A computational review of spectral sequences and applications

Ana Romero
Universidad de La Rioja (Spain)

XVI Encuentro de Álgebra Computacional y Aplicaciones
Zaragoza, 4 July 2018

Introduction

Why did I decide to talk about spectral sequences?

Why did I decide to talk about spectral sequences?

- They are a useful tool that can been used in many different situations.

Introduction

Why did I decide to talk about spectral sequences?

- They are a useful tool that can been used in many different situations.
- Many people do not know them

Why did I decide to talk about spectral sequences?

- They are a useful tool that can been used in many different situations.
- Many people do not know them or are afraid of using them.

Introduction

Why did I decide to talk about spectral sequences?

- They are a useful tool that can been used in many different situations.
- Many people do not know them or are afraid of using them.
- Many people think they are not a computational tool.

Introduction

Why did I decide to talk about spectral sequences?

- They are a useful tool that can been used in many different situations.
- Many people do not know them or are afraid of using them.
- Many people think they are not a computational tool.
- I have worked with spectral sequences for a long time...

Introduction

Introduction

Consider the **chain complex**

$$C_* : \quad \cdots \leftarrow C_{n-1} \xleftarrow{d_n} C_n \xleftarrow{d_{n+1}} C_{n+1} \leftarrow \cdots$$

Introduction

Consider the **chain complex**

$$C_* : \quad \cdots \leftarrow C_{n-1} \xleftarrow{d_n} C_n \xleftarrow{d_{n+1}} C_{n+1} \leftarrow \cdots$$

The *n*-**homology group** of C_* is defined as

$$H_n(C_*) := \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}$$

Introduction

Consider the **chain complex**

$$C_* : \quad \cdots \leftarrow C_{n-1} \xleftarrow{d_n} C_n \xleftarrow{d_{n+1}} C_{n+1} \leftarrow \cdots$$

The *n*-**homology group** of C_* is defined as

$$H_n(C_*) := \frac{\text{Ker } d_n}{\text{Im } d_{n+1}}$$

Given a simplicial set X, a chain complex $C_*(X)$ can be constructed such that the homology groups of X are defined as

$$H_n(X) := H_n(C_*(X))$$

Definition of spectral sequence

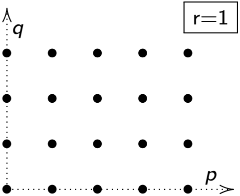
Definition of spectral sequence

Definition

A *spectral sequence* $E = (E^r, d^r)_{r \geq 1}$ is a family of bigraded $\mathbb{Z}$-modules $E^r = \{E^r_{p,q}\}$, each provided with a differential $d^r = \{d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) = \operatorname{Ker} d^r / \operatorname{Im} d^r \cong E^{r+1}$ for every $r \geq 1$.

Definition

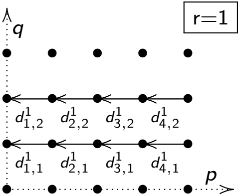
A *spectral sequence* $E = (E^r, d^r)_{r\geq 1}$ is a family of bigraded $\mathbb{Z}$-modules $E^r = \{E^r_{p,q}\}$, each provided with a differential $d^r = \{d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) = \operatorname{Ker} d^r / \operatorname{Im} d^r \cong E^{r+1}$ for every $r \geq 1$.



Definition of spectral sequence

Definition

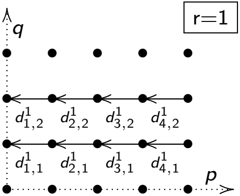
A *spectral sequence* $E = (E^r, d^r)_{r\geq 1}$ is a family of bigraded $\mathbb{Z}$-modules $E^r = \{E^r_{p,q}\}$, each provided with a differential $d^r = \{d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) = \operatorname{Ker} d^r / \operatorname{Im} d^r \cong E^{r+1}$ for every $r \geq 1$.

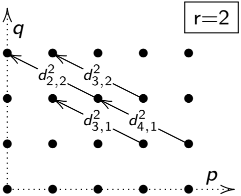


Definition of spectral sequence

Definition

A *spectral sequence* $E = (E^r, d^r)_{r \geq 1}$ is a family of bigraded $\mathbb{Z}$-modules $E^r = \{E^r_{p,q}\}$, each provided with a differential $d^r = \{d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) = \operatorname{Ker} d^r / \operatorname{Im} d^r \cong E^{r+1}$ for every $r \geq 1$.

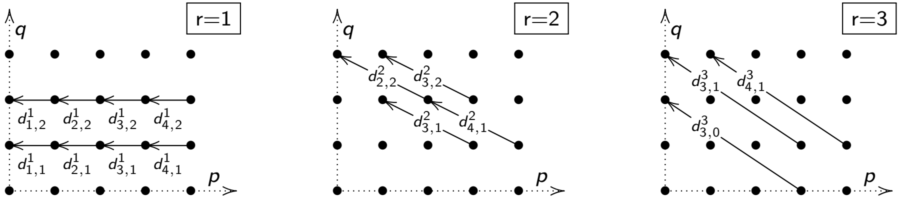




Definition of spectral sequence

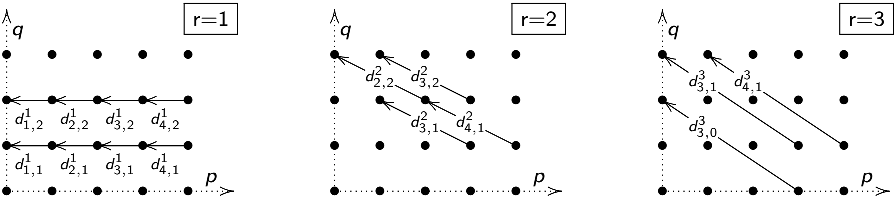
Definition

A *spectral sequence* $E = (E^r, d^r)_{r\geq 1}$ is a family of bigraded $\mathbb{Z}$-modules $E^r = \{E^r_{p,q}\}$, each provided with a differential $d^r = \{d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) = \operatorname{Ker} d^r / \operatorname{Im} d^r \cong E^{r+1}$ for every $r \geq 1$.



Definition

A *spectral sequence* $E = (E^r, d^r)_{r\geq 1}$ is a family of bigraded $\mathbb{Z}$-modules $E^r = \{E^r_{p,q}\}$, each provided with a differential $d^r = \{d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) = \operatorname{Ker} d^r / \operatorname{Im} d^r \cong E^{r+1}$ for every $r \geq 1$.

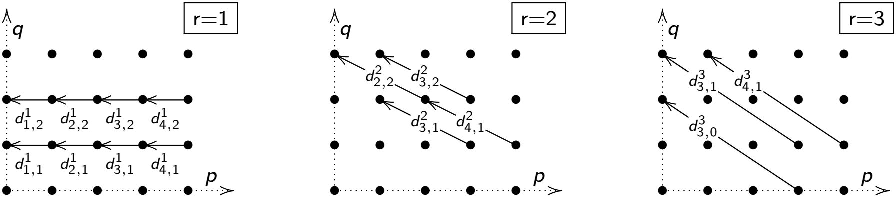


Since $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$ for each $r \geq 1$, one can define the *final groups* of the spectral sequence as $E^\infty_{p,q} = \bigcap_{r\geq 1} E^r_{p,q}$.

Definition of spectral sequence

Definition

A *spectral sequence* $E = (E^r, d^r)_{r\geq 1}$ is a family of bigraded $\mathbb{Z}$-modules $E^r = \{E^r_{p,q}\}$, each provided with a differential $d^r = \{d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) = \operatorname{Ker} d^r / \operatorname{Im} d^r \cong E^{r+1}$ for every $r \geq 1$.



Since $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$ for each $r \geq 1$, one can define the *final groups* of the spectral sequence as $E^\infty_{p,q} = \bigcap_{r\geq 1} E^r_{p,q}$. Under good conditions (very frequently), the spectral sequence *stabilizes*.

Why are spectral sequences useful?

Why are spectral sequences useful?

They usually *converge* to *interesting* things (frequently homology or homotopy groups).

Why are spectral sequences useful?

They usually *converge* to *interesting* things (frequently homology or homotopy groups).

Definition

Let $H_* = \{H_n\}_{n\in\mathbb{N}}$ be a graded group. A spectral sequence $(E^r, d^r)_{r\geq 1}$ is said to *converge* to H_* (denoted by $E^1 \Rightarrow H_*$) if there is a filtration F of H_* and for each (p, q) one has an isomorphism $E^\infty_{p,q} \cong \frac{F_p H_{p+q}}{F_{p-1} H_{p+q}}$

Why are spectral sequences useful?

They usually *converge* to *interesting* things (frequently homology or homotopy groups).

Definition

Let $H_* = \{H_n\}_{n\in\mathbb{N}}$ be a graded group. A spectral sequence $(E^r, d^r)_{r\geq 1}$ is said to *converge* to H_* (denoted by $E^1 \Rightarrow H_*$) if there is a filtration F of H_* and for each (p, q) one has an isomorphism $E^\infty_{p,q} \cong \frac{F_p H_{p+q}}{F_{p-1} H_{p+q}}$

Examples:

Why are spectral sequences useful?

They usually *converge* to *interesting* things (frequently homology or homotopy groups).

Definition

Let $H_* = \{H_n\}_{n\in\mathbb{N}}$ be a graded group. A spectral sequence $(E^r, d^r)_{r\geq 1}$ is said to *converge* to H_* (denoted by $E^1 \Rightarrow H_*$) if there is a filtration F of H_* and for each (p,q) one has an isomorphism $E^\infty_{p,q} \cong \frac{F_p H_{p+q}}{F_{p-1}H_{p+q}}$

Examples:

- The *Serre spectral sequence* converges to the homology groups of the total space of a fibration.

Why are spectral sequences useful?

They usually *converge* to *interesting* things (frequently homology or homotopy groups).

Definition

Let $H_* = \{H_n\}_{n\in\mathbb{N}}$ be a graded group. A spectral sequence $(E^r, d^r)_{r\geq 1}$ is said to *converge* to H_* (denoted by $E^1 \Rightarrow H_*$) if there is a filtration F of H_* and for each (p, q) one has an isomorphism $E^\infty_{p,q} \cong \frac{F_p H_{p+q}}{F_{p-1} H_{p+q}}$

Examples:

- The *Serre spectral sequence* converges to the homology groups of the total space of a fibration.
- The *Eilenberg-Moore spectral sequence* converges to the homology groups of the loop space of a simplicial set.

Why are spectral sequences useful?

They usually *converge* to *interesting* things (frequently homology or homotopy groups).

Definition

Let $H_* = \{H_n\}_{n\in\mathbb{N}}$ be a graded group. A spectral sequence $(E^r, d^r)_{r\geq 1}$ is said to *converge* to H_* (denoted by $E^1 \Rightarrow H_*$) if there is a filtration F of H_* and for each (p,q) one has an isomorphism $E^\infty_{p,q} \cong \frac{F_p H_{p+q}}{F_{p-1}H_{p+q}}$

Examples:

- The *Serre spectral sequence* converges to the homology groups of the total space of a fibration.
- The *Eilenberg-Moore spectral sequence* converges to the homology groups of the loop space of a simplicial set.
- The *Adams spectral sequence* converges to the homotopy groups of a simplicial set X.

Why are spectral sequences useful?

Why are spectral sequences useful?

Theorem (Serre spectral sequence)

Let $G \hookrightarrow E \to B$ be a fibration with a simply connected base space B. Then a first quadrant spectral sequence $E = (E^r, d^r)_{r \geq 1}$ can be defined with $E^2_{p,q} = H_p(B, H_q(G))$ and $E^1 \Rightarrow H_*(E)$.

Why are spectral sequences useful?

Theorem (Serre spectral sequence)

Let $G \hookrightarrow E \to B$ be a fibration with a simply connected base space B. Then a first quadrant spectral sequence $E = (E^r, d^r)_{r \geq 1}$ can be defined with $E^2_{p,q} = H_p(B, H_q(G))$ and $E^1 \Rightarrow H_*(E)$.

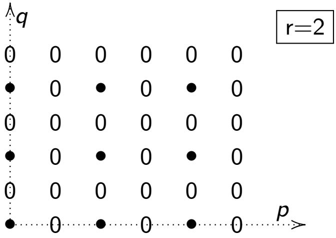
Suppose $H_i(G)$ and $H_i(B)$ are zero for odd i and free abelian for even i. The entries $E^2_{p,q}$ of the E^2 page are then zero unless p and q are even.

Why are spectral sequences useful?

Theorem (Serre spectral sequence)

Let $G \hookrightarrow E \to B$ be a fibration with a simply connected base space B. Then a first quadrant spectral sequence $E = (E^r, d^r)_{r \geq 1}$ can be defined with $E^2_{p,q} = H_p(B, H_q(G))$ and $E^1 \Rightarrow H_*(E)$.

Suppose $H_i(G)$ and $H_i(B)$ are zero for odd i and free abelian for even i. The entries $E^2_{p,q}$ of the E^2 page are then zero unless p and q are even.

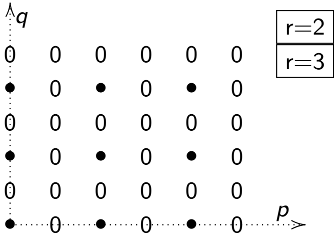


Why are spectral sequences useful?

Theorem (Serre spectral sequence)

Let $G \hookrightarrow E \to B$ be a fibration with a simply connected base space B. Then a first quadrant spectral sequence $E = (E^r, d^r)_{r \geq 1}$ can be defined with $E^2_{p,q} = H_p(B, H_q(G))$ and $E^1 \Rightarrow H_*(E)$.

Suppose $H_i(G)$ and $H_i(B)$ are zero for odd i and free abelian for even i. The entries $E^2_{p,q}$ of the E^2 page are then zero unless p and q are even.

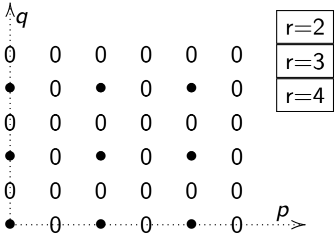


Why are spectral sequences useful?

Theorem (Serre spectral sequence)

Let $G \hookrightarrow E \to B$ be a fibration with a simply connected base space B. Then a first quadrant spectral sequence $E = (E^r, d^r)_{r \geq 1}$ can be defined with $E^2_{p,q} = H_p(B, H_q(G))$ and $E^1 \Rightarrow H_*(E)$.

Suppose $H_i(G)$ and $H_i(B)$ are zero for odd i and free abelian for even i. The entries $E^2_{p,q}$ of the E^2 page are then zero unless p and q are even.

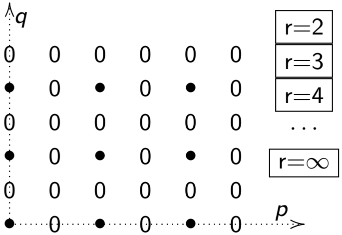


Why are spectral sequences useful?

Theorem (Serre spectral sequence)

Let $G \hookrightarrow E \to B$ be a fibration with a simply connected base space B. Then a first quadrant spectral sequence $E = (E^r, d^r)_{r \geq 1}$ can be defined with $E^2_{p,q} = H_p(B, H_q(G))$ and $E^1 \Rightarrow H_*(E)$.

Suppose $H_i(G)$ and $H_i(B)$ are zero for odd i and free abelian for even i. The entries $E^2_{p,q}$ of the E^2 page are then zero unless p and q are even.



Problems of spectral sequences

Problems of spectral sequences

- The problem of differentials

Problems of spectral sequences

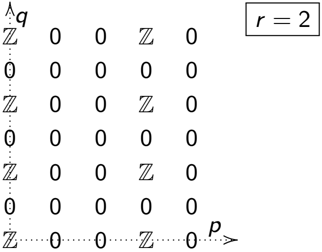
- The problem of differentials

  Example: $K(\mathbb{Z},2) \hookrightarrow X \to S^3$, where $H_{2i}(K(\mathbb{Z},2)) = \mathbb{Z}$ and $H_{2i+1}(K(\mathbb{Z},2)) = 0$ for all i.

Problems of spectral sequences

- The problem of differentials

  Example: $K(\mathbb{Z},2) \hookrightarrow X \to S^3$, where $H_{2i}(K(\mathbb{Z},2)) = \mathbb{Z}$ and $H_{2i+1}(K(\mathbb{Z},2)) = 0$ for all i.



Problems of spectral sequences

- The problem of differentials

Example: $K(\mathbb{Z},2) \hookrightarrow X \to S^3$, where $H_{2i}(K(\mathbb{Z},2)) = \mathbb{Z}$ and $H_{2i+1}(K(\mathbb{Z},2)) = 0$ for all i.

q

$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0

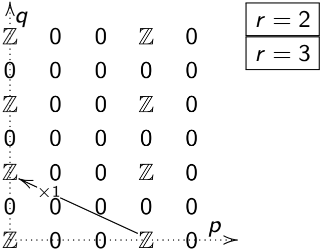
p

$r = 2$

$r = 3$

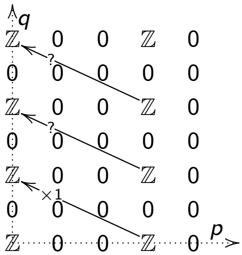
- The problem of differentials

Example: $K(\mathbb{Z},2) \hookrightarrow X \to S^3$, where $H_{2i}(K(\mathbb{Z},2)) = \mathbb{Z}$ and $H_{2i+1}(K(\mathbb{Z},2)) = 0$ for all i.



- The problem of differentials

Example: $K(\mathbb{Z},2) \hookrightarrow X \to S^3$, where $H_{2i}(K(\mathbb{Z},2)) = \mathbb{Z}$ and $H_{2i+1}(K(\mathbb{Z},2)) = 0$ for all i.

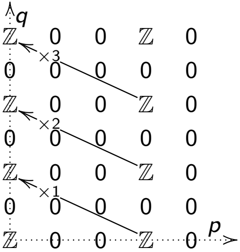


$r = 2$

$r = 3$

- The problem of differentials

Example: $K(\mathbb{Z},2) \hookrightarrow X \to S^3$, where $H_{2i}(K(\mathbb{Z},2)) = \mathbb{Z}$ and $H_{2i+1}(K(\mathbb{Z},2)) = 0$ for all i.

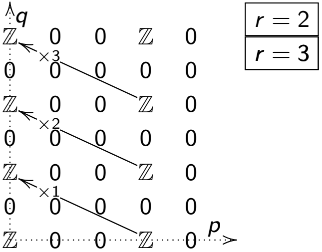


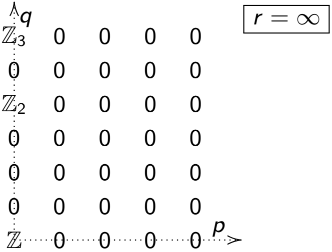
$r = 2$

$r = 3$

- The problem of differentials

Example: $K(\mathbb{Z},2) \hookrightarrow X \to S^3$, where $H_{2i}(K(\mathbb{Z},2)) = \mathbb{Z}$ and $H_{2i+1}(K(\mathbb{Z},2)) = 0$ for all i.





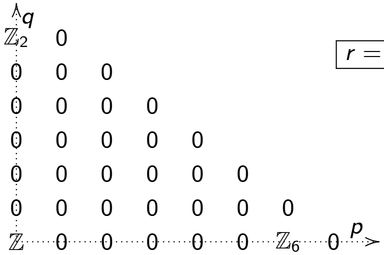
Problems of spectral sequences

Problems of spectral sequences

- The extension problem

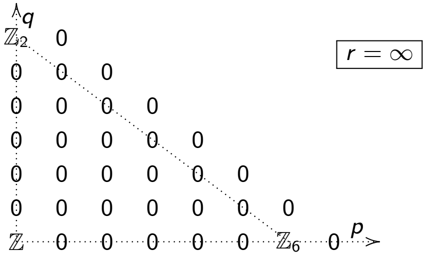
Problems of spectral sequences

- The extension problem

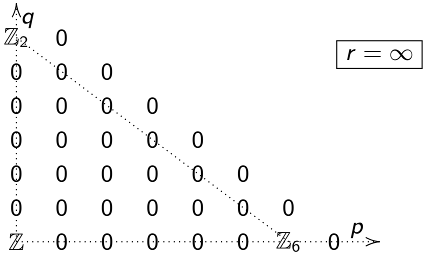


$r = \infty$

- The extension problem



- The extension problem



We obtain a short exact sequence:

$$0 \leftarrow \mathbb{Z}_6 \leftarrow H_6(E) \leftarrow \mathbb{Z}_2 \leftarrow 0$$

but now there are two possible extensions: the trivial one $\mathbb{Z}_2 \oplus \mathbb{Z}_6$ and the twisted one $\mathbb{Z}_{12}$.

Spectral sequences of filtered complexes

Spectral sequences of filtered complexes

Definition

An *increasing filtration* F of a chain complex $C_* = (C_n, d_n)_{n \in \mathbb{N}}$ is a family of sub-chain complexes $\ldots \subseteq F_{p-1}C_* \subseteq F_pC_* \subseteq F_{p+1}C_* \subseteq \ldots$

Spectral sequences of filtered complexes

Definition

An *increasing filtration* F of a chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ is a family of sub-chain complexes $\ldots \subseteq F_{p-1}C_* \subseteq F_pC_* \subseteq F_{p+1}C_* \subseteq \ldots$

Theorem

Let C_* be a chain complex with a filtration. There exists a spectral sequence with

$$E^r_{p,q} = \frac{Z^r_{p,q} + F_{p-1}C_{p+q}}{d_{p+q+1}(Z^{r-1}_{p+r-1,q-r+2}) + F_{p-1}C_{p+q}}$$

where $Z^r_{p,q}$ is $Z^r_{p,q} = \{a \in F_pC_{p+q} | \ d_{p+q}(a) \in F_{p-r}C_{p+q-1}\} \subseteq F_pC_{p+q}$, and $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is the morphism induced by $d_{p+q} : C_{p+q} \to C_{p+q-1}$. This spectral sequence converges to $H_*(C)$.

S. MacLane. *Homology*. Springer, 1963.

Programs computing spectral seq. of filtered complexes

Programs computing spectral seq. of filtered complexes

Remark: when the initial chain complex is of finite type, the groups $E^r_{p,q}$ (of all levels!) can be determined by means of diagonalization algorithms on some matrices

Remark: when the initial chain complex is of finite type, the groups $E^r_{p,q}$ (of all levels!) can be determined by means of diagonalization algorithms on some matrices without knowing the differential maps.

Programs computing spectral seq. of filtered complexes

Remark: when the initial chain complex is of finite type, the groups $E^r_{p,q}$ (of all levels!) can be determined by means of diagonalization algorithms on some matrices without knowing the differential maps.

- First implementation: new module for the Kenzo system.

A. R., J. Rubio, F. Sergeraert. *Computing spectral sequences*. Journal of Symbolic Computation 41 (10), 1059–1079, 2006.

Programs computing spectral seq. of filtered complexes

Remark: when the initial chain complex is of finite type, the groups $E^r_{p,q}$ (of all levels!) can be determined by means of diagonalization algorithms on some matrices without knowing the differential maps.

- First implementation: new module for the Kenzo system.

  A. R., J. Rubio, F. Sergeraert. *Computing spectral sequences.* Journal of Symbolic Computation 41 (10), 1059–1079, 2006.

- Other implementations (in GAP, for finite type complexes):

Programs computing spectral seq. of filtered complexes

Remark: when the initial chain complex is of finite type, the groups $E^r_{p,q}$ (of all levels!) can be determined by means of diagonalization algorithms on some matrices without knowing the differential maps.

- First implementation: new module for the Kenzo system.

  A. R., J. Rubio, F. Sergeraert. *Computing spectral sequences*. Journal of Symbolic Computation 41 (10), 1059–1079, 2006.

- Other implementations (in GAP, for finite type complexes):

  M. Barakat. *Spectral Filtrations via Generalized Morphisms*. Preprint, 2009. `https://arxiv.org/abs/0904.0240`

Programs computing spectral seq. of filtered complexes

Remark: when the initial chain complex is of finite type, the groups $E^r_{p,q}$ (of all levels!) can be determined by means of diagonalization algorithms on some matrices without knowing the differential maps.

- First implementation: new module for the Kenzo system.

A. R., J. Rubio, F. Sergeraert. *Computing spectral sequences*. Journal of Symbolic Computation 41 (10), 1059–1079, 2006.

- Other implementations (in GAP, for finite type complexes):

M. Barakat. *Spectral Filtrations via Generalized Morphisms*. Preprint, 2009. https://arxiv.org/abs/0904.0240

G. Ellis, P. Smith. *Computing group cohomology rings from the Lyndon-Hochschild-Serre spectral sequence*. Journal of Symbolic Computation 46 (4), 360–370, 2011.

Programs computing spectral seq. of filtered complexes

Remark: when the filtered chain complex is not of finite type, the groups $E^r_{p,q}$ and the differential maps $d^r_{p,q}$ cannot be *directly* determined.

Remark: when the filtered chain complex is not of finite type, the groups $E^r_{p,q}$ and the differential maps $d^r_{p,q}$ cannot be *directly* determined.

By using the *effective homology* theory, implemented in the Kenzo system, it is also possible to determine spectral sequences of chain complexes which are not of finite type.

Programs computing spectral seq. of filtered complexes

Programs computing spectral seq. of filtered complexes

Our programs work in a similar way to the method that Kenzo uses to determine homology groups of a given chain complex:

Programs computing spectral seq. of filtered complexes

Our programs work in a similar way to the method that Kenzo uses to determine homology groups of a given chain complex:

- If a filtered complex C_* is of finite type, its spectral sequence can be determined by means of diagonalization algorithms on some matrices.
- Otherwise, a pair of *reductions* $C_* \Lleftarrow \hat{C}_* \Rrightarrow D_*$ from the initial chain complex C_* to another one D_* of finite type (also filtered) is constructed, such that (thanks to some theoretical results) the spectral sequences of C_* and D_* are isomorphic after some level.

Programs computing spectral seq. of filtered complexes

Our programs work in a similar way to the method that Kenzo uses to determine homology groups of a given chain complex:

- If a filtered complex C_* is of finite type, its spectral sequence can be determined by means of diagonalization algorithms on some matrices.
- Otherwise, a pair of *reductions* $C_* \Lleftarrow \hat{C}_* \Rrightarrow D_*$ from the initial chain complex C_* to another one D_* of finite type (also filtered) is constructed, such that (thanks to some theoretical results) the spectral sequences of C_* and D_* are isomorphic after some level. The pair of reductions $C_* \Lleftarrow \hat{C}_* \Rrightarrow D_*$ is called the *effective homology* of C_*

Programs computing spectral seq. of filtered complexes

Our programs work in a similar way to the method that Kenzo uses to determine homology groups of a given chain complex:

- If a filtered complex C_* is of finite type, its spectral sequence can be determined by means of diagonalization algorithms on some matrices.
- Otherwise, a pair of *reductions* $C_* \twoheadleftarrow \hat{C}_* \twoheadrightarrow D_*$ from the initial chain complex C_* to another one D_* of finite type (also filtered) is constructed, such that (thanks to some theoretical results) the spectral sequences of C_* and D_* are isomorphic after some level. The pair of reductions $C_* \twoheadleftarrow \hat{C}_* \twoheadrightarrow D_*$ is called the *effective homology* of C_* and D_* is said to be *effective*.

Our programs work in a similar way to the method that Kenzo uses to determine homology groups of a given chain complex:

- If a filtered complex C_* is of finite type, its spectral sequence can be determined by means of diagonalization algorithms on some matrices.
- Otherwise, a pair of *reductions* $C_* \twoheadleftarrow \hat{C}_* \twoheadrightarrow D_*$ from the initial chain complex C_* to another one D_* of finite type (also filtered) is constructed, such that (thanks to some theoretical results) the spectral sequences of C_* and D_* are isomorphic after some level. The pair of reductions $C_* \twoheadleftarrow \hat{C}_* \twoheadrightarrow D_*$ is called the *effective homology* of C_* and D_* is said to be *effective*.

Our programs determine the groups $E^r_{p,q}$'s and the differential maps $d^r_{p,q}$'s for every level r.

Our programs work in a similar way to the method that Kenzo uses to determine homology groups of a given chain complex:

- If a filtered complex C_* is of finite type, its spectral sequence can be determined by means of diagonalization algorithms on some matrices.
- Otherwise, a pair of *reductions* $C_* \twoheadleftarrow \hat{C}_* \twoheadrightarrow D_*$ from the initial chain complex C_* to another one D_* of finite type (also filtered) is constructed, such that (thanks to some theoretical results) the spectral sequences of C_* and D_* are isomorphic after some level. The pair of reductions $C_* \twoheadleftarrow \hat{C}_* \twoheadrightarrow D_*$ is called the *effective homology* of C_* and D_* is said to be *effective*.

Our programs determine the groups $E^r_{p,q}$'s and the differential maps $d^r_{p,q}$'s for every level r.

A. R., J. Rubio, F. Sergeraert. *Computing spectral sequences*. Journal of Symbolic Computation 41 (10), 1059–1079, 2006.

Programs computing spectral seq. of filtered complexes

Programs computing spectral seq. of filtered complexes

Given a chain complex C_*, how is it possible to determine its effective homology?

Programs computing spectral seq. of filtered complexes

Given a chain complex C_*, how is it possible to determine its effective homology?

- If a chain complex C_* is effective, then we can choose the trivial effective homology $C_* \Lleftarrow C_* \Rrightarrow C_*$.

Given a chain complex C_*, how is it possible to determine its effective homology?

- If a chain complex C_* is effective, then we can choose the trivial effective homology $C_* \Lleftarrow C_* \Rrightarrow C_*$.

- In some cases, some theoretical results are available providing an equivalence between some chain complex C_* and an *effective* chain complex. Example: $C_*(K(\mathbb{Z},1)) \Rrightarrow C_*(S^1)$.

Programs computing spectral seq. of filtered complexes

Given a chain complex C_*, how is it possible to determine its effective homology?

- If a chain complex C_* is effective, then we can choose the trivial effective homology $C_* \Lleftarrow C_* \Rrightarrow C_*$.

- In some cases, some theoretical results are available providing an equivalence between some chain complex C_* and an *effective* chain complex. Example: $C_*(K(\mathbb{Z},1)) \Rrightarrow C_*(S^1)$.

- Given some topological spaces $X_1, \ldots, X_n$ and a topological constructor Φ which produces a new topological space X; if effective homology versions of the spaces $X_1, \ldots, X_n$ are known, then it should be possible to construct the effective homology of the space X.

Given a chain complex C_*, how is it possible to determine its effective homology?

- If a chain complex C_* is effective, then we can choose the trivial effective homology $C_* \Lleftarrow C_* \Rrightarrow C_*$.
- In some cases, some theoretical results are available providing an equivalence between some chain complex C_* and an *effective* chain complex. Example: $C_*(K(\mathbb{Z},1)) \Rrightarrow C_*(S^1)$.
- Given some topological spaces $X_1, \ldots, X_n$ and a topological constructor Φ which produces a new topological space X; if effective homology versions of the spaces $X_1, \ldots, X_n$ are known, then it should be possible to construct the effective homology of the space X.

  Particular algorithms are needed for all different constructors.

Programs computing spectral seq. of filtered complexes

Given a chain complex C_*, how is it possible to determine its effective homology?

- If a chain complex C_* is effective, then we can choose the trivial effective homology $C_* \Lleftarrow C_* \Rrightarrow C_*$.

- In some cases, some theoretical results are available providing an equivalence between some chain complex C_* and an *effective* chain complex. Example: $C_*(K(\mathbb{Z},1)) \Rrightarrow C_*(S^1)$.

- Given some topological spaces $X_1, \ldots, X_n$ and a topological constructor Φ which produces a new topological space X; if effective homology versions of the spaces $X_1, \ldots, X_n$ are known, then it should be possible to construct the effective homology of the space X.

  Particular algorithms are needed for all different constructors. Examples: total space of a fibration, loop space of a simplicial set...

Applications

Applications of programs computing spectral sequences of filtered complexes:

- We can compute the spectral sequence associated with a bicomplex.

Applications

Applications of programs computing spectral sequences of filtered complexes:

- We can compute the spectral sequence associated with a bicomplex.
- We can compute the classical spectral sequences of Serre and Eilenberg-Moore,

Applications

Applications of programs computing spectral sequences of filtered complexes:

- We can compute the spectral sequence associated with a bicomplex.
- We can compute the classical spectral sequences of Serre and Eilenberg-Moore, defined by means of filtered complexes,

Applications

Applications of programs computing spectral sequences of filtered complexes:

- We can compute the spectral sequence associated with a bicomplex.
- We can compute the classical spectral sequences of Serre and Eilenberg-Moore, defined by means of filtered complexes, even when the spaces are not of finite type and (some) differential maps cannot be easily deduced.

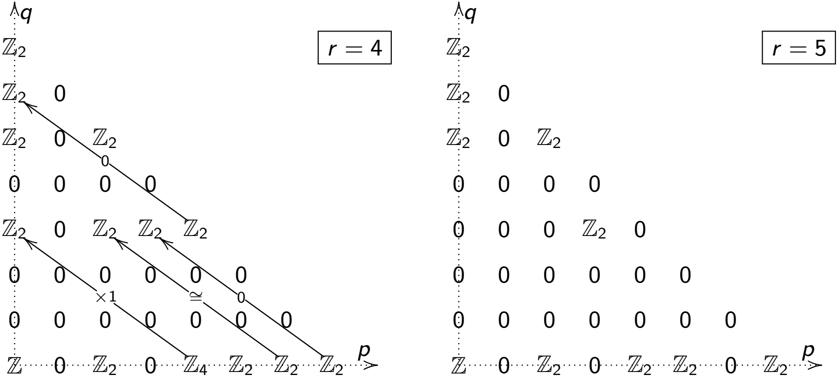
Applications: Serre spectral sequence

Applications: Serre spectral sequence

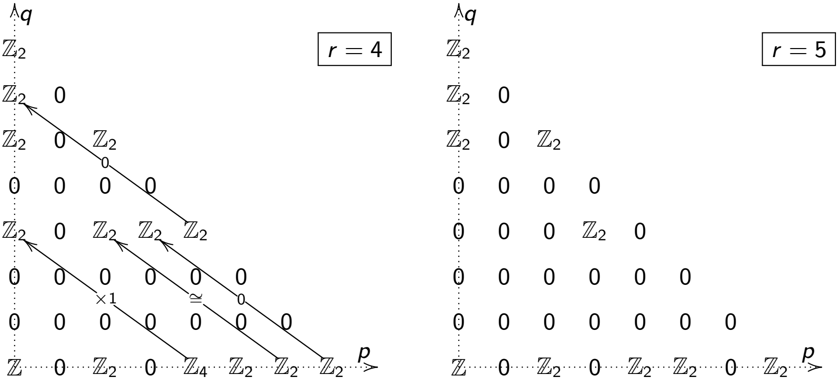
One can compute the Serre spectral sequence associated with $X_3 = K(\mathbb{Z}_2, 3) \times_{k_3} K(\mathbb{Z}_2, 2)$, total space of a fibration $K(\mathbb{Z}_2, 3) \hookrightarrow X_3 \to K(\mathbb{Z}_2, 2)$.

Applications: Serre spectral sequence

One can compute the Serre spectral sequence associated with $X_3 = K(\mathbb{Z}_2, 3) \times_{k_3} K(\mathbb{Z}_2, 2)$, total space of a fibration $K(\mathbb{Z}_2, 3) \hookrightarrow X_3 \to K(\mathbb{Z}_2, 2)$.



One can compute the Serre spectral sequence associated with $X_3 = K(\mathbb{Z}_2, 3) \times_{k_3} K(\mathbb{Z}_2, 2)$, total space of a fibration $K(\mathbb{Z}_2, 3) \hookrightarrow X_3 \to K(\mathbb{Z}_2, 2)$.



The object $X3$ is already of finite type, but its effective homology gives us an associated effective chain complex which is much smaller.

Applications: homotopy of suspended classifying spaces

The homotopy groups of suspended classifying spaces $\Sigma K(G,1)$ can be computed by means of the Serre spectral sequence associated with some fibrations involved in the Postnikov tower of these spaces, or directly using the effective homology of these fibrations.

Applications: homotopy of suspended classifying spaces

The homotopy groups of suspended classifying spaces $\Sigma K(G, 1)$ can be computed by means of the Serre spectral sequence associated with some fibrations involved in the Postnikov tower of these spaces, or directly using the effective homology of these fibrations.

Our algorithms have made it possible to determine the homotopy groups of spaces $\Sigma K(G, 1)$ for different groups G, and our calculations have found an error in the paper

R. Mikhailov, J. Wu. *On homotopy groups of the suspended classifying spaces*. Algebraic and Geometric Topology 10, 565-625, 2010.

Applications: homotopy of suspended classifying spaces

The homotopy groups of suspended classifying spaces $\Sigma K(G, 1)$ can be computed by means of the Serre spectral sequence associated with some fibrations involved in the Postnikov tower of these spaces, or directly using the effective homology of these fibrations.

Our algorithms have made it possible to determine the homotopy groups of spaces $\Sigma K(G, 1)$ for different groups G, and our calculations have found an error in the paper

R. Mikhailov, J. Wu. *On homotopy groups of the suspended classifying spaces*. Algebraic and Geometric Topology 10, 565-625, 2010.

Mikhailov and Wu say that:

> *Theorem 5.4*: *Let A_4 be the 4-th alternating group.*
> *Then $\pi_4(\Sigma K(A_4, 1)) = \mathbb{Z}_4$*

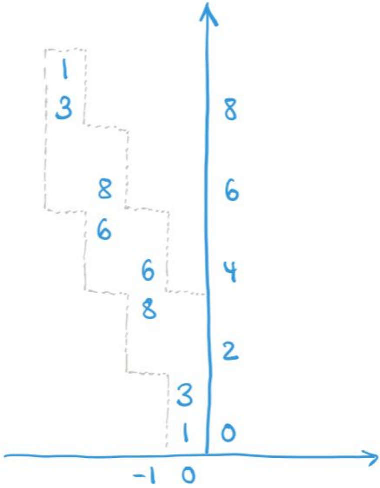
but we have obtained $\pi_4(\Sigma K(A_4, 1)) = \mathbb{Z}_{12}$

Applications: homology of groups

```
From: Markus Szymik
jue 26/01/2017 22:14
To: Ana Romero

Dear Ana,

I hope this finds you well.

I am interested in computing homology with computers...

Let's take two elementary abelian 2-groups A = (Z/2)^a and
B = (Z/2)^b and a map K( A, 1 ) ---> K( B, 2 )
...
and I would like to see the Eilenberg--Moore spectral
sequence for the fiber of this map...
```

Private communication with Prof. Markus Szymik, NTNU Norwegian University of Science and Technology, Trondheim.

Applications: homology of groups



Applications: homology of finite topological spaces

Ph.D. of Julián Cuevas Rozo, started in November 2017 (supervised by L. Lambán and A. R.):

"Effective computation of invariants of finite topological spaces"

Applications: homology of finite topological spaces

Ph.D. of Julián Cuevas Rozo, started in November 2017 (supervised by L. Lambán and A. R.):

"Effective computation of invariants of finite topological spaces"

Several techniques:

Applications: homology of finite topological spaces

Ph.D. of Julián Cuevas Rozo, started in November 2017 (supervised by L. Lambán and A. R.):

"Effective computation of invariants of finite topological spaces"

Several techniques:

- Beat and weak points

Applications: homology of finite topological spaces

Ph.D. of Julián Cuevas Rozo, started in November 2017 (supervised by L. Lambán and A. R.):

"Effective computation of invariants of finite topological spaces"

Several techniques:

- Beat and weak points
- Discrete vector fields

Applications: homology of finite topological spaces

Ph.D. of Julián Cuevas Rozo, started in November 2017 (supervised by L. Lambán and A. R.):

"Effective computation of invariants of finite topological spaces"

Several techniques:

- Beat and weak points
- Discrete vector fields
- Spectral sequence of a filtered complex associated with a poset defined in

N. Cianci, M. Ottina. *On homology of finite topological spaces*. Topology and its Applications 217, 1-19, 2017.

Bousfield-Kan spectral sequence

Bousfield-Kan spectral sequence

Associated with a simplicial set X,

Bousfield-Kan spectral sequence

Associated with a simplicial set X, not defined by means of a filtered complex,

Bousfield-Kan spectral sequence

Associated with a simplicial set X, not defined by means of a filtered complex, under good conditions converges to $\pi_*(X)$.

Bousfield-Kan spectral sequence

Associated with a simplicial set X, not defined by means of a filtered complex, under good conditions converges to $\pi_*(X)$.

Its definition is much more complicated and involves different mathematical structures such as

- Cosimplicial spaces
- Loop spaces
- Inverse limits
- Towers of fibrations

Bousfield-Kan spectral sequence

Associated with a simplicial set X, not defined by means of a filtered complex, under good conditions converges to $\pi_*(X)$.

Its definition is much more complicated and involves different mathematical structures such as

- Cosimplicial spaces
- Loop spaces
- Inverse limits
- Towers of fibrations

There does not exist a formal expression for the groups $E^r_{p,q}$'s as in the case of the spectral sequence associated with a filtered complex.

Bousfield-Kan spectral sequence

We have developed an algorithm computing all the components of the Bousfield-Kan spectral sequence.

A. R., F. Sergeraert. *A Bousfield-Kan Algorithm for Computing the Effective Homotopy of a Space*. Foundations of Computational Mathematics 17(5), 1335-1366, 2017.

Bousfield-Kan spectral sequence

We have developed an algorithm computing all the components of the Bousfield-Kan spectral sequence.

A. R., F. Sergeraert. *A Bousfield-Kan Algorithm for Computing the Effective Homotopy of a Space*. Foundations of Computational Mathematics 17(5), 1335-1366, 2017.

To this aim, we have developed first an *effective homotopy theory*.

A. R., F. Sergeraert. *Effective homotopy of fibrations*. Appl. Algebra Eng. Commun. Comput. 23(1-2), 85-100, 2012.

Bousfield-Kan spectral sequence

We have developed an algorithm computing all the components of the Bousfield-Kan spectral sequence.

A. R., F. Sergeraert. *A Bousfield-Kan Algorithm for Computing the Effective Homotopy of a Space*. Foundations of Computational Mathematics 17(5), 1335-1366, 2017.

To this aim, we have developed first an *effective homotopy theory*.

A. R., F. Sergeraert. *Effective homotopy of fibrations*. Appl. Algebra Eng. Commun. Comput. 23(1-2), 85-100, 2012.

Other *ingredients* have been necessary.

A. R. *Computing the first stages of the Bousfield-Kan spectral sequence*. Appl. Algebra Eng. Commun. Comput. 21(3), 227-248, 2010.

A. R., F. Sergeraert. *Programming before theorizing, a case study*. Proceedings ISSAC 2012, 289-296.

A. R., F. Sergeraert: *A Combinatorial Tool for Computing the Effective Homotopy of Iterated Loop Spaces*. Discrete and Computational Geometry 53(1), 1-15, 2015.

Persistent homology

Let K be a filtered simplicial complex:

$$\emptyset = K^0 \subseteq K^1 \subseteq K^2 \subseteq \cdots \subseteq K^m = K$$

Persistent homology

Let K be a filtered simplicial complex:

$$\emptyset = K^0 \subseteq K^1 \subseteq K^2 \subseteq \cdots \subseteq K^m = K$$

The filtration produces a sequence of groups and homomorphisms:

$$0 = H_n(K^0) \to H_n(K^1) \to \cdots \to H_n(K^m) = H_n(K)$$

Persistent homology

Let K be a filtered simplicial complex:

$$\emptyset = K^0 \subseteq K^1 \subseteq K^2 \subseteq \cdots \subseteq K^m = K$$

The filtration produces a sequence of groups and homomorphisms:

$$0 = H_n(K^0) \to H_n(K^1) \to \cdots \to H_n(K^m) = H_n(K)$$

Definition

The n-th *persistent homology groups* of K are defined as $H_n^{i,j} = \mathrm{Im}(f_n^{i,j} : H_n(K^i) \to H_n(K^j))$, for $0 \leq i \leq j \leq m$.

Persistent homology

Let K be a filtered simplicial complex:

$$\emptyset = K^0 \subseteq K^1 \subseteq K^2 \subseteq \cdots \subseteq K^m = K$$

The filtration produces a sequence of groups and homomorphisms:

$$0 = H_n(K^0) \to H_n(K^1) \to \cdots \to H_n(K^m) = H_n(K)$$

Definition

The n-th *persistent homology groups* of K are defined as $H_n^{i,j} = \mathrm{Im}(f_n^{i,j} : H_n(K^i) \to H_n(K^j))$, for $0 \leq i \leq j \leq m$.

If γ *is born* at K^i and *dies* entering K^j, we denote $\mathrm{pers}(\gamma) = j - i$.

Persistent homology

Let K be a filtered simplicial complex:

$$\emptyset = K^0 \subseteq K^1 \subseteq K^2 \subseteq \cdots \subseteq K^m = K$$

The filtration produces a sequence of groups and homomorphisms:

$$0 = H_n(K^0) \to H_n(K^1) \to \cdots \to H_n(K^m) = H_n(K)$$

Definition

The n-th *persistent homology groups* of K are defined as $H_n^{i,j} = \mathrm{Im}(f_n^{i,j} : H_n(K^i) \to H_n(K^j))$, for $0 \leq i \leq j \leq m$.

If γ *is born* at K^i and *dies* entering K^j, we denote $\mathrm{pers}(\gamma) = j - i$. If the homology is computed with field coefficients, the groups $H_n^{i,j}$ are determined by their ranks, denoted $\beta_n^{i,j}$.

Persistent homology

Let K be a filtered simplicial complex:

$$\emptyset = K^0 \subseteq K^1 \subseteq K^2 \subseteq \cdots \subseteq K^m = K$$

The filtration produces a sequence of groups and homomorphisms:

$$0 = H_n(K^0) \to H_n(K^1) \to \cdots \to H_n(K^m) = H_n(K)$$

Definition

The n-th *persistent homology groups* of K are defined as $H_n^{i,j} = \mathrm{Im}(f_n^{i,j} : H_n(K^i) \to H_n(K^j))$, for $0 \leq i \leq j \leq m$.

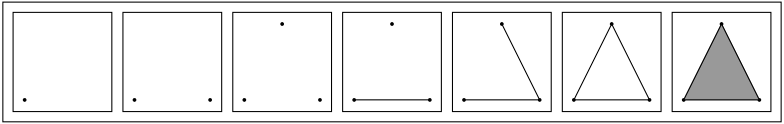
If γ *is born* at K^i and *dies* entering K^j, we denote $\mathrm{pers}(\gamma) = j - i$. If the homology is computed with field coefficients, the groups $H_n^{i,j}$ are determined by their ranks, denoted $\beta_n^{i,j}$. This allows one to represent all groups $H_n^{i,j}$ by means of a *barcode diagram*.

Persistent homology

Example:

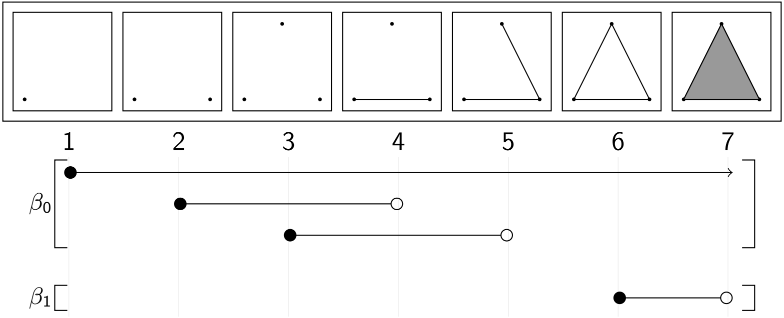
Persistent homology

Example:



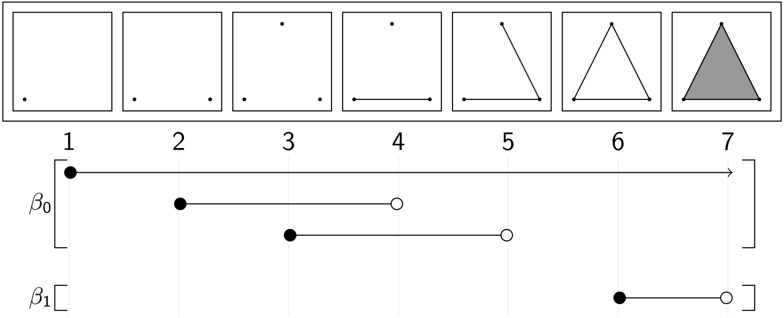
Persistent homology

Example:



Persistent homology

Example:



Roughly speaking:

- Homology detects topological features (connected components, holes, and so on).
- Persistent homology describes the evolution of topological features looking at the different steps of the filtration.

Programs computing persistent homology

Programs computing persistent homology

- The groups $H_n^{i,j}$ can also be expressed in terms of the subgroups involved in the spectral sequence:

$$H_n^{i,j} = \frac{Z^i_{i,n-i}}{d_{n+1}(Z^{j-i}_{j,n-j+1})}$$

- The groups $H_n^{i,j}$ can also be expressed in terms of the subgroups involved in the spectral sequence:

$$H_n^{i,j} = \frac{Z^i_{i,n-i}}{d_{n+1}(Z^{j-i}_{j,n-j+1})}$$

- In this way, it has been very easy to adapt our programs for spectral sequences in order to compute $H_n^{i,j}$ for finite (filtered) simplicial complexes.

Programs computing persistent homology

- The groups $H_n^{i,j}$ can also be expressed in terms of the subgroups involved in the spectral sequence:

$$H_n^{i,j} = \frac{Z_{i,n-i}^{i}}{d_{n+1}(Z_{j,n-j+1}^{j-i})}$$

- In this way, it has been very easy to adapt our programs for spectral sequences in order to compute $H_n^{i,j}$ for finite (filtered) simplicial complexes.

- Our programs are also valid in the integer case and this makes it possible to solve the possible extension problems.

Programs computing persistent homology

- The groups $H_n^{i,j}$ can also be expressed in terms of the subgroups involved in the spectral sequence:

$$H_n^{i,j} = \frac{Z^i_{i,n-i}}{d_{n+1}(Z^{j-i}_{j,n-j+1})}$$

- In this way, it has been very easy to adapt our programs for spectral sequences in order to compute $H_n^{i,j}$ for finite (filtered) simplicial complexes.

- Our programs are also valid in the integer case and this makes it possible to solve the possible extension problems.

- They can also be applied in the infinite case, where the effective homology method can be used to determine the groups $H_n^{i,j}$ by means of an *equivalence* between the initial chain complex C_* and an auxiliary chain complex of finite type.

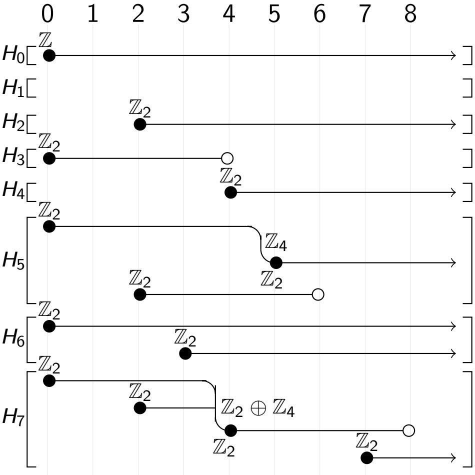
Programs computing persistent homology

Programs computing persistent homology

New definition of *barcode diagram* for integer persistence, solving extension problems:

Programs computing persistent homology

New definition of *barcode diagram* for integer persistence, solving extension problems:



Multipersistence

Multipersistence

Multidimensional filtration of a simplicial complex:

$$
\begin{array}{ccccccc}
K_{m1} & \hookrightarrow & K_{m2} & \hookrightarrow & \cdots & \hookrightarrow & K_{mm'} \\
\uparrow & & \uparrow & & & & \uparrow \\
\cdots & & \cdots & & & & \cdots \\
\uparrow & & \uparrow & & & & \uparrow \\
K_{21} & \hookrightarrow & K_{22} & \hookrightarrow & \cdots & \hookrightarrow & K_{2m'} \\
\uparrow & & \uparrow & & & & \uparrow \\
K_{11} & \hookrightarrow & K_{12} & \hookrightarrow & \cdots & \hookrightarrow & K_{1m'}
\end{array}
$$

Multipersistence

Multidimensional filtration of a simplicial complex:

$$
\begin{array}{ccccccc}
K_{m1} & \hookrightarrow & K_{m2} & \hookrightarrow & \cdots & \hookrightarrow & K_{mm'} \\
\uparrow & & \uparrow & & & & \uparrow \\
\cdots & & \cdots & & & & \cdots \\
\uparrow & & \uparrow & & & & \uparrow \\
K_{21} & \hookrightarrow & K_{22} & \hookrightarrow & \cdots & \hookrightarrow & K_{2m'} \\
\uparrow & & \uparrow & & & & \uparrow \\
K_{11} & \hookrightarrow & K_{12} & \hookrightarrow & \cdots & \hookrightarrow & K_{1m'}
\end{array}
$$

Multipersistence groups:

$$H_n^{P,Q} := \mathrm{Im}(f_n^{P,Q} : H_n(K_P) \to H_n(K_Q)), \quad P, Q \in \mathbb{Z}^2, P \preceq Q.$$

Generalized spectral sequences

Generalized spectral sequences

The notion of spectral sequence of a filtered complex has been recently generalized by B. Matschke for a filtration indexed over a *poset* I, i.e. a collection of sub-chain complexes $\{F_i C_*\}_{i \in I}$ with $F_i C_* \subseteq F_j C_*$ if $i \leq j$, as a set of groups, for all $z \leq s \leq p \leq b$ in I and for each degree n:

$$S_n[z, s, p, b] = \frac{F_p C_n \cap d_n^{-1}(F_z C_{n-1})}{d_{n+1}(F_b C_{n+1}) + F_s C_n}$$

Generalized spectral sequences

The notion of spectral sequence of a filtered complex has been recently generalized by B. Matschke for a filtration indexed over a *poset* I, i.e. a collection of sub-chain complexes $\{F_i C_*\}_{i \in I}$ with $F_i C_* \subseteq F_j C_*$ if $i \leq j$, as a set of groups, for all $z \leq s \leq p \leq b$ in I and for each degree n:

$$S_n[z,s,p,b] = \frac{F_p C_n \cap d_n^{-1}(F_z C_{n-1})}{d_{n+1}(F_b C_{n+1}) + F_s C_n}$$

and differential maps $d_n : S_n[z_2, s_2, p_2, b_2] \to S_{n-1}[z_1, s_1, p_1, b_1]$.

Generalized spectral sequences

The notion of spectral sequence of a filtered complex has been recently generalized by B. Matschke for a filtration indexed over a *poset* I, i.e. a collection of sub-chain complexes $\{F_i C_*\}_{i \in I}$ with $F_i C_* \subseteq F_j C_*$ if $i \leq j$, as a set of groups, for all $z \leq s \leq p \leq b$ in I and for each degree n:

$$S_n[z,s,p,b] = \frac{F_p C_n \cap d_n^{-1}(F_z C_{n-1})}{d_{n+1}(F_b C_{n+1}) + F_s C_n}$$

and differential maps $d_n : S_n[z_2, s_2, p_2, b_2] \to S_{n-1}[z_1, s_1, p_1, b_1]$.

As *classical* spectral sequences, they usually converge to *interesting things*.

Generalized spectral sequences

The notion of spectral sequence of a filtered complex has been recently generalized by B. Matschke for a filtration indexed over a *poset* I, i.e. a collection of sub-chain complexes $\{F_i C_*\}_{i \in I}$ with $F_i C_* \subseteq F_j C_*$ if $i \leq j$, as a set of groups, for all $z \leq s \leq p \leq b$ in I and for each degree n:

$$S_n[z, s, p, b] = \frac{F_p C_n \cap d_n^{-1}(F_z C_{n-1})}{d_{n+1}(F_b C_{n+1}) + F_s C_n}$$

and differential maps $d_n : S_n[z_2, s_2, p_2, b_2] \to S_{n-1}[z_1, s_1, p_1, b_1]$.

As *classical* spectral sequences, they usually converge to *interesting things*.

- Generalized Serre spectral sequence for a tower of fibrations.

Generalized spectral sequences

The notion of spectral sequence of a filtered complex has been recently generalized by B. Matschke for a filtration indexed over a *poset* I, i.e. a collection of sub-chain complexes $\{F_iC_*\}_{i\in I}$ with $F_iC_* \subseteq F_jC_*$ if $i \leq j$, as a set of groups, for all $z \leq s \leq p \leq b$ in I and for each degree n:

$$S_n[z,s,p,b] = \frac{F_pC_n \cap d_n^{-1}(F_zC_{n-1})}{d_{n+1}(F_bC_{n+1}) + F_sC_n}$$

and differential maps $d_n : S_n[z_2,s_2,p_2,b_2] \to S_{n-1}[z_1,s_1,p_1,b_1]$.

As *classical* spectral sequences, they usually converge to *interesting things*.

- Generalized Serre spectral sequence for a tower of fibrations.
- Generalized Eilenberg-Moore spectral sequence for a pull-back diagram of fibrations.

Programs computing generalized spectral sequences

Programs computing generalized spectral sequences

- By adapting our programs for classical spectral sequences, we can also compute the groups $S_n[z, s, p, b]$ and the differential maps for all indexes.

Programs computing generalized spectral sequences

- By adapting our programs for classical spectral sequences, we can also compute the groups $S_n[z, s, p, b]$ and the differential maps for all indexes.

- Our programs are also valid in the integer case and this makes it possible to solve the possible extension problems.

Programs computing generalized spectral sequences

- By adapting our programs for classical spectral sequences, we can also compute the groups $S_n[z, s, p, b]$ and the differential maps for all indexes.

- Our programs are also valid in the integer case and this makes it possible to solve the possible extension problems.

- They can also be applied in the infinite case, where the effective homology method can be used to determine the groups $S_n[z, s, p, b]$ by means of a pair of reductions between the initial chain complex C_* and an auxiliary chain complex of finite type.

Programs computing generalized spectral sequences

- By adapting our programs for classical spectral sequences, we can also compute the groups $S_n[z, s, p, b]$ and the differential maps for all indexes.

- Our programs are also valid in the integer case and this makes it possible to solve the possible extension problems.

- They can also be applied in the infinite case, where the effective homology method can be used to determine the groups $S_n[z, s, p, b]$ by means of a pair of reductions between the initial chain complex C_* and an auxiliary chain complex of finite type.

A. Guidolin, A. R. *Effective Computation of Generalized Spectral Sequences*. To appear in Proceedings ISSAC 2018.

Thank you!