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Homology of groups: introduction

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• Algebra:

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Several constructions:

- Topology: we identify H_i(G) with the homology of a canonical topological space K ≡ K(G, 1) associated with G, with π₁(K) = G and π_i(K) = 0 for all i > 1.
- Algebra: $H_i(G)$ are computed by means of *resolutions*.

Resolutions (Algebra)

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Definition

A resolution F_* for a group G is an acyclic chain complex of $\mathbb{Z}G$ -modules

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} F_{-1} = \mathbb{Z} \longrightarrow 0$$

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Theorem

Let G be a group and F_* , F'_* two free resolutions of G. Then

 $H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*) \cong H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F'_*) \cong H_n(K(G,1)) \quad \text{for all } n \in \mathbb{N}$

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Definition

Given a group G, the homology groups $H_n(G)$ are defined as $H_n(G) = H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*)$, $n \in \mathbb{N}$, where F_* is any free resolution for G.

Resolutions (Algebra)

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For some particular cases, small (or minimal) resolutions can be directly constructed.

For instance, let $G = C_m$ with generator t. The resolution F_*

$$\cdots \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

produces

$$H_i(G) = \begin{cases} \mathbb{Z} & \text{if } i = 0\\ \mathbb{Z}/m\mathbb{Z} & \text{if } i \text{ is odd}\\ 0 & \text{if } i \text{ is even}, i > 0 \end{cases}$$

Effective homology (Topology)

Image: A matrix and a matrix

Definition

A reduction ρ between two chain complexes C_* and D_* (denoted by $\rho: C_* \Longrightarrow D_*$) is a triple $\rho = (f, g, h)$ $h \underbrace{c_* \underbrace{f}_{\sigma} }_{\sigma} D_*$

satisfying the following relations:

1)
$$fg = Id_{D_*};$$

2) $d_Ch + hd_C = Id_{C_*} - gf;$
3) $fh = 0; \quad hg = 0; \quad hh = 0$

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1)
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2) $d_C h + hd_C = Id_{C_*} - gf;$
3) $fh = 0; hg = 0; hh = 0$

If $C_* \Rightarrow D_*$, then $C_* \cong D_* \oplus A_*$, with A_* acyclic, which implies that $H_n(C_*) \cong H_n(D_*)$ for all n.

Effective homology (Topology)

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Definition

A (strong chain) equivalence ε between C_* and D_* , $\varepsilon : C_* \iff D_*$, is a triple $\varepsilon = (B_*, \rho, \rho')$ where B_* is a chain complex, $\rho : B_* \Rightarrow C_*$ and $\rho' : B_* \Rightarrow D_*$. $B_* \Rightarrow D_*$ $B_* \Rightarrow D_*$ $B_* \Rightarrow D_*$ $\frac{42}{30} \Rightarrow \frac{21}{15}$

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Definition

An object with effective homology is a quadruple $(X, C_*(X), HC_*, \varepsilon)$ where HC_* is an effective chain complex and $\varepsilon : C_*(X) \iff HC_*$.

This implies that $H_n(X) \cong H_n(HC_*)$ for all *n*.

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Algorithm computing the effective homology of a group

Given G a group, F_* a (small) free $\mathbb{Z}G$ -resolution with a *contracting* homotopy $h_n : F_n \to F_{n+1}$.

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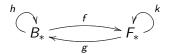
It is well known that there exists a morphism of chain complexes of $\mathbb{Z}G$ -modules $f : B_* \to F_*$ which is a homotopy equivalence.

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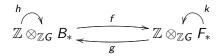
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It is well known that there exists a morphism of chain complexes of $\mathbb{Z}G$ -modules $f: B_* \to F_*$ which is a homotopy equivalence. An algorithm has been designed constructing the explicit expressions of f and the corresponding maps g, h and k

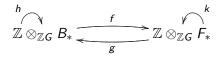


Algorithm computing the effective homology of a group

Applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}G}$ — we obtain an equivalence of chain complexes (of \mathbb{Z} -modules):



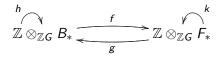
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In order to obtain a strong chain equivalence we make use of the mapping cylinder construction.

$$\mathbb{Z} \otimes_{\mathbb{Z}G} B_* \overset{\rho'}{\Leftarrow} \mathsf{Cylinder}(f)_* \overset{
ho}{\twoheadrightarrow} \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$$

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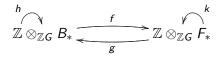


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Finally we observe that the left chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} B_*$ is equal to $C_*(\mathcal{K}(G,1))$.

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$$\mathbb{Z} \otimes_{\mathbb{Z}G} B_* \stackrel{\rho'}{\twoheadleftarrow} \text{Cylinder}(f)_* \stackrel{\rho}{\twoheadrightarrow} \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$$

Finally we observe that the left chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} B_*$ is equal to $C_*(\mathcal{K}(G,1))$. Moreover, if the initial resolution F_* is of finite type (and small), then the right chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} F_* \equiv E_*$ is effective.

Algorithm computing the effective homology of a group

Algorithm

Input: a group G and a free resolution F_* of finite type with contracting homotopy. Output: the effective homology of K(G, 1), that is, a (strong chain) equivalence $C_*(K(G, 1)) \iff E_*$ where E_* is an effective chain complex.

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- Implemented in Common Lisp, enhancing the Kenzo system (developed by F. Sergeraert and some coworkers).
- It allows to compute homology of groups and, what is more important, to use the space K(G, 1) in other constructions allowing new computations.

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Applications and examples

Computations with $K(C_p, n)'s$

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> (setf KC71 (K-G-1 C7))
[K82 Abelian-Simplicial-Group]
> (efhm KC71)
[K119 Homotopy-Equivalence K82 <= K109 => K76]

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The classifying space constructor \overline{W} gives us $\overline{W}(K(G,1)) = K(G,2)$.

```
> (setf KC72 (classifying-space KC71))
[K120 Abelian-Simplicial-Group]
> (efhm KC72)
[K259 Homotopy-Equivalence K120 <= K249 => K245]
> (homology KC72 3 6)
Homology in dimension 3 :
---done---
Homology in dimension 4 :
Component Z/7Z
---done---
Homology in dimension 5 :
---done---
```

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Applications and examples

Homology of 2-types

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Let $G = C_3$, $A = \mathbb{Z}/3\mathbb{Z}$ with trivial G-action, and $[f] \in H^3(G, A) = \mathbb{Z}/3\mathbb{Z}$ a non-trivial cohomology class.

Homology of 2-types

Let $G = C_3$, $A = \mathbb{Z}/3\mathbb{Z}$ with trivial *G*-action, and $[f] \in H^3(G, A) = \mathbb{Z}/3\mathbb{Z}$ a non-trivial cohomology class. This corresponds to a 2-type with $\pi_1 = G$ and $\pi_2 = A$, which can be seen as $X = K(A, 2) \times_f K(G, 1)$.

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```
> (setf K-C3-1 (K-Cm-n 3 1))
[K261 Abelian-Simplicial-Group]
> (setf chml-clss (chml-clss K-C3-1 3))
[K308 Cohomology-Class on K288 of degree 3]
> (setf tau (zp-whitehead 3 K-C3-1 chml-clss))
[K323 Fibration K261 -> K309]
> (setf x (fibration-total tau))
[K329 Kan-Simplicial-Set]
> (efhm x)
[K541 Homotopy-Equivalence K329 <= K531 => K527]
> (homology x 5)
Homology in dimension 5 :
Component Z/3Z
----dome---
```

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Applications and examples

Central extensions

Central extensions

$$E = \langle x, y, z | x^{p} = y^{p} = z^{p^{n-2}} = [x, z] = [y, z] = 1; [x, y] = z^{p^{n-3}} > 0$$

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can be seen as a central extension of the groups $A = \langle z | z^{p^{n-2}} = 1 \rangle \cong C_{p^{n-2}}$ and $G = \langle x, y | x^p = y^p = [x, y] = 1 \rangle \cong C_p \oplus C_p$.

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Central extensions

```
> (setf KE1 (K-G-1 E))
[K776 Simplicial-Group]
> (efhm KE1)
[K884 Homotopy-Equivalence K776 <= K870 => K866]
> (homology KE1 3)
Homology in dimension 3 :
Component Z/9Z
Component Z/3Z
Component Z/3Z
Component Z/3Z
---done---
```

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Discrete Morse theory

Discrete Morse theory

Definition

Let $C_* = (C_p, d_p)_{p \in \mathbb{Z}}$ a free chain complex with distinguished \mathbb{Z} -basis $\beta_p \subset C_p$. A *discrete vector field* V on C_* is a collection of pairs $V = \{(\sigma_i; \tau_i)\}_{i \in I}$ satisfying the conditions:

- Every σ_i is some element of β_p, in which case τ_i ∈ β_{p+1}. The degree p depends on i and in general is not constant.
- Every component σ_i is a *regular face* of the corresponding τ_i .
- Each generator (cell) of C_{*} appears at most one time in V.

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Theorem

Let $C_* = (C_p, d_p)_{p \in \mathbb{Z}}$ be a free chain complex and $V = \{(\sigma_i; \tau_i)\}_{i \in I}$ be an admissible discrete vector field on C_* . Then the vector field V defines a canonical reduction $\rho = (f, g, h) : (C_p, d_p) \Rightarrow (C_p^c, d_p')$ where $C_p^c = \mathbb{Z}[\beta_p^c]$ is the free \mathbb{Z} -module generated by the critical p-cells.

Discrete Morse theory and homology of $K(C_p, 1)$'s

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Theorem

An admissible discrete vector field V can be defined on $C_*(K(C_p, 1))$:

- The critical cells are the unique 0-simplex [], the (2k)-simplex $[p-1, 1, \ldots, p-1, 1]$ for each $k \ge 1$ and the (2k + 1)-simplex $[1, p-1, 1, \ldots, p-1, 1]$ for each $k \ge 0$.
- ② The source cells are the k-simplices [a₁,..., a_{k-2r}, p − 1, 1, ..., p − 1, 1] with a_{k-2r} > 1 and 0 ≤ r < k/2.</p>
- One target cells are the (k + 1)-simplices [a₁,..., a_{k-2r}, 1, p − 1, 1,..., p − 1, 1] with a_{k-2r}
- The pairing [source cell ↔ target cell] associates to the source cell [a₁,..., a_{k-2r}, p 1, 1, ..., p 1, 1] the target cell [a₁,..., a_{k-2r} 1, 1, p 1, 1, ..., p 1, 1].

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Discrete Morse theory and homology of $K(C_p, 1)$'s

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- ② The source cells are the k-simplices [a₁,..., a_{k-2r}, p − 1, 1, ..., p − 1, 1] with a_{k-2r} > 1 and 0 ≤ r < k/2.</p>
- One target cells are the (k + 1)-simplices [a₁,..., a_{k-2r}, 1, p − 1, 1,..., p − 1, 1] with a_{k-2r}
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This gives us a reduction $C_*(K(C_p, 1)) \Rightarrow \mathbb{Z}[\beta^c]$, where $\mathbb{Z}[\beta^c]$ is the free chain complex generated by the critical cells, which is an effective (and small) chain complex.

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Discrete Morse theory and homology of fibrations

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Theorem (F. Sergeraert, 2010)

Let $F \hookrightarrow E \to B$ a fibration, where E can be seen as a twisted Cartesian product $E = F \times_{\tau} B$. Then one can define an admissible discrete vector field on E which defines a homological reduction:

 $C_*(F \times_{\tau} B) \Longrightarrow \mathbb{Z}[\beta^c] \cong C_*(F) \otimes_t C_*(B)$

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This improves the original reduction $C_*(F \times_{\tau} B) \Rightarrow C_*(F) \otimes_t C_*(B)$ given by the twisted Eilenberg-Zilber theorem.

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This improves the original reduction $C_*(F \times_{\tau} B) \Rightarrow C_*(F) \otimes_t C_*(B)$ given by the twisted Eilenberg-Zilber theorem. Combined with the effective homology of a (twisted) tensor product, one obtains an equivalence

$$C_*(F imes_{ au} B) \iff EFF'_*$$

where EFF'_* is an effective chain complex.

Conclusions and further work

Image: A matrix

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- Discrete vector fields can be considered to improve our algorithms, obtaining in some cases direct reductions instead of equivalences or decreasing the complexity of the constructions.

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- Moreover, effective homology makes it possible to use the spaces K(G,1)'s in other interesting constructions allowing new results (for instance, homology of 2-types).
- Discrete vector fields can be considered to improve our algorithms, obtaining in some cases direct reductions instead of equivalences or decreasing the complexity of the constructions.
- One should try to obtain discrete vector fields for the effective homology of other groups (different from C_p 's) and introduce them in other parts of our constructions.

- Topology can also be used to compute homology of groups by means of the effective homology method.
- Moreover, effective homology makes it possible to use the spaces K(G,1)'s in other interesting constructions allowing new results (for instance, homology of 2-types).
- Discrete vector fields can be considered to improve our algorithms, obtaining in some cases direct reductions instead of equivalences or decreasing the complexity of the constructions.
- One should try to obtain discrete vector fields for the effective homology of other groups (different from C_p 's) and introduce them in other parts of our constructions.
- With respect to computation time, our first experiments have not been as good as expected. Work should be done trying to improve it.