Effective Homotopy: a first approach

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Computation of homotopy groups

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Applications of homotopy groups of spheres:

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Applications of homotopy groups of spheres:

- The winding number (corresponding to an integer of $\pi_1(S^1) = \mathbb{Z}$) can be used to prove the fundamental theorem of algebra.
- $\pi_{n-1}(S^{n-1}) = \mathbb{Z}$ implies the Brouwer fixed point theorem.
- The stable homotopy groups of spheres are important in singularity theory.
- $\pi_{n+3}(S^n) = \mathbb{Z}_{24}$ implies Rokhlin's theorem that the signature of a compact smooth spin 4-manifold is divisible by 16.
- The stable homotopy groups of spheres are used in the classification of possible smooth structures on a topological or piecewise linear manifold.
- The Kervaire invariant problem about the existence of manifolds of Kervaire invariant 1 in dimensions $2k - 2$ can be reduced to a question about stable homotopy groups of spheres.
- The stable homotopy groups of the spheres have been interpreted in terms of the plus construction applied to the classifying space of the symmetric group (K-theory).

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- Adams and Bousfield-Kan spectral sequences

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Examples:

- if X has effective homology, $\Omega(X)$ has effective homology;
- if $F \hookrightarrow E \rightarrow B$ is a fibration and F and B have effective homology, then E has effective homology.

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It allows the computation of homology groups of complicated spaces: total spaces of fibrations, arbitrarily iterated loop spaces (Adams' problem), classifying spaces...

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It can also determine some homotopy groups of spaces by means of the Postnikov tower technique.

The homotopical problem of a Kan simplicial set

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Definition

A *simplicial set* X is a graded set $X = \{X_n\}_{n\in\mathbb{N}}$ with maps $\partial_i : X_n \to X_{n-1}$ and $\eta_i : X_n \to X_{n+1}$, $0 \leq i \leq q$, which satisfy the *simplicial identities*.

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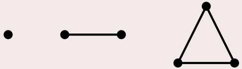

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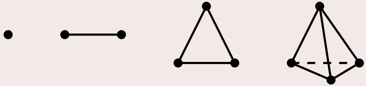

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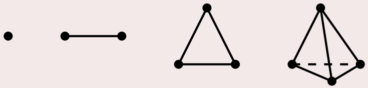

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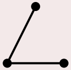

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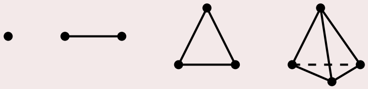

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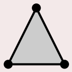

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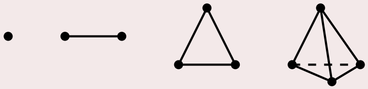

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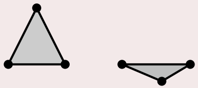

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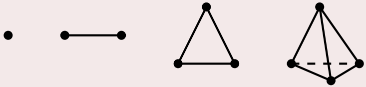

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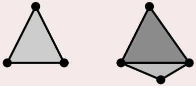

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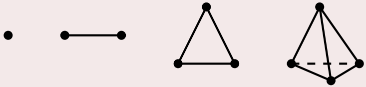

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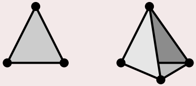

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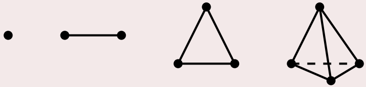

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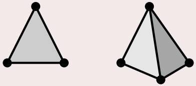

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Two n-simplices x and y of X are said to be *homotopic*, written $x \sim y$, if $\partial_i x = \partial_i y$ for $0 \leq i \leq n$, and there exists an $(n+1)$-simplex z such that $\partial_n z = x, \partial_{n+1} z = y$, and $\partial_i z = \eta_{n-1} \partial_i x = \eta_{n-1} \partial_i y$ for $0 \leq i < n$.

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Let $\star \in X_0$ be a base point. $S_n(X)$ is the set of all $x \in X_n$ such that $\partial_i x = \star$ for every $0 \leq i \leq n$, called the *n-spheres* of X.

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Definition

Given a Kan simplicial set X and a base point $\star \in X_0$, we define

$$\pi_n(X, \star) = S_n(X)/(\sim)$$

The set $\pi_n(X, \star)$ admits a group structure for $n \geq 1$ and it is Abelian for $n \geq 2$. It is called the *n-homotopy group* of X.

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$$\begin{tikzcd} \operatorname{Ker} f \arrow[r, phantom, "\subseteq"] \arrow[r, bend left, "h_n"] & S_n(X) \arrow[r, shift left, "f_n"] & \pi_n \arrow[l, shift left, "g_n"] \end{tikzcd}$$

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Proposition

Let X be a Kan simplicial set and $(\pi_n, f_n, g_n, h_n)_{n \geq 0}$ a solution for the homotopical problem of X. Then, for each $n \geq 1$, the homotopy group $\pi_n(X) = S_n(X)/(\sim)$ is isomorphic to the given group π_n.

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- Some spaces have *trivial* effective homotopy. For instance, Eilenberg-MacLane spaces $X(\pi, n)$ for finitely generated groups π.
- Given some Kan simplicial sets $X_1, \ldots, X_n$, with effective homotopy and a topological constructor Φ which produces a new simplicial set X, one should obtain a solution for the homotopical problem of X. First example: fibrations.

Solution for the homotopical problem of a fibration

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Definition

Let $f : E \rightarrow B$ be a simplicial map. f is said to be a *Kan fibration* if for every collection of $n+1$ n-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ of E which satisfy the compatibility condition $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j, i \neq k$ and $j \neq k$, and for every $(n+1)$-simplex y of B such that $\partial_i y = f(x_i), i \neq k$, there exists an $(n+1)$-simplex x of E such that $\partial_i x = x_i$ for $i \neq k$ and $f(x) = y$.

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Theorem

An algorithm can be written down:
Input:

- *A constructive Kan fibration $f : E \to B$ where B is a constructive Kan complex (which implies F and E are also constructive Kan simplicial sets).*
- *Respective $SHmtP_F$ and $SHmtP_B$ for the simplicial sets F and B.*

Output: *A $SHmtP_E$ for the Kan simplicial set E.*

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The long exact sequence of homotopy

$$\cdots \xrightarrow{f_*} \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \xrightarrow{\text{inc}_*} \pi_n(E) \xrightarrow{f_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \xrightarrow{\text{inc}_*} \cdots$$

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produces a short exact sequence

$$0 \longrightarrow \mathrm{Coker}[\pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F)] \xrightarrow{i} \pi_n(E) \xrightarrow{j} \mathrm{Ker}[\pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F)] \longrightarrow 0$$

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Proof:

The long exact sequence of homotopy

$$\cdots \xrightarrow{f_*} \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \xrightarrow{\text{inc}_*} \pi_n(E) \xrightarrow{f_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \xrightarrow{\text{inc}_*} \cdots$$

produces a short exact sequence

$$0 \longrightarrow \text{Coker}[\pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F)] \xrightarrow{i} \pi_n(E) \xrightarrow{j} \text{Ker}[\pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F)] \longrightarrow 0$$

and this implies that $\pi_n(E)$ can be expressed as $\pi_n(E) \cong \text{Coker} \times_\chi \text{Ker}$ for a cohomology class $\chi \in H^2(\text{Ker}, \text{Coker})$ classifying the extension.

Solution for the homotopical problem of a fibration

Proof:

The long exact sequence of homotopy

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This cohomology class and the maps f_n, g_n and h_n are determined by means of the Kan properties of X and the fibration and the solutions for the homotopical problems of F and B.

Examples and applications

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- Fibrations of Eilenberg-MacLane spaces

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- Bousfield-Kan spectral sequence

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  defined by means of a tower of fibrations

$$
\begin{tikzcd}
\cdots \arrow[r, "f_4"] & \mathrm{Tot}_3\,\mathcal{R}X \arrow[r, "f_3"] & \mathrm{Tot}_2\,\mathcal{R}X \arrow[r, "f_2"] & \mathrm{Tot}_1\,\mathcal{R}X \arrow[r, "f_1"] & \mathrm{Tot}_0\,\mathcal{R}X \cong RX \\
& F_3 \arrow[u, "i"] & F_2 \arrow[u, "i"] & F_1 \arrow[u, "i"] &
\end{tikzcd}
$$

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- Bousfield-Kan spectral sequence
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With the effective homotopy of the spaces $\mathrm{Tot}_n\,\mathcal{R}X$ one can determine the Bousfield-Kan spectral sequence and the homotopy groups of a (1-reduced) simplicial set X.