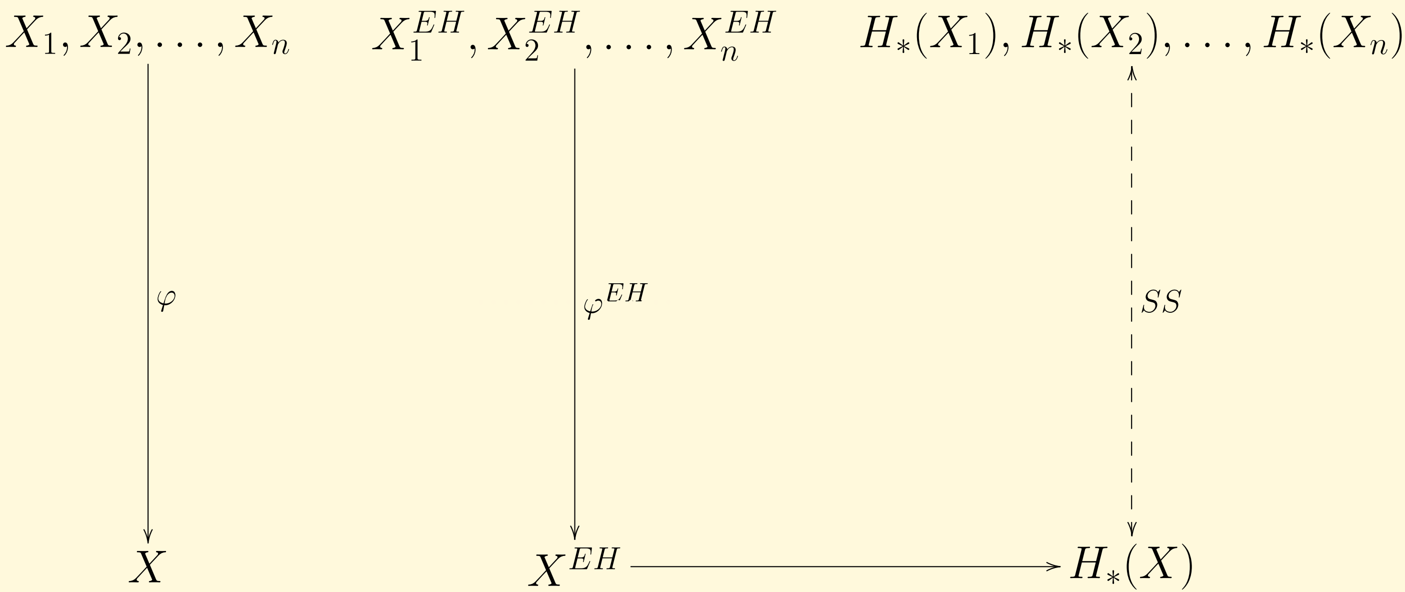

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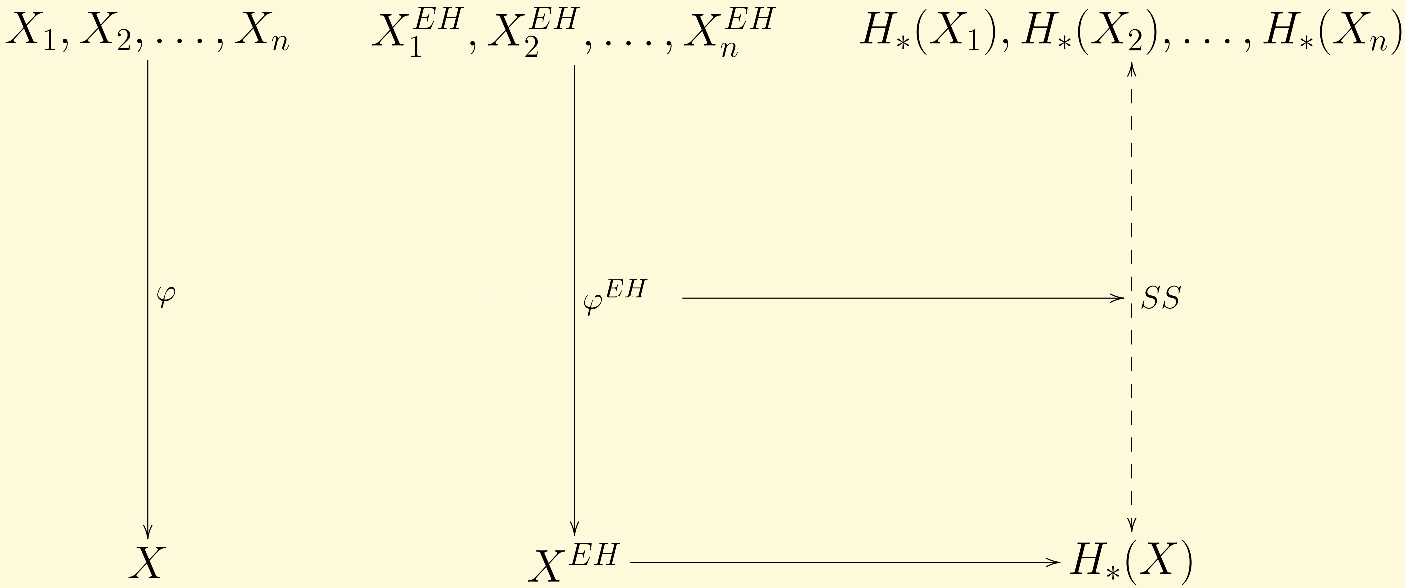

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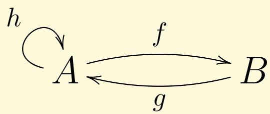


satisfying the following relations:

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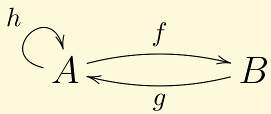


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**Remark.** If $A \Rrightarrow B$, then $A = B \oplus C$, with $C$ acyclic, which implies that $H_*(A) \cong H_*(B)$.

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$$H_6(X_6) = \mathbb{Z}_{12}$$

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This algorithm has been implemented as a set of programs (about 2500 lines) enhancing the Kenzo system, that determine the groups $E^r_{p,q}$ and also the differential maps $d^r_{p,q}$ for every level $r$.

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- Otherwise, the effective homology is needed to compute it by means of an analogous spectral sequence deduced of an appropriate filtration on the associated effective complex.

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