From Homological Perturbation to Spectral Sequences: a Case Study

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Spectral sequences

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Definition. A *Spectral Sequence* $E = \{E^r, d^r\}$ is a family of $\mathbb{Z}$-bigraded modules E^1, E^2,..., each provided with a differential $d^r = \{d^r_{p,q}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$, $r = 1, 2, \ldots$

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"If we think of a spectral sequence as a black box, then the input is a differential bigraded module, usually $E^1_{*,*}$, and, with each turn of the handle, the machine computes a successive homology according to a sequence of differentials. If some differential is unknown, then some other (any other) principle is needed to proceed. [...] In the nontrivial cases, it is often a deep geometric idea that is caught up in the knowledge of a differential."

John McCleary, User's guide to spectral sequences (Publish or Perish, 1985)

An example of spectral sequence

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A fibration $G \hookrightarrow E \longrightarrow B$

G is the fiber space, B the base space, and $E = B \times_\tau G$ the total space.

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A fibration $G \hookrightarrow E \to B$

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Theorem (Serre spectral sequence). *Let $G \hookrightarrow E \to B$ be a fibration with a base space B simply connected. Then a first quadrant spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}_{r\geq 2}$ is defined with $E^2_{p,q} = H_p(B, H_q(G))$ and $E^r_{p,q} \Rightarrow H_{p+q}(E)$.*

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Using this spectral sequence, Serre computed many sphere homotopy groups.

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$$\boxed{r = 2}$$

$$
\begin{array}{c|ccccc}
q & & & & & \\
 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 \\
\hline
 & & & & & p
\end{array}
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 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 \\
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 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 \\
\hline
 & & & & & p
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0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll, "\times 1"'] & 0
\end{tikzcd}$$

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$$\begin{tikzcd}[row sep=tiny]
\mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll, "?"'] & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll, "?"'] & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll, "\times 1"'] & 0
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$$
\begin{tikzcd}[row sep=tiny, column sep=small]
\mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll, "\times 3"'] & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll, "\times 2"'] & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll, "\times 1"'] & 0
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\hline
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\end{array}
\qquad \boxed{r=2} \quad \boxed{r=3}
$$

$\times 3$ $\quad$ $\times 2$ $\quad$ $\times 1$

$$
\begin{array}{c|ccccc}
q & & & & & \\
 & \mathbb{Z}_3 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z}_2 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
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\hline
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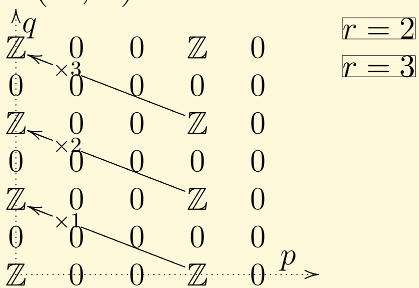

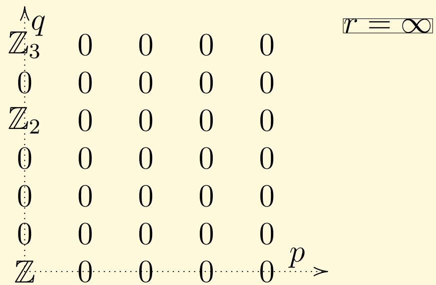

The Hurewicz theorem and the long exact sequence of homotopy imply that $\pi_4(S^3) = \pi_4(X_4) = H_4(X_4) = \mathbb{Z}_2$.

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- Then, a new fibration $F_3 \hookrightarrow X_5 \to X_4$ is considered to determine $\pi_5(S^3)$, where $F_3 = K(\mathbb{Z}_2, 3)$ is chosen because $\pi_4(X_4) = \mathbb{Z}_2$.

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  Again some extra information is needed to compute the differentials, and we obtain $\pi_5(S^3) = \pi_5(X_4) = \pi_5(X_5) = H_5(X_5) = \mathbb{Z}_2$.

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- Similarly, Serre used a new fibration $F_4 \hookrightarrow X_6 \longrightarrow X_5$, with $F_4 = K(\mathbb{Z}_2, 4)$, to compute $\pi_6(S^3)$.

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  Using his spectral sequence, he proved $\pi_6(S^3)$ has 12 elements, but he was unable to choose between the two possible options $\mathbb{Z}_{12}$ and $\mathbb{Z}_2 + \mathbb{Z}_6$.

Effective homology

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$$\begin{array}{c} X_1, X_2, \ldots, X_n \\ \Big\downarrow \varphi \\ X \end{array}$$

Effective homology

$$
\begin{tikzcd}
X_1, X_2, \ldots, X_n \arrow[d, "\varphi"] & X_1^{EH}, X_2^{EH}, \ldots, X_n^{EH} \\
X &
\end{tikzcd}
$$

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Spectral sequences vs. effective homology

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$$H_*(X_1), H_*(X_2), \ldots, H_*(X_n)$$

Spectral sequences vs. effective homology

$$\begin{array}{ccc} X_1, X_2, \ldots, X_n & \qquad\qquad & H_*(X_1), H_*(X_2), \ldots, H_*(X_n) \\ \Big\downarrow \varphi & & \\ X & & H_*(X)??? \end{array}$$

Spectral sequences vs. effective homology

$$
\begin{tikzcd}
X_1, X_2, \ldots, X_n \arrow[d, "\varphi"] & & H_*(X_1), H_*(X_2), \ldots, H_*(X_n) \arrow[d, "SS"] \\
X & & H_*(X)
\end{tikzcd}
$$

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\begin{tikzcd}
X_1, X_2, \ldots, X_n \arrow[d, "\varphi"] & & H_*(X_1), H_*(X_2), \ldots, H_*(X_n) \arrow[d, dashed, leftrightarrow, "SS"] \\
X & & H_*(X)
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X & X^{EH} \arrow[r] & H_*(X)
\end{tikzcd}$$

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\end{tikzcd}
$$

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Definition. A *reduction* ρ between two chain complexes A and B (denoted by $\rho : A \Rrightarrow B$) is a triple $\rho = (f, g, h)$

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

satisfying the following relations:

$fg = \mathrm{id}_B$; $gf + d_A h + h d_A = \mathrm{id}_A$;

$fh = 0$; $hg = 0$; $hh = 0$.

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Remark. If $A \Rrightarrow B$, then $A = B \oplus C$, with C acyclic, which implies that $H_*(A) \cong H_*(B)$.

Effective homology

Effective homology

Definition. A *(strong chain) equivalence* between the complexes A and B ($A \lll\!\!\Longleftarrow\!\!\!\Longrightarrow\!\!\ggg B$) is a triple (D, ρ, ρ') where D is a chain complex, $\rho : D \Rrightarrow A$ and $\rho' : D \Rrightarrow B$.

$$A \lll D \ggg B$$

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$$A \Lleftarrow D \Rrightarrow B$$

Definition. An *object with effective homology* is a triple (X, EC, ε) where EC is an effective chain complex and $C(X) \overset{\varepsilon}{\Lleftarrow\!\!\Rrightarrow} EC$.

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Remark. This implies that $H_*(X) \cong H_*(EC)$.

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Example: X_5, total space of the fibration $F_3 \hookrightarrow X_5 \rightarrow X_4$

The Kenzo system

The Kenzo system

First, we construct the space X_5

```
>(setf s3 (sphere 3))
[K1 Simplicial-Set]
> (setf f3 (z-whitehead s3 (chml-clss s3 3)))
[K37 Fibration K1 -> K25]
> (setf x4 (fibration-total f3))
[K43 Simplicial-Set]
> (setf f4 (z2-whitehead x4 (chml-clss x4 4)))
[K292 Fibration K43 -> K278]
> (setf x5 (fibration-total f4))
[K298 Simplicial-Set]
```

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```
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[K292 Fibration K43 -> K278]
> (setf x5 (fibration-total f4))
[K298 Simplicial-Set]
```

We can ask for the effective homology of X_5:

```
> (efhm x5)
[K608 Homotopy-Equivalence K298 <= K598 => K594]
```

The Kenzo system

The Kenzo system

For the homology groups of X_5 in dimensions 5 and 6 we obtain

```
> (homology x5 5)
Homology in dimension 5 :
Component Z/2Z
---done---
> (homology x5 6)
Homology in dimension 6 :
Component Z/6Z
---done---
```

which means $H_5(X_5) = \mathbb{Z}_2$ and $H_6(X_5) = \mathbb{Z}_6$

Basic Perturbation Lemma (BPL)

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Remark. If δ is a perturbation on (C, d_C), then $(C, d_C + \delta)$ is also a chain complex.

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Problem. Given a reduction $(C, d_C) \Rrightarrow (D, d_D)$ and a perturbation δ of the chain complex (C, d_C), how to determine a new reduction

$$(C, d_C + \delta) \Rrightarrow (D, ???)$$

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Theorem (Basic Perturbation Lemma). *Let $\rho = (f, g, h)$ be a reduction $\rho : (C, d_C) \Rrightarrow (D, d_D)$ and δ a* perturbation *of d_C, where the composite function $h \circ \delta$ is* locally nilpotent*. Then a new reduction $\rho' : (C, d_C + \delta) \Rrightarrow (D, d_D + \bar{\delta})$, $\rho' = (f', g', h')$, can be constructed.*

Basic Perturbation Lemma (BPL)

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Proof. 1. $\bar{\delta} = f \circ \delta \circ \phi \circ g = f \circ \psi \circ \delta \circ g$,

2. $f' = f \circ \psi = f \circ (Id - \delta \circ \phi \circ h)$,

3. $g' = \phi \circ g$,

4. $h' = \phi \circ h = h \circ \psi$,

where the operators ϕ and ψ are defined by

$$\phi = \sum_{i=0}^{\infty} (-1)^i (h \circ \delta)^i; \quad \psi = \sum_{i=0}^{\infty} (-1)^i (\delta \circ h)^i = Id - \delta \circ \phi \circ h,$$

(the series are convergent thanks to the locally nilpotency of $h \circ \delta$) $\square$

Effective homology of the total space of a fibration

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A fibration $G \hookrightarrow E \longrightarrow B$

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A fibration $G \hookrightarrow E \to B$

B and G are objects with effective homology

$$\begin{tikzcd}[column sep=small] & DB \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ C(B) & & EB \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & DG \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ C(G) & & EG \end{tikzcd}$$

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$$\begin{tikzcd} C(B \times \ G) \arrow[dr, Rightarrow] & \\ & C(B) \otimes \ C(G) \end{tikzcd}$$

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Effective homology of the total space of a fibration

A fibration $G \hookrightarrow E \rightarrow B$

B and G are objects with effective homology

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Effective homology of $E = B \times_\tau G$

$$\begin{tikzcd} C(B \times_\tau G) \arrow[dr, Rightarrow] & \\ & C(B) \otimes_t C(G) \end{tikzcd} \qquad \begin{tikzcd} DB \otimes \ DG \arrow[dr, Rightarrow] & \\ & EB \otimes \ EG \end{tikzcd}$$

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- To compute the effective homology of X_4, the effective homologies of F_2 and S^3 are necessary.
- The simplicial set S^3 is already of finite type.
- And finally, the effective homology of $F_2 = K(\mathbb{Z}, 2)$ is also computable.

Computing spectral sequences

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Theorem. *Let C be a filtered chain complex with effective homology (HC, ε), with $\varepsilon = (D, \rho, \rho')$, $\rho = (f, g, h)$, and $\rho' = (f', g', h')$. Let us supose that filtrations are also defined in the chain complexes HC and D. If the maps f, f', g, and g' are morphisms of filtered complexes (i.e., they are compatible with the filtrations) and both homotopies h and h' have order $\leq t$ (i.e. $h(F_pD), h'(F_pD) \subset F_{p+t}D \quad \forall p \in \mathbb{Z}$), then the spectral sequences of the complexes C and HC are isomorphic for $r > t$:*

$$E(C)^r_{p,q} \cong E(HC)^r_{p,q} \quad \forall r > t$$

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- If the filtered complex is effective, then the formal expression of the groups $E^r_{p,q}$ can be computed through elementary methods with integer matrices.
- Otherwise, the effective homology is needed to compute the $E^r_{p,q}$ by means of an analogous spectral sequence deduced of an appropriate filtration on the associated effective complex.

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First of all, the space X_5 and its effective equivalent object are filtered

```
>(change-chcm-to-fltrchcm x5 fbrt-flin `(fbrt-flin))
[K298 Filtered-Simplicial-Set]
>(change-chcm-to-fltrchcm (rbcc (efhm x5)) tnpr-flin
`(tnpr-flin))
[K594 Filtered-Chain-Complex]
```

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'(tnpr-flin))
[K594 Filtered-Chain-Complex]
```

Computation of some groups:

```
>(print-spct-sqn-cmpns x5 4 8 0)
Spectral sequence E^4_{8,0}
Component Z/4Z
>(print-spct-sqn-cmpns x5 4 4 3)
Spectral sequence E^4_{4,3}
Component Z/2Z
```

Computing spectral sequences

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The differential maps can also be obtained

```
>(spct-sqn-dffr x5 4 8 0 '(1))
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>(spct-sqn-cnvg-level x5 8)
9
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```
>(spct-sqn-cnvg-level x5 8)
9
```

And finally, we can determine the filtration of the homology groups

```
>(homology-fltr x5 6 5)
Filtration F_5 H_6
Component Z/2Z
>(homology-fltr x5 6 6)
Filtration F_6 H_6
Component Z/6Z
```