Spectral sequences for computing persistent homology of digital images

Ana Romero[1], Jónathan Heras[2], Gadea Mata[1], Julio Rubio[1], and Francis Sergeraert[3]

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Persistence in digital images

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$$0 = H_n(K^0) \overset{f_n^{0,1}}{\to} H_n(K^1) \overset{f_n^{1,2}}{\to} \cdots \overset{f_n^{m-1,m}}{\to} H_n(K^m) = H_n(K)$$

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If γ *is born* at K^i and *dies* entering K^j, we denote $\operatorname{pers}(\gamma) = j - i$.

Persistent homology

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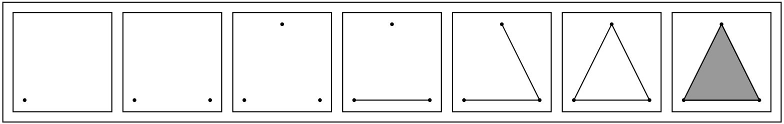

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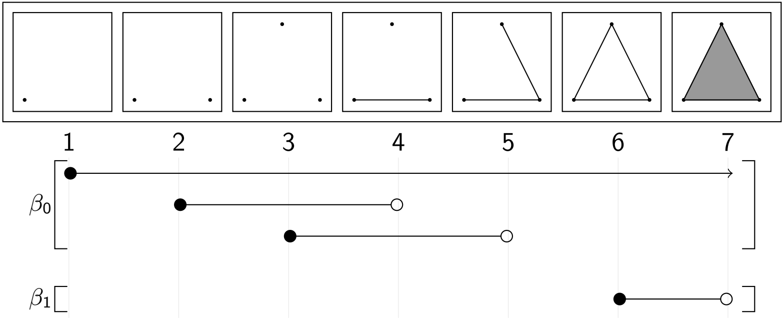

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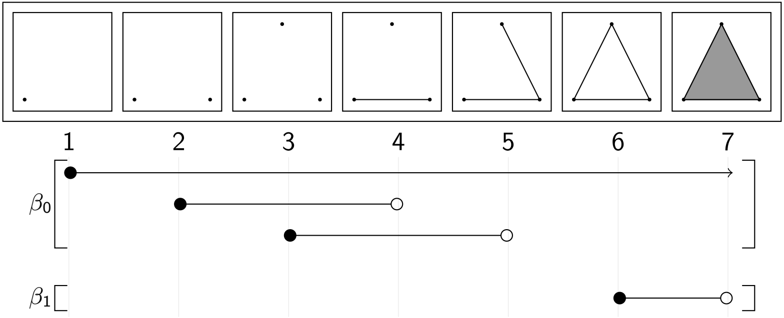

Roughly speaking:

- Homology detects topological features (connected components, holes, and so on).
- Persistent homology describes the evolution of topological features looking at consecutive thresholds.

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A *spectral sequence* $E = (E^r, d^r)_{r \geq 1}$ is a family of bigraded $\mathbb{Z}$-modules $E^r = \{E^r_{p,q}\}$, each provided with a differential $d^r = \{d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$ for every $r \geq 1$.

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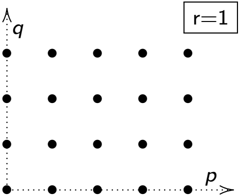

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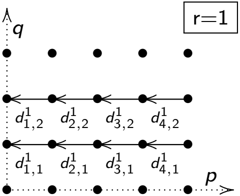

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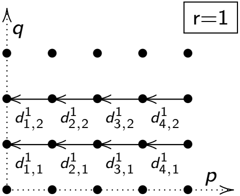

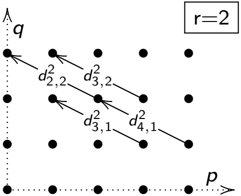

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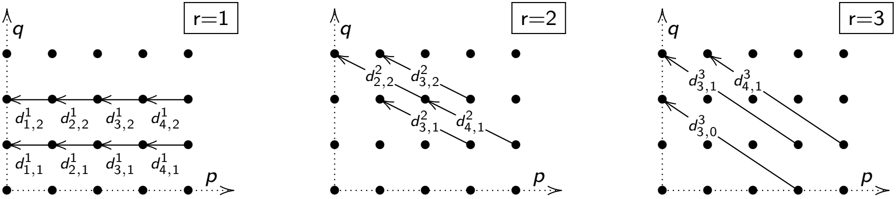

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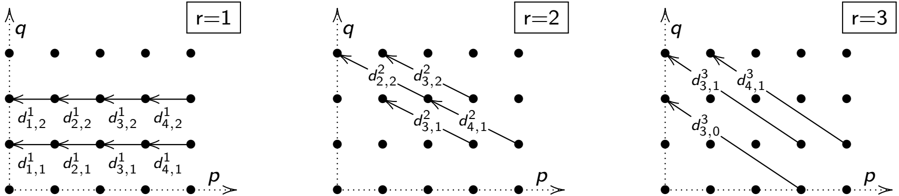

Since $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$ for each $r \geq 1$, one can define the *final groups* of the spectral sequence as $E^\infty_{p,q} = \bigcap_{r\geq 1} E^r_{p,q}$.

Spectral sequences

Spectral sequences

Theorem

Let C be a chain complex with a filtration. There exists a spectral sequence with

$$E^r_{p,q} = \frac{Z^r_{p,q} + C^{p-1}_{p+q}}{d_{p+q+1}(Z^{r-1}_{p+r-1,q-r+2}) + C^{p-1}_{p+q}}$$

where $Z^r_{p,q}$ *is* $Z^r_{p,q} = \{a \in C^p_{p+q} \mid d_{p+q}(a) \in C^{p-r}_{p+q-1}\} \subseteq C^p_{p+q}$, *and* $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ *is the morphism induced by* $d_{p+q} : C_{p+q} \to C_{p+q-1}$. *This spectral sequence converges to* $H_*(C)$, *that is, there are natural isomorphisms*

$$E^\infty_{p,q} \cong \frac{H^p_{p+q}(C)}{H^{p-1}_{p+q}(C)}$$

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The so called "Spectral sequence theorem" in the book *Computational topology: an introduction* by H. Edelsbrunner and J. Harer claims that:

"Erroneous" theorem

The total rank of the groups of dimension $p+q$ in the level $r \geq 1$ of the associated spectral sequence equals the number of points in the $(p+q)$-th persistence diagram whose persistence is r or larger, that is,

$$\sum_{p=1}^{m} \operatorname{rank} E^r_{p,q} = \operatorname{card}\{a \in \operatorname{Dgm}_{p+q}(f) | \operatorname{pers}(a) \geq r\}$$

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However, the formula is erroneous because in the left side there can be more elements than in the right side; the formula should be therefore an inequality.

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The total rank of the images of the differential maps in the level $r \geq 1$ of the spectral sequence equals the number of points in the $(p+q)$-th persistence diagram whose persistence is r:

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A. Romero, J. Heras, J. Rubio and F. Sergeraert. Defining and computing persistent $\mathbb{Z}$-homology in the general case. Preprint, 2013.

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- The new programs use the effective homology technique and allow the Kenzo user to determine the different components of spectral sequences of filtered complexes even in some cases where the chain complex has infinite type.

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$$H_n^{i,j} = \frac{\text{Ker}\, d_n \cap C_n^i}{d_{n+1}(Z_{j,n-j+1}^{j-i})} = \frac{Z_{i,n-i}^i}{d_{n+1}(Z_{j,n-j+1}^{j-i})}$$

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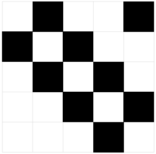
- The programs are also valid in the integer case and this makes it possible to solve the possible extension problems.
- Our programs can also be applied in the infinite case, where the effective homology method can be used to determine the groups $H_n^{i,j}$ by means of a *reduction* of the initial chain complex C to an auxiliary chain complex of finite type.

Digital Algebraic Topology

Digital Image



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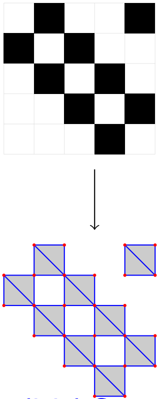

Simplicial Complex

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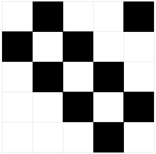

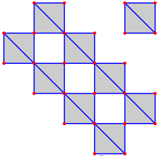

Simplicial Complex

$C_0 = \text{vertices}$
$C_1 = \text{edges}$
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Chain Complex

Digital Algebraic Topology

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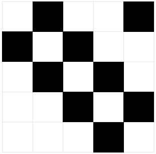

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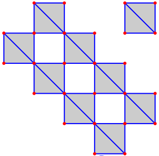

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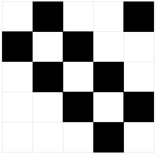

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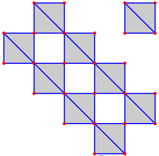

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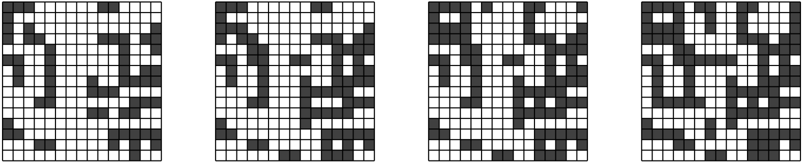

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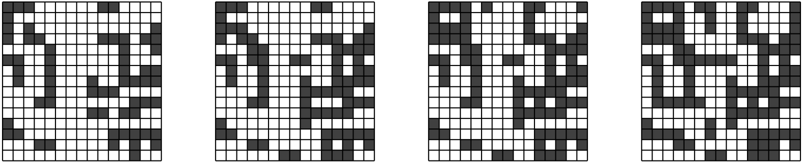

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Persistent Homology H^{1,4}_0
Component Z
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Let $C_* = (C_p, d_p)_{p\in\mathbb{Z}}$ be a free chain complex with distinguished $\mathbb{Z}-$basis $\beta_p \subset C_p$. A *discrete vector field* on C_* is a collection of pairs $V = \{(\sigma_i, \tau_i)\}_{i\in\beta}$ satisfying the conditions:

1. Every σ_i is some element of β_p, in which case the other corresponding component $\tau_i \in \beta_{p+1}$. The degree p depends on i and in general is not constant.
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A *V-path of degree p* is a sequence $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0 \leq k < m}$ satisfying:

1. Every pair $((\sigma_{i_k}, \tau_{i_k}))$ is a component of V and the cell τ_{i_k} is a p-cell
2. For every $0 < k < m$, the component σ_{i_k} is a face of $\tau_{i_{k-1}}$, non necessarily regular, but different from $\sigma_{i_{k-1}}$

Definition

A discrete vector field V is *admissible* if for every $p \in \mathbb{Z}$, a function $\lambda_p : \beta_p \to \mathbb{Z}$ is provided satisfying the property: every V-path starting from $\sigma \in \beta_p$ has a length bounded by $\lambda_p(\sigma)$.

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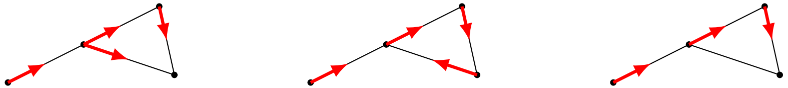

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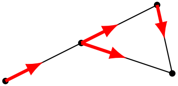
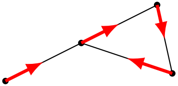
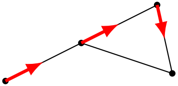


Dvf x





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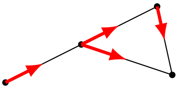

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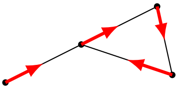
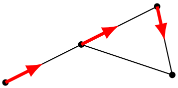
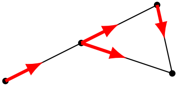


Dvf ✓

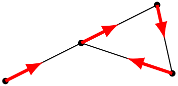
Admissible x



Example: an admissible discrete vector field



Dvf x



Dvf ✓

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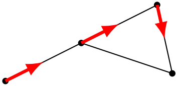

Dvf ✓

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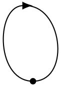

Discrete Vector Fields and effective homology

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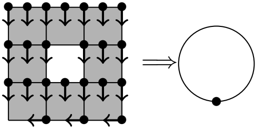

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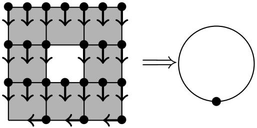

A. Romero and F. Sergeraert. Discrete Vector Fields and Fundamental Algebraic Topology, 2010. http://arxiv.org/abs/1005.5685v1.

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An admissible discrete vector field V for M is nothing but a set of integer pairs $\{(a_i, b_i)\}$ satisfying these conditions:

1. $1 \leq a_i \leq m$ and $1 \leq b_i \leq n$
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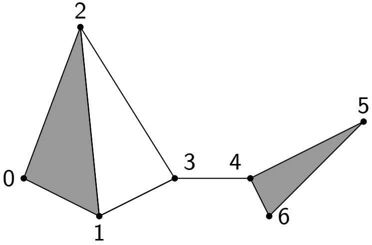

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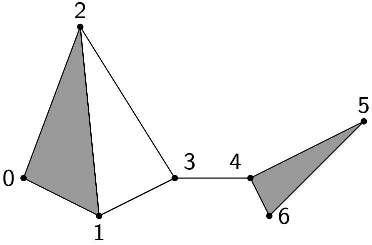

	{0, 1}	{0, 2}	{1, 2}	{1, 3}	{2, 3}	{3, 4}	{4, 5}	{4, 6}	{5, 6}
{0}	1	1	0	0	0	0	0	0	0
{1}	1	0	1	1	0	0	0	0	0
{2}	0	1	1	0	1	0	0	0	0
{3}	0	0	0	1	1	1	0	0	0
{4}	0	0	0	0	0	1	1	1	0
{5}	0	0	0	0	0	0	1	0	1
{6}	0	0	0	0	0	0	0	1	1

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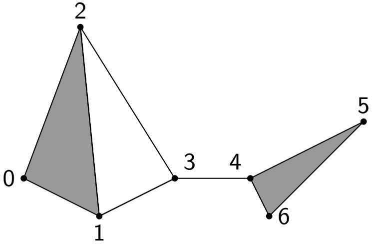

	{0, 1}	{0, 2}	{1, 2}	{1, 3}	{2, 3}	{3, 4}	{4, 5}	{4, 6}	{5, 6}
{0}	**1**	1	0	0	0	0	0	0	0
{1}	1	0	**1**	1	0	0	0	0	0
{2}	0	1	1	0	**1**	0	0	0	0
{3}	0	0	0	1	1	**1**	0	0	0
{4}	0	0	0	0	0	1	**1**	1	0
{5}	0	0	0	0	0	0	1	0	**1**
{6}	0	0	0	0	0	0	0	1	1

Discrete vector fields and persistence

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- To obtain such a *filtered* discrete vector field, we apply our algorithm separately to the differential submatrices corresponding to each filtration index.
- This allows us to compute persistent homology groups of big digital images by means of a reduced chain complex.

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Vertices	9082	150
Edges	20364	86
Triangles	11352	6

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$H_0^{1,1}$	57sec	0, 5sec
$H_0^{1,2}$	33min	10sec
$H_0^{1,3}$	-	$0, 4sec$

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- The relation between persistent homology and spectral sequences allows us to reuse our previous programs for spectral sequences to compute persistent homology groups. The programs are valid for the integer case and for infinite simplicial sets with effective homology. In particular they can be used for digital images.
- Discrete vector fields and effective homology can be used to reduce the size of digital images before computing persistent homology.