Effective Computation of Generalized Spectral Sequences

Andrea Guidolin[1] and **Ana Romero**[2]

[1]Basque Center for Applied Mathematics (Spain)
[2]University of La Rioja (Spain)

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A **filtration** of the chain complex C_* is a sequence $(F_p C_*)_{p \in \mathbb{Z}}$

$$\ldots \subseteq F_{p-1} C_* \subseteq F_p C_* \subseteq \ldots \subseteq C_*$$

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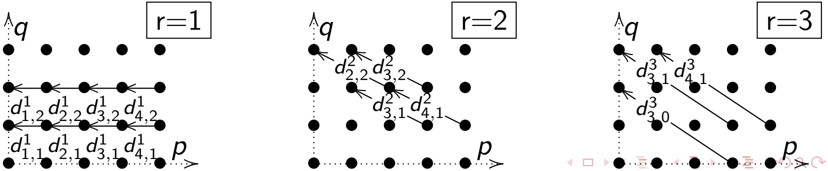

Generalized filtrations and spectral systems

The notion of spectral sequence of a filtered complex has been recently generalized by B. Matschke for a filtration indexed over a *poset* I, i.e. a collection of sub-chain complexes $\{F_i C_*\}_{i \in I}$ with $F_i C_* \subseteq F_j C_*$ if $i \leq j$, as a set of groups, for all $z \leq s \leq p \leq b$ in I and for each degree n:

$$S_n[z,s,p,b] = \frac{F_p C_n \cap d_n^{-1}(F_z C_{n-1}) + F_s C_n}{d_{n+1}(F_b C_{n+1}) + F_s C_n}$$

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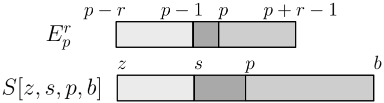
Example: $\mathbb{Z}$-filtration $(F_p)_{p \in \mathbb{Z}}$, indices $z \leq s \leq p \leq b$ in $\mathbb{Z}$:



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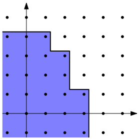
A **downset** of $\mathbb{Z}^m$ is a subset $p \subseteq \mathbb{Z}^m$ such that if $P \in p$ and $Q \leq P$ in $\mathbb{Z}^m$ then $Q \in p$.



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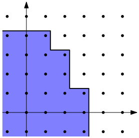
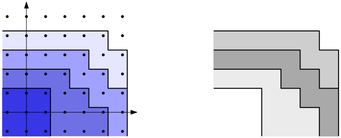


We denote $D(\mathbb{Z}^m)$ the collection of all downsets of $\mathbb{Z}^m$, which is a poset with respect to the inclusion $\subseteq$.



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Theorem (Serre, 1951)

Let $G \hookrightarrow E \to B$ be a **fibration** and suppose the base B is 1-reduced. There is a spectral sequence converging to $H_*(E)$ whose second page is given by $E^2_{p,q} = H_p(B; H_q(G))$.

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Theorem (Matschke, 2013)

Consider a **tower of fibrations**

$$\begin{array}{ccccc} E & \longrightarrow & N & \longrightarrow & B \\ \uparrow & & \uparrow & & \\ G & & M & & \end{array}$$

and suppose the base B is 1-reduced. There exists a $D(\mathbb{Z}^2)$-spectral system converging to $H_*(E)$ whose second page is given by

$$S^*_n(P;2) = H_{p_2}(B; H_{p_1}(M; H_{n-p_1-p_2}(G))), \quad P = (p_1, p_2) \in \mathbb{Z}^2.$$

Multidimensional persistence and persistence of I-filtrations

Multidimensional filtrations (or $\mathbb{Z}^m$-filtrations) of simplicial complexes:

$$\begin{tikzcd}
K_{1N'} \arrow[r, hook] & K_{2N'} \arrow[r, hook] & \cdots \arrow[r, hook] & K_{NN'} \\
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Associated invariant: **rank invariant**

$$\beta_n^{P,Q} := \dim_{\mathbb{F}} \mathrm{Im}(H_n(K_P) \to H_n(K_Q)), \qquad P, Q \in \mathbb{Z}^m, \quad P \leq Q.$$

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Similarly, for an I-filtration $(F_i)_{i \in I}$, we define **rank invariant** the collection of integers

$$\beta_n(v, w) := \dim_{\mathbb{F}} \operatorname{Im}(H_n(F_v) \to H_n(F_w)), \qquad v, w \in I, \quad v \leq w.$$

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The result is a *basis-divisors* description of the group, that is:

- a list of combinations $(c_1, \ldots, c_{\alpha+k})$
- a list of torsion coefficients $(b_1, \ldots, b_k, 0, \overset{\alpha}{\ldots}, 0)$.

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To compute the differential map
$d : S_2 \equiv S[z_2, s_2, p_2, b_2] \to S_1 \equiv S[z_1, s_1, p_1, b_1]$ applied to an element
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- We compute the coefficients of $d(y)$ with respect to the set of generators of S_1.
- We reduce them considering the corresponding divisors.

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Theorem

Let $\rho = (f, g, h) : C_* \Rightarrow D_*$ be a reduction between the I-filtered chain complexes (C_*, F) and (D_*, F'), and suppose that f and g are compatible with the filtrations. Then, given four indices $z \leq s \leq p \leq b$ in I, the map f induces an isomorphism $f^{z,s,p,b} : S[z,s,p,b]_n \to S'[z,s,p,b]_n$ whenever the homotopy $h : (C_*, F) \to (C_{*+1}, F)$ satisfies the conditions

$$h(F_z) \subseteq F_s \qquad \text{and} \qquad h(F_p) \subseteq F_b.$$

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Let $F = (F_i)_{i \in I}$ be an I-filtration of (C_*, β), and let $V = \{(\sigma_j; \tau_j)\}_{j \in J}$ be an admissible discrete vector field on (C_*, β) such that, for all $j \in J$, the cells σ_j and τ_j appear together in the filtration. Then there exists a reduction $\rho =: C_* \Rrightarrow C_*^c$, where C_*^c is the **critical** chain complex (generated by the cells which do not appear in the vector field), which is compatible with the filtrations.

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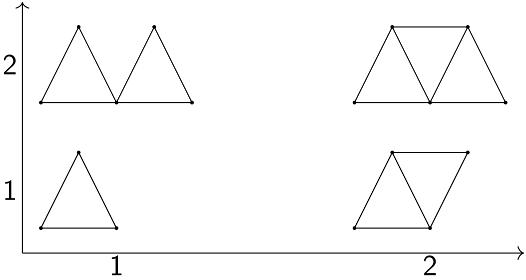
Corollary

Under the same hypotheses, the generalized spectral sequences associated with the I-filtrations of C_* and C_*^c are isomorphic.

Example



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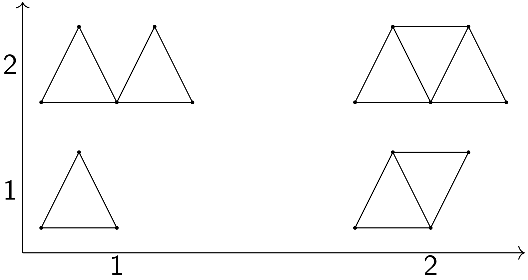


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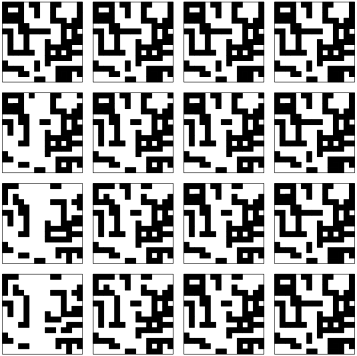
Discrete vector fields: example

Filtration over $\mathbb{Z}^2$ of a **digital image**:



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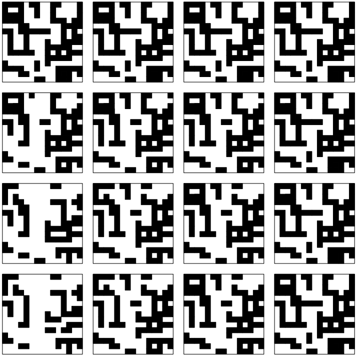

Associated simplicial complex: 203 vertices, 408 edges and 208 triangles.

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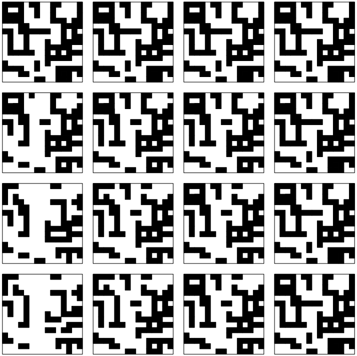

Associated simplicial complex: 203 vertices, 408 edges and 208 triangles.
Reduced chain complex: 21 vertices, 23 edges and 5 triangles.

Generalized Serre spectral sequence: example

First stages of the Postnikov tower for computing the homotopy groups of the sphere S^3, given by the following tower of fibrations:

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Thank you!