Effective Homotopy of Fibrations

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Constructive homological algebra methods, implementations and applications
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Computation of homotopy groups

Theoretical algorithm by Edgar Brown (1957)

Postnikov tower

Adams and Bousfield-Kan spectral sequences
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Effective homology

A technique which provides algorithms for the computation of homology groups of complicated spaces. One of the fundamental ideas is the notion of a solution for the homological problem of a chain complex, which consists in four algorithms describing in a constructive way the homology groups of a space, which is said to have effective homology. For some chain complexes, their effective homology can be directly computed. From spaces with effective homology one can obtain more complicated chain complexes which are also objects with effective homology.
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\[ \varphi \]

\[ X \]
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\[
\begin{array}{c}
\downarrow \\
\varphi \\
\downarrow \\
X
\end{array}
\]

\[ X_1^{EH}, X_2^{EH}, \ldots, X_n^{EH} \]
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Examples:
Effective homology

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Examples:

- if \( X \) has effective homology, \( \Omega(X) \) has effective homology;
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Examples:
- if \( X \) has effective homology, \( \Omega(X) \) has effective homology;
- if \( F \hookrightarrow E \twoheadrightarrow B \) is a fibration and \( F \) and \( B \) have effective homology, then \( E \) has effective homology.
The Kenzo system

A Common Lisp program implementing the effective homology method.
It allows the computation of homology groups of complicated spaces: total spaces of fibrations, arbitrarily iterated loop spaces (Adams' problem), classifying spaces...
It has made it possible to compute some homology groups so far unreachable.
It can also determine some homotopy groups of spaces by means of the Postnikov tower technique.
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The homotopical problem of a Kan simplicial set
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Definition

A simplicial set $X$ is a graded set $X = \{X_n\}_{n \in \mathbb{N}}$ with maps $\partial_i : X_n \to X_{n-1}$ and $\eta_i : X_n \to X_{n+1}$, $0 \leq i \leq q$, which satisfy the simplicial identities.
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A simplicial set $X$ is a Kan simplicial set if for every collection of $n+1$ $n$-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ which satisfy the compatibility condition $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j$, $i \neq k$, and $j \neq k$, there exists an $(n + 1)$-simplex $x \in X_{n+1}$ such that $\partial_i x = x_i$ for every $i \neq k$.
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\begin{figure}
\centering
\includegraphics[width=\textwidth]{diagram}
\end{figure}
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\[
\partial_i \partial_{i+1} x = \partial_{i+1} \partial_i x \quad \text{for all} \quad i < j, \quad i \neq k, \quad j \neq k.
\]

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![Diagram of simplicial set]

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\begin{align*}
\partial_i x_j &= \partial_{j-1} x_i \\
\eta_i x &= x_i
\end{align*}

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![Diagram of simplicial identities](image)

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Two \( n \)-simplices \( x \) and \( y \) of \( X \) are said to be homotopic, written \( x \sim y \), if \( \partial_i x = \partial_i y \) for \( 0 \leq i \leq n \), and there exists an \((n+1)\)-simplex \( z \) such that \( \partial_n z = x \), \( \partial_{n+1} z = y \), and \( \partial_i z = \eta_{n-1} \partial_i x = \eta_{n-1} \partial_i y \) for \( 0 \leq i < n \).

Definition

Let \( \star \in X_0 \) be a base point. \( S_n(X) \) is the set of all \( x \in X_n \) such that \( \partial_i x = \star \) for every \( 0 \leq i \leq n \), called the \( n \)-spheres of \( X \).

Definition

Given a Kan simplicial set \( X \) and a base point \( \star \in X_0 \), we define \( \pi_n(X, \star) \equiv \pi_n(X) = S_n(X) / (\sim) \). The set \( \pi_n(X, \star) \) admits a group structure for \( n \geq 1 \) and it is Abelian for \( n \geq 2 \). It is called the \( n \)-homotopy group of \( X \).
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The homotopical problem of a Kan simplicial set

In general the formal definition of $\pi_n(X)$ does not produce an algorithm computing these groups. In some good situations some theoretical reasoning could lead, for instance, to affirm $\pi_6(X) = \mathbb{Z}_{12}$. This means: there exists an isomorphism between the group $\pi_6(X)$ and $\mathbb{Z}_{12}$. But in general this proof is not constructive, an isomorphism is rarely made explicit. If we then intend to determine some unknown homology or homotopy group $H$ after calculations involving $\pi_6(X)$, if the result about the computed $\pi_6(X)$ is not constructive, then often the alleged algorithm $\pi_6(X) \to H$ in fact fails. The effective homotopy method tries to solve this problem by defining in a precise way what a constructive solution for the homotopy group of a space is.
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A solution for the homotopical problem (SHmtP) posed by a constructive Kan simplicial set $X$ is a graded 4-tuple $(\pi_n, f_n, g_n, h_n)_{n \geq 0}$.
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- $\pi_n$ is a (standard presentation of a) finitely generated group. It will be isomorphic to the desired homotopy group $\pi_n(X)$. 

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- \(f_n\) is an algorithm \(f_n : S_n(X) \rightarrow \pi_n\) satisfying \(f_n g_n = \text{Id}_{\pi_n}\) and \(f_n(x) = 0\) for all \(x \in S_n(X)\) with \(x \sim \star\).
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A *solution for the homotopical problem* (SHmtP) posed by a *constructive* Kan simplicial set $X$ is a graded 4-tuple $(\pi_n, f_n, g_n, h_n)_{n \geq 0}$

$$h_n \quad \text{Ker } f \quad \subseteq \quad S_n(X) \quad \longleftrightarrow \quad f_n \quad \text{g}_n \quad \rightarrow \quad \pi_n$$

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- $h_n$ is an algorithm $h_n : \text{ker } f_n \rightarrow X_{n+1}$ satisfying $\partial_i h_n = \ast$ for all $0 \leq i \leq n$ and $\partial_{n+1} h_n = \text{Id}_{\text{ker } f_n}$. 

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We say that $X$ has effective homotopy.
Proposition

Let $X$ be a Kan simplicial set and $(\pi_n, f_n, g_n, h_n)_{n \geq 0}$ a solution for the homotopical problem of $X$. Then, for each $n \geq 1$, the homotopy group $\pi_n(X) = S_n(X) \div \sim$ is isomorphic to the given group $\pi_n$.

Proof:

The isomorphism is given by a map $\phi: \pi_n(X) = S_n(X) \div \sim \rightarrow \pi_n$ defined as $\phi[x] = f_n(x)$ and its inverse $\psi: \pi_n \rightarrow \pi_n(X) = S_n(X) \div \sim$ constructed as the composition of $g_n$ with the projection to the corresponding quotient. Both maps are well-defined morphisms of groups and provide an explicit isomorphism between the "formal" group $\pi_n(X) = S_n(X) \div \sim$ and the explicit group $\pi_n$. 

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Let $X$ be a Kan simplicial set and $(\pi_n, f_n, g_n, h_n)_{n \geq 0}$ a solution for the homotopical problem of $X$. Then, for each $n \geq 1$, the homotopy group $\pi_n(X) = S_n(X)/(\sim)$ is isomorphic to the given group $\pi_n$.

**Proof:**

The isomorphism is given by a map $\phi : \pi_n(X) = S_n(X)/(\sim) \to \pi_n$ defined as $\phi[x] = f_n(x)$. 

A. Romero and F. Sergeraert
Luminy, January 2011
Proposition

Let $X$ be a Kan simplicial set and $(\pi_n, f_n, g_n, h_n)_{n \geq 0}$ a solution for the homotopical problem of $X$. Then, for each $n \geq 1$, the homotopy group $\pi_n(X) = S_n(X)/(\sim)$ is isomorphic to the given group $\pi_n$.

Proof:

The isomorphism is given by a map $\phi : \pi_n(X) = S_n(X)/(\sim) \to \pi_n$ defined as $\phi[x] = f_n(x)$ and its inverse $\psi : \pi_n \to \pi_n(X) = S_n(X)/(\sim)$ constructed as the composition of $g_n$ with the projection to the corresponding quotient.
The homotopical problem of a Kan simplicial set

**Proposition**

Let $X$ be a Kan simplicial set and $(\pi_n, f_n, g_n, h_n)_{n \geq 0}$ a solution for the homotopical problem of $X$. Then, for each $n \geq 1$, the homotopy group $\pi_n(X) = S_n(X)/(\sim)$ is isomorphic to the given group $\pi_n$.

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The isomorphism is given by a map $\phi : \pi_n(X) = S_n(X)/(\sim) \rightarrow \pi_n$ defined as $\phi[x] = f_n(x)$ and its inverse $\psi : \pi_n \rightarrow \pi_n(X) = S_n(X)/(\sim)$ constructed as the composition of $g_n$ with the projection to the corresponding quotient. Both maps are well-defined morphisms of groups and provide an explicit isomorphism between the "formal" group $\pi_n(X) = S_n(X)/(\sim)$ and the explicit group $\pi_n$. 
The homotopical problem of a Kan simplicial set

Problem: how can we determine a solution for the homotopical problem of a given Kan simplicial set \( X \)? For some spaces it can be directly determined.

\[
\pi_n(K(\pi, n)) = K(\pi, n) \sim \pi
\]

and

\[
S_i(K(\pi, n)) = \{\star\} \sim 0 \quad \text{for} \quad i \neq n.
\]

We define:

\[
\pi_n = \pi \quad \text{and} \quad \pi_i = 0 \quad \text{for each} \quad i \neq n.
\]

\( g_i : \pi_i \to S_i(K(\pi, n)) \) is the identity morphism in dimension \( n \) and the null map for \( i \neq n \).

\( f_i : S_i(K(\pi, n)) \to \pi_i \) is again the identity if \( i = n \) and null if \( i \neq n \).

\( \ker f_i = \{\star\} \) for every \( i \) and therefore \( h_i \) is always null.
The homotopical problem of a Kan simplicial set

Problem: how can we determine a solution for the homotopical problem of a given Kan simplicial set $X$?
Effective Homotopy of Fibrations

The homotopical problem of a Kan simplicial set

Problem: how can we determine a solution for the homotopical problem of a given Kan simplicial set $X$?

- For some spaces it can be \textit{directly} determined.
The homotopical problem of a Kan simplicial set

Problem: how can we determine a solution for the homotopical problem of a given Kan simplicial set $X$?

○ For some spaces it can be *directly* determined.
  ○ $K(\pi, n)$ for finitely generated groups $\pi$
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Problem: how can we determine a solution for the homotopical problem of a given Kan simplicial set $X$?

- For some spaces it can be directly determined.
  - $K(\pi, n)$ for finitely generated groups $\pi$
  - The standard simplex $\Delta$
The homotopical problem of a Kan simplicial set

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  - The standard simplex $\Delta$

$\Delta$ is known to be contractible, and in fact $S_n(\Delta) = \{\ast\}$ for all $n$. Again it is not difficult to give the definition of the solution for the homotopical problem.
The homotopical problem of a Kan simplicial set

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- Given some Kan simplicial sets $X_1, \ldots, X_n$, with effective homotopy and a topological constructor $\Phi$ which produces a new simplicial set $X$, one should obtain a solution for the homotopical problem of $X$.
The homotopical problem of a Kan simplicial set

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- Given some Kan simplicial sets $X_1, \ldots, X_n$, with effective homotopy and a topological constructor $\Phi$ which produces a new simplicial set $X$, one should obtain a solution for the homotopical problem of $X$.
- Given a simplicial Abelian group $G$ with effective homotopy, we have used Discrete Vector Fields to develop an algorithm constructing a solution for the homotopical problem of $\Omega(G)$. 

The homotopical problem of a Kan simplicial set

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- Given some Kan simplicial sets $X_1, \ldots, X_n$, with effective homotopy and a topological constructor $\Phi$ which produces a new simplicial set $X$, one should obtain a solution for the homotopical problem of $X$.
  - Given a simplicial Abelian group $G$ with effective homotopy, we have used Discrete Vector Fields to develop an algorithm constructing a solution for the homotopical problem of $\Omega(G)$.
  - Our first important example: given a constructive Kan fibration $F \hookrightarrow E \rightarrow B$ where $F$ and $B$ have effective homotopy, $E$ is also an object with effective homotopy.
Solution for the homotopical problem of a fibration

Definition

Let \( f : E \to B \) be a simplicial map. \( f \) is said to be a Kan fibration if for every collection of \( n+1 \) simplices \( x_0, x_1, \ldots, x_k-1, x_k+1, \ldots, x_n+1 \) of \( E \) which satisfy the compatibility condition

\[
\partial_i x_j = \partial_{j-1} x_i \quad \text{for all } i < j, i \neq k \text{ and } j \neq k,
\]

and for every \((n+1)\)-simplex \( y \) of \( B \) such that \( \partial_i y = f(x_i) \), \( i \neq k \), there exists an \((n+1)\)-simplex \( x \) of \( E \) such that \( \partial_i x = x_i \) for \( i \neq k \) and \( f(x) = y \).
Solution for the homotopical problem of a fibration

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Let $f : E \to B$ be a simplicial map. $f$ is said to be a Kan fibration if for every collection of $n + 1$ $n$-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ of $E$ which satisfy the compatibility condition $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j, i \neq k$ and $j \neq k$, and for every $(n + 1)$-simplex $y$ of $B$ such that $\partial_i y = f(x_i), i \neq k$, there exists an $(n + 1)$-simplex $x$ of $E$ such that $\partial_i x = x_i$ for $i \neq k$ and $f(x) = y$. 
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$$
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$$
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\[
\begin{array}{ccc}
E & \xrightarrow{f} & B \\
\end{array}
\]

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Definition

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Definition

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\[ \begin{array}{ccc} E & \overset{f}{\longrightarrow} & B \\ x & \overset{f}{\longrightarrow} & y \\ \downarrow \partial_i, i \neq k & & \downarrow \partial_i, i \neq k \\ x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} & \overset{f}{\longrightarrow} & \partial_0 y, \ldots, \partial_{k-1} y, \partial_{k+1} y, \ldots \partial_{n+1} y \end{array} \]
Solution for the homotopical problem of a fibration

Theorem
An algorithm can be written down:
Input:
A constructive Kan fibration $f : E \to B$ where $B$ is a constructive Kan complex (which implies $F$ and $E$ are also constructive Kan simplicial sets).
Respective $\text{SHmtP}_F$ and $\text{SHmtP}_B$ for the simplicial sets $F$ and $B$.
Output:
A $\text{SHmtP}_E$ for the Kan simplicial set $E$.

Proof:
The long exact sequence of homotopy
\[
\cdots \to f_* \pi_{n+1}(B) \to \partial_* \pi_n(F) \to \text{inc}_* \pi_n(E) \to f_* \pi_n(B) \to \partial_* \pi_{n-1}(F) \to \text{inc}_* \pi_{n-1}(E) \to \cdots
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**Proof:**

The long exact sequence of homotopy

$$\cdots \xrightarrow{f_*} \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \xrightarrow{\text{inc}_*} \pi_n(E) \xrightarrow{f_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \xrightarrow{\text{inc}_*} \cdots$$
Effective Homotopy of Fibrations

Solution for the homotopical problem of a fibration

produces a short exact sequence

\[ 0 \rightarrow \text{Coker} \left[ \pi_{n+1}(B) \right] \partial \rightarrow \pi_n(F) \rightarrow \pi_n(E) \rightarrow \text{Ker} \left[ \pi_{n-1}(F) \right] \rightarrow 0 \]

and this implies that \( \pi_n(E) \) can be expressed as \( \pi_n(E) \cong \text{Coker} \times \chi \text{Ker} \) for an unknown cohomology class \( \chi \in H^2(\text{Ker}, \text{Coker}) \) classifying the extension.

This cohomology class can be determined if we make the short exact sequence constructive by defining a section \( \sigma : \text{Ker} \rightarrow \pi_n(E) \) and a retraction \( \rho : \pi_n(E) \rightarrow \text{Coker} \) (which are set-theoretic maps) such that \( \rho \circ i = \text{Id}_{\text{Coker}} \), \( \rho \circ j + \sigma \circ j = \text{Id}_{\pi_n(E)} \), and \( j \circ \sigma = \text{Id}_{\text{Ker}} \).

Once the maps \( \sigma \) and \( \rho \) have been defined it is easy to give a constructive definition of the cohomology class and construct the algorithms \( f_n, g_n \) and \( h_n \).

The most difficult part of the proof is the definition of \( \sigma \) and \( \rho \), which are based in a suitable game with the Kan properties of \( F, B \) and the fibration \( f \) and on the SHmtP of \( F \) and \( B \).
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\[
0 \longrightarrow \text{Coker}[\pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F)] \xrightarrow{i} \pi_n(E) \xrightarrow{j} \text{Ker}[\pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F)] \longrightarrow 0
\]
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Solution for the homotopical problem of a fibration produces a short exact sequence

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This cohomology class can be determined if we make the short exact sequence constructive by defining a section \( \sigma : \text{Ker} \to \pi_n(E) \) and a retraction \( \rho : \pi_n(E) \to \text{Coker} \) (which are set-theoretic maps).
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The most difficult part of the proof is the definition of \( \sigma \) and \( \rho \), which are based in a suitable game with the Kan properties of \( F \), \( B \) and the fibration \( f \) and on the SHmtP of \( F \) and \( B \).
Examples and applications

Given \( F = K(\pi, n) \) and \( B = K(\pi', m) \) where both \( \pi \) and \( \pi' \) are finitely generated groups, and a constructive Kan fibration \( F \hookrightarrow E \twoheadrightarrow B \), our theorem allows one to construct a SHmtP of the total space \( E \).

If \( n \neq m \) the long exact sequence of homotopy gives directly \( \pi^* (E) \).

The interest of our application can be seen when \( n = m \). Then one has

\[ 0 \rightarrow \pi \rightarrow \pi_n(E) \rightarrow \pi' \rightarrow 0 \]

which implies \( \pi_n(E) \) is an extension of \( \pi' \) by \( \pi \), but several extensions could be possible.

Our algorithms determine \( \pi^* (E) \).

Our results can also be applied when \( F \) and \( B \) are direct sums of \( K(\pi, n) \)'s. Several non-null groups appear then in the long exact sequence and it can happen that this information is not sufficient to determine \( \pi^* (E) \).
Fibrations of Eilenberg-MacLane spaces
Examples and applications

Fibrations of Eilenberg-MacLane spaces

Given $F = K(\pi, n)$ and $B = K(\pi', m)$ where both $\pi$ and $\pi'$ are finitely generated groups, and a constructive Kan fibration $F \hookrightarrow E \rightarrow B$, our theorem allows one to construct a SHmtP of the total space $E$.

If $n \neq m$ the long exact sequence of homotopy gives directly $\pi_*(E)$. The interest of our application can be seen when $n = m$. Then one has $0 \rightarrow \pi \rightarrow \pi_n(E) \rightarrow \pi' \rightarrow 0$ which implies $\pi_n(E)$ is an extension of $\pi'$ by $\pi$, but several extensions could be possible. Our algorithms determine $\pi_*(E)$. Our results can also be applied when $F$ and $B$ are direct sums of $K(\pi, n)$'s. Several non-null groups appear then in the long exact sequence and it can happen that this information is not sufficient to determine $\pi_*(E)$.
Fibrations of Eilenberg-MacLane spaces

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If \( n \neq m \) the long exact sequence of homotopy gives directly \( \pi_\ast(E) \).
Fibrations of Eilenberg-MacLane spaces

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If $n \neq m$ the long exact sequence of homotopy gives directly $\pi_*(E)$. The interest of our application can be seen when $n = m$. Then one has

$$0 \longrightarrow \pi \longrightarrow \pi_n(E) \longrightarrow \pi' \longrightarrow 0$$

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Fibrations of Eilenberg-MacLane spaces

Given $F = K(\pi, n)$ and $B = K(\pi', m)$ where both $\pi$ and $\pi'$ are finitely generated groups, and a constructive Kan fibration $F \hookrightarrow E \twoheadrightarrow B$, our theorem allows one to construct a SHmtP of the total space $E$.

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which implies $\pi_n(E)$ is an extension of $\pi'$ by $\pi$, but several extensions could be possible. Our algorithms determine $\pi_\ast(E)$. 
Fibrations of Eilenberg-MacLane spaces

Given $F = K(\pi, n)$ and $B = K(\pi', m)$ where both $\pi$ and $\pi'$ are finitely generated groups, and a constructive Kan fibration $F \hookrightarrow E \rightarrow B$, our theorem allows one to construct a SHmtP of the total space $E$.

If $n \neq m$ the long exact sequence of homotopy gives directly $\pi_*(E)$. The interest of our application can be seen when $n = m$. Then one has

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Our results can also be applied when $F$ and $B$ are direct sums of $K(\pi, n)$’s. Several non-null groups appear then in the long exact sequence and it can happen that this information is not sufficient to determine $\pi_*(E)$. 
Examples and applications

It is defined by means of a tower of fibrations

\[ \text{F}_3 \xrightarrow{\text{f}_1} \text{F}_2 \xrightarrow{\text{f}_2} \text{F}_1 \xrightarrow{\text{f}_3} \ldots \xrightarrow{\text{f}_4} \text{Y} \]

and under good conditions converges to \( \pi^* (X) \).

The different groups \( E^{p,q}_r \) can be determined if \( \pi^* (Y^n) \) and \( \pi^* (F^n) \) are constructively known.
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Bousfield-Kan spectral sequence of a simplicial set $X$
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The first space in the tower is $Y_0 = RX$, the free simplicial Abelian group generated by $X$, which satisfies $\pi_\ast(RX) \cong \tilde{H}_\ast(X)$. If $X$ has effective homology, then $RX$ has effective homotopy. If we apply the constructor $R$ to $RX$ we obtain $R^2X$ which satisfies $\pi_\ast(R^2X) \cong \tilde{H}_\ast(RX)$. We have developed an algorithm computing the effective homology of $RX$ from the effective homology of $X$, and therefore $R^2X$ is also an object with effective homotopy. The first fiber is $F_1 = \Omega(R^2X \cap \text{Ker } \eta_0)$.
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Effective Homotopy of Fibrations

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A. Romero and F. Sergeraert
Luminy, January 2011
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$$\pi_*(F_1) = \pi_*(\Omega(R^2 X \cap \ker \eta^0)) \cong \pi_{*+1}(R^2 X \cap \ker \eta^0) \cong \pi_{*+1}(R^2 X) \cap \ker \eta^0$$
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so that one can construct a SHmtP of $F_1$. 

On the other hand, by using Discrete Vector Fields we have proved that $f_n$ are constructive Kan fibrations. Applying our Theorem, the space $Y_1$ has effective homotopy. In a similar way one can prove that all $F_n$'s have effective homotopy, and iterating the process one see that all the spaces in the fibrations are objects with effective homotopy. An algorithm is obtained computing all the components of the Bousfield-Kan spectral sequence.
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Conclusions and further work

Following the ideas of the effective homology method, we have started a new effective homotopy theory. We have introduced the main definition of a solution for the homotopical problem of a simplicial set and we have given an algorithm computing the effective homotopy of the total space of a fibration.

As applications, we can compute the homotopy groups of fibrations of Eilenberg-MacLane spaces and we have developed an algorithm computing all levels of the Bousfield-Kan spectral sequence. Similar algorithms could be obtained producing a SHmtP of the base space (respectively the fiber space) from the SHmtP of the total space and the fiber (resp. the base). Furthermore, other constructions in Algebraic Topology should be studied, as already done in the effective homology framework.

All our algorithms should be implemented in Common Lisp enhancing the Kenzo system.
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