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Computing the first stages of the Bousfield-Kan spectral sequence

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Abstract In this paper, an algorithm computing the terms E^1 and E^2 of the Bousfield-Kan spectral sequence of a 1-reduced simplicial set X is defined. In order to compute the ordinary description of the first level E^1, some elementary operations of Homological Algebra are sufficient. On the contrary, to compute the stage E^2 it is necessary to know more information about the previous groups, in particular with respect to the generators. This additional information can be reached by computing the *effective homology* of RX, RX being the free simplicial Abelian group generated by X. The algorithm to get the effective homology of RX from the effective homology of X can be considered the main result in our paper. Moreover, we include a combinatorial proof of the convergence of the Bousfield-Kan spectral sequence, based on the *tapered* nature of the stage E^1.

Keywords Constructive algebraic topology · Bousfield-Kan spectral sequence · Computation of homotopy groups · Symbolic computation

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1 Introduction

The Bousfield-Kan spectral sequence first appeared in [5]. It presents the Adams spectral sequence [2] in the setting of combinatorial topology and makes its

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algebraic properties more accessible. The Adams spectral sequence and its satellite spectral sequences are the main tools to compute homotopy groups, in particular stable and unstable sphere homotopy groups. The Adams spectral sequence and the others did allow topologists to compute many homotopy groups, but no constructive version of this spectral sequence is yet available; in other words no routine translation work allows a programmer to implement this spectral sequence on a theoretical or concrete machine to produce an algorithm computing homotopy groups.

A spectral sequence is a family of "pages" (E^r, d^r) of differential bigraded modules, each page being made of the homology groups of the preceding one. In many cases, only the first levels are given, and then some extra information is necessary to determine the successive differential maps. This implies that a spectral sequence can only be *computed* in some simple situations and in general it is not an *algorithm*.

The effective homology method [17] provides on the contrary real algorithms for the computation of homology and homotopy groups of complicated spaces, and it has been concretely implemented in the Kenzo system [9], a Common Lisp program devoted to Symbolic Computation in Algebraic Topology. In some cases, this technique *replaces* some common spectral sequences such as those of Serre or Eilenberg-Moore: when the usual inputs of these spectral sequences are organized as *objects with effective homology*, general algorithms are produced computing for example the homology groups of the total space of a fibration, of an arbitrarily iterated loop space (Adams' problem), of a classifying space, etc. Furthermore, we have shown in a previous paper [16] that the effective homology technique can also be used to obtain as a by-product the different components of spectral sequences associated with filtered complexes (which include those of Serre and Eilenberg-Moore), obtaining in this way real algorithms.

In this work we try to use the effective homology method to construct algorithms for computing the Bousfield-Kan spectral sequence, which is not defined by means of filtered complexes. As a first step, we have developed an effective homology version of the constructor R associating to every simplicial set X the free simplicial Abelian group RX. This construction can then be used to obtain the first two levels of the Bousfield-Kan spectral sequence, the "pages" E^1 and E^2. The general algorithm computing the higher levels is not finished yet.

This paper is organized as follows. Section 2 begins with some elementary ideas about spectral sequences and then a brief description of the construction of the Bousfield-Kan spectral sequence is given, including some necessary definitions and results. In Sect. 3 we present a proof of the convergence of the spectral sequence based on elementary operations of Homological Algebra, which can also be used to compute the first level E^1, as explained in Sect. 4. In order to determine the second page E^2, these results are not sufficient and the effective homology method is necessary; we introduce it in Sect. 5 and then we compute in particular the effective homology of the free simplicial Abelian group RX. This is the main result of the paper and we use it in Sect. 6 to construct an algorithm computing *effective* versions of the levels E^1 and E^2 of the spectral sequence. The paper ends with a section of conclusions and further work.

2 Definition of the Bousfield-Kan spectral sequence

For later reference, we begin this section with some elementary facts about spectral sequences, mainly extracted from [12] and [14].

Definition 1 Let R be a ring. A *bigraded R-module* is a family of R-modules $E = \{E_{p,q}\}_{p,q\in\mathbb{Z}}$. A *differential* $d : E \to E$ of bidegree $(-r, r-1)$ is a family of morphisms of R-modules $d_{p,q} : E_{p,q} \to E_{p-r,q+r-1}$ for each $p, q \in \mathbb{Z}$, such that $d_{p-r,q+r-1} \circ d_{p,q} = 0$. The pair (E, d) is called a *differential bigraded module*.

The relations $d_{p-r,q+r-1} \circ d_{p,q} = 0$ allow us to define the *homology* of E as the bigraded R-module $H(E, d) = H(E) = \{H_{p,q}(E)\}_{p,q\in\mathbb{Z}}$ with

$$H_{p,q}(E) = \frac{\operatorname{Ker} d_{p,q}}{\operatorname{Im} d_{p+r,q-r+1}}$$

Definition 2 A *spectral sequence* $E = (E^r, d^r)_{r\geq 1}$ is a sequence of bigraded R-modules $E^r = \{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$, each provided with a differential $d^r = \{d^r_{p,q}\}_{p,q\in\mathbb{Z}}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$ for every $r \geq 1$.

In some situations, only the first levels of a spectral sequence are given; in other words, only $(E^r, d^r)_{1\leq r\leq k}$ are known (very frequently $k = 1$ or 2). Since each E^{r+1} in the spectral sequence is (up to isomorphism) the bigraded homology module of the preceding differential bigraded module (E^r, d^r), the stage k in the spectral sequence, given by $E^k = \{E^k_{p,q}\}$ and $d^k = \{d^k_{p,q}\}$, allows us to build the bigraded module at the level $k+1$, $E^{k+1} = \{E^{k+1}_{p,q}\}$. But then our information is not sufficient to define the next differential d^{k+1}, which therefore must be independently defined too. In this way, a finite number of stages of the spectral sequence does not allow us to compute the *whole* spectral sequence, some extra information is necessary to determine the successive differential maps.

Definition 3 A spectral sequence $E = (E^r, d^r)_{r\geq 1}$ is a *first quadrant spectral sequence* if for all $r \geq 1$ one has $E^r_{p,q} = 0$ when $p < 0$ or $q < 0$. A *second quadrant spectral sequence* E is one with $E^r_{p,q} = 0$ if $p > 0$ or $q < 0$.

From now on in this paper we will deal with second quadrant spectral sequences, but we will represent them in the first quadrant by changing the sign of the first degree p. In other words, we put the module $E^r_{p,q}$ with $p \leq 0$ and $q \geq 0$ at the point $(-p, q)$ (which is in the first quadrant), and denote it by $E^r_{-p,q}$. The differential maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p+r,q+r-1}$ have then shift $(r, r-1)$. Furthermore, we consider only the particular case $R = \mathbb{Z}$, the most important for the aimed applications; yet we continue to denote it by R, a traditional notation in this context.

Definition 4 A (*second quadrant*) spectral sequence $E = (E^r, d^r)_{r\geq 1}$ is said to be *convergent* if for every $p, q \in \mathbb{Z}$ there exists $r_{p,q} \geq 1$ such that $d^r_{p,q} = 0 = d^r_{p-r,q-r+1}$ for all $r \geq r_{p,q}$.

If $E = (E^r, d^r)_{r\geq 1}$ is convergent, one has $E^r_{p,q} = E^{r_{p,q}}_{p,q}$ for all $r \geq r_{p,q}$, and one can define the groups $E^\infty_{p,q} = E^{r_{p,q}}_{p,q}$, which are called the *final groups* of the spectral sequence.

Definition 5 Let $H_* = \{H_n\}_{n\in\mathbb{N}}$ be a graded module. A (decreasing) *filtration* F is a family of sub-graded modules $F_pH_* \subseteq H_*$ for each $p \in \mathbb{Z}$ such that

$$\cdots \supseteq F_{p-1}H_n \supseteq F_pH_n \supseteq F_{p+1}H_n \supseteq \cdots \quad \text{for all } n \in \mathbb{N}$$

Definition 6 Let $H_* = \{H_n\}_{n\in\mathbb{N}}$ be a graded module. A spectral sequence $(E^r, d^r)_{r\geq 1}$ is said to *converge* to H_* (denoted by $E^1 \Rightarrow H_*$) if there exists a filtration F of H_* and for each pair (p, q) one has isomorphisms

$$E^{\infty}_{p,q} \cong \frac{F_pH_{q-p}}{F_{p+1}H_{q-p}}$$

The collection $H_* = \{H_n\}_{n\in\mathbb{N}}$ is called the *abutment* of the spectral sequence.

The Bousfield-Kan spectral sequence was introduced in [5] trying to generalize the Adams spectral sequence, related with the computation of the homotopy groups of a simplicial set X, denoted $\pi_*(X)$. Different constructions have been considered to obtain the Bousfield-Kan spectral sequence, all of them based on the following definitions and results, which can be found in [4].

Definition 7 Let X be a simplicial set with a base point $\star \in X_0$, then RX is defined as the simplicial Abelian group

$$RX = \frac{R[X]}{R[\star]}$$

where $R[X]$ denotes the simplicial $\mathbb{Z}$-module freely generated by the simplices of X, and $R[\star]$ is the simplicial submodule generated by the base point $\star$ and its degeneracies (which are also represented by $\star$).

The most important property which RX satisfies is the following one.

Proposition 1 *Given X a pointed simplicial set, there exists a canonical isomorphism*

$$\pi_*(RX) \cong \widetilde{H}_*(X; \mathbb{Z})$$

where $\widetilde{H}_*(X; \mathbb{Z})$ *denotes the reduced homology groups of X with coefficients in* $\mathbb{Z}$.

Definition 8 A *cosimplicial space* $\mathcal{X}$ consists of

– for every integer $n \geq 0$, a simplicial set $\mathcal{X}^n$;
– for every pair of integers (i, n) such that $0 \leq i \leq n$, *coface* and *codegeneracy* operators $\partial^i : \mathcal{X}^{n-1} \to \mathcal{X}^n$ and $\eta^i : \mathcal{X}^{n+1} \to \mathcal{X}^n$ (both of them simplicial maps) satisfying the *cosimplicial identities*:

$$\begin{array}{ll}
\partial^j \partial^i = \partial^i \partial^{j-1} & \text{if } i < j \\
\eta^j \eta^i = \eta^{i-1} \eta^j & \text{if } i > j \\
\eta^j \partial^i = \partial^i \eta^{j-1} & \text{if } i < j \\
\eta^j \partial^i = \mathrm{Id} & \text{if } i = j, j+1 \\
\eta^j \partial^i = \partial^{i-1} \eta^j & \text{if } i > j+1
\end{array}$$

Definition 9 An augmentation of a cosimplicial space $\mathcal{X}$ consists of a simplicial set $\mathcal{X}^{-1}$ and a morphism $\partial^0 : \mathcal{X}^{-1} \to \mathcal{X}^0$ such that $\partial^1 \partial^0 = \partial^0 \partial^0 : \mathcal{X}^{-1} \to \mathcal{X}^1$.

In other words, a cosimplicial space $\mathcal{X}$ consists of a bigraded family $\mathcal{X} = \{\mathcal{X}_q^p\}_{p,q \in \mathbb{N}}$ with face, coface, degeneracy and codegeneracy maps $\partial_i : \mathcal{X}_q^p \to \mathcal{X}_{q-1}^p$, $\partial^j : \mathcal{X}_q^{p-1} \to \mathcal{X}_q^p$, $\eta_i : \mathcal{X}_q^p \to \mathcal{X}_{q+1}^p$, and $\eta^j : \mathcal{X}_q^{p+1} \to \mathcal{X}_q^p$, for $0 \le i \le q$ and $0 \le j \le p$. The face and degeneracy operators ∂_i and η_i must satisfy the simplicial identities (see [13]), while for ∂^j and η^j the cosimplicial identities of Definition 8 hold. Furthermore, ∂_i and η_i commute with both coface and codegeneracy maps ∂^j and η^j.

Every cosimplicial space has an associated spectral sequence, as explained in the following theorem.

Theorem 1 [4] *Given a cosimplicial space $\mathcal{X}$, a second quadrant spectral sequence $E = (E^r, d^r)_{r \ge 1}$ is canonically defined, whose first level E^1 satisfies*

$$E^1_{p,q} = \pi'_q(\mathcal{X}^p) = \pi_q(\mathcal{X}^p) \cap \mathrm{Ker}\, \eta^0 \cap \cdots \cap \mathrm{Ker}\, \eta^{p-1}$$

where the maps $\eta^j \equiv \pi_q(\eta^j) : \pi_q(\mathcal{X}^p) \to \pi_q(\mathcal{X}^{p-1})$ are induced by the codegeneracy operators $\eta^j : \mathcal{X}^p \to \mathcal{X}^{p-1}$, and where the differential map $d^1_{p,q} : E^1_{p,q} \to E^1_{p+1,q}$ is given by the coboundary map $\delta_q^{p+1} = \sum_{j=0}^{p+1} (-1)^j \partial^j$. If $\mathcal{X}$ is augmented, and under some favorable conditions, this spectral sequence converges to the homotopy groups $\pi_*(\mathcal{X}^{-1})$.

An important example of a cosimplicial space is the cosimplicial resolution of a simplicial set, which plays an essential role in the definition of the Bousfield-Kan spectral sequence.

Definition 10 Let X be a pointed simplicial set, the *cosimplicial resolution* of X (with respect to the ring $R = \mathbb{Z}$) is the augmented cosimplicial space $\mathcal{R}X$ given by

- for each cosimplicial degree p, the column $\mathcal{R}X^p$ is the simplicial Abelian group $R^{p+1}X$ obtained by applying $p+1$ times the functor R (Definition 7) to the simplicial set X (with the corresponding face and degeneracy maps);
- the coface and codegeneracy operators are defined as

$$\partial^j : \mathcal{R}X^{p-1} = R^p X \longrightarrow \mathcal{R}X^p = R^{p+1}X, \quad \partial^j = R^j \Phi R^{p-j}$$
$$\eta^j : \mathcal{R}X^{p+1} = R^{p+2} X \longrightarrow \mathcal{R}X^p = R^{p+1}X, \quad \eta^j = R^j \Psi R^{p-j}$$

  where the maps $\Phi : X \to RX$ and $\Psi : R^2X \to RX$ are given by $\Phi(x) = 1 * x$ for all $x \in X$ and $\Psi(1 * y) = y$ for all $y \in RX$;
- the augmentation is given by the map $\Phi : X \to RX$.

It is worth emphasizing that each column $\mathcal{R}X^p = R^{p+1}X$ is a simplicial Abelian group, which implies that for each $q \geq 0$ the set $\mathcal{R}X^p_q$ is an Abelian group, and the face operators $\partial_i : \mathcal{R}X^p_q \to \mathcal{R}X^p_{q-1}$ and the degeneracies $\eta_i : \mathcal{R}X^p_q \to \mathcal{R}X^p_{q+1}$ are group morphisms. On the other hand, one can observe that the codegeneracy maps $\eta^j : \mathcal{R}X^{p+1}_q \to \mathcal{R}X^p_q$ are also morphisms of groups for all $j \geq 0$, but $\partial^j : \mathcal{R}X^{p-1}_q \to \mathcal{R}X^p_q$ is a group morphism only if $j \geq 1$. For $j = 0$, the coface $\partial^0 : \mathcal{R}X^{p-1}_q \to \mathcal{R}X^p_q$ is not a morphism of groups. For this reason, the cosimplicial space $\mathcal{R}X$ is said to be *grouplike*.

The spectral sequence associated with this particular cosimplicial space is called the *Bousfield-Kan* spectral sequence of the simplicial set X.

Theorem 2 (Bousfield-Kan spectral sequence)[4] *Let X be a simplicial set with base point $\star \in X_0$. There exists a canonical second quadrant spectral sequence $E = (E^r, d^r)_{r\geq 1}$, whose term E^1 is given by*

$$E^1_{p,q} = \pi'_q(\mathcal{R}X^p) = \pi_q(R^{p+1}X) \cap \operatorname{Ker} \eta^0 \cap \cdots \cap \operatorname{Ker} \eta^{p-1}$$

and with differential map d^1 induced by the coboundary map $\delta = \sum(-1)^j \partial^j$. Under suitable hypotheses (for instance, if X is 1-reduced) this spectral sequence converges to the homotopy groups $\pi_*(X)$.

We find it convenient to remark here that this theorem does not allow us to *compute* directly the Bousfield-Kan spectral sequence associated with a simplicial set X. The columns $R^{p+1}X$ have the homotopy type of products of Eilenberg-MacLane spaces, and Cartan's algorithm [7] computes the corresponding $\pi_q(R^{p+1}X) \cong \widetilde{H}_q(R^pX)$. But to our knowledge, in our very general framework, so far no *effective* method (algorithm!) is known allowing one to determine the coface operators between these homology groups. We will prove that the methods of effective homology give such an algorithm. The key point is a version *with effective homology* of the functor R.

In the following sections we will develop algorithms for computing the first two levels E^1 and E^2. As a first step, in the next section we present a proof of the convergence of the spectral sequence, based on the particular nature of the first stage, which will also be used in Sect. 4 to determine a formal description of the groups $E^1_{p,q}$ (without the corresponding generators).

3 Convergence of the Bousfield-Kan spectral sequence

The initial page of the Bousfield-Kan spectral sequence of a simplicial set X consists of the homotopy groups of the columns of the cosimplicial space $\mathcal{R}X$, which are *horizontally normalized*:

$$E^1_{p,q} = \pi'_q(\mathcal{R}X^p) = \pi_q(R^{p+1}X) \cap \operatorname{Ker} \eta^0 \cap \cdots \cap \operatorname{Ker} \eta^{p-1}$$

where the maps $\eta^j \equiv \pi_q(\eta^j) : \pi_q(R^{p+1}X) \to \pi_q(R^pX)$ are induced by the codegeneracy operators $\eta^j : \mathcal{R}X^p = R^{p+1}X \to \mathcal{R}X^{p-1} = R^pX$, for each $0 \leq j \leq p-1$.

We include in this section a theorem which gives a very important property of these *normalized groups*: if the initial simplicial set X is 1-reduced, then

$$\pi'_q(\mathcal{R}X^p) = 0 \quad \text{if } q < 2p + 2$$

This feature implies that the bigraded module $E^1 = \{E^1_{p,q}\}_{p,q\in\mathbb{Z}}$ is *tapered* (that is, $E^1_{p,q} = 0$ if $q < 2p$), which ensures the spectral sequence is convergent. This property is maybe already known, but we have not been able to find a reference. The proof that we propose is completely elementary, but a little complex: the complete proof (see [15]) has more than ten pages. For this reason we include in this paper only the general ideas; the details are then a sequence of elementary lemmas. Although the convergence of the spectral sequence was one of the initial results by Bousfield and Kan, our proof has the advantage of describing in a detailed way the structure of the level E^1, having this description itself its own interest, as we will see in Sect. 4.

Theorem 3 *Let X be a 1-reduced pointed simplicial set, and $E = (E^r, d^r)_{r\geq 1}$ the associated Bousfield-Kan spectral sequence. Then E^1 satisfies*

$$E^1_{p,q} = 0 \quad \text{if } q < 2p + 2$$

Proof (*Sketch of the proof*) Following the indexation (p, q) of a page E^r of the spectral sequence, we denote

$$\pi_{p,q} \equiv \pi_q(R^{p+1}X)$$

which is the *vertical* homotopy group in the position (p, q) *before* the horizontal normalization. In particular $\pi_{0,q} = \pi_q(RX) \cong H_q(X)$ for $q \geq 1$ is an *initial* group; our proof of the theorem consists in showing that these groups $H_q(X)$ are sufficient to produce the page E^1 of the spectral sequence, following a recursive process which must be well understood.

We give a description of $\pi_{p,q}$ of the following form

$$\pi_{p,q} \cong \bigoplus_{G\in Gen_q} (\pi_G)^{n_G(p)}$$

where

- Gen_q is the set of *genealogies of degree* q, a notion which will be defined later;
- π_G is the Abelian group canonically associated with a genealogy G;
- n_G is the *enumeration function* describing for each p how many components π_G take part of the description of $\pi_{p,q}$.

In other words, each group $\pi_{p,q}$ has a structure depending just on the ordinate q, since it is a sum of components π_G for $G \in Gen_q$ a genealogy of degree q; the unique factor which depends on p is the *number* of components, described by the enumeration function n_G.

A genealogy $G \in Gen_q$ is a formal description of a recursive process which allows us to express the groups $\pi_{p,q} \equiv \pi_q(R^{p+1}X)$ in terms of the initial groups $\pi_{0,i} = \pi_i(RX) \cong H_i(X)$ for $i \leq q$. First of all, we remark that the homotopy groups of $R^{p+1}X$ are isomorphic to the (reduced) homology groups of the previous column R^pX, in other words,

$$\pi_{p,q} \cong \widetilde{H}_q(R^pX) \cong H_q(R^pX), \quad q \geq 1$$

Furthermore, R^pX is a (1-reduced) simplicial Abelian group and therefore (see [13]) it is homotopy equivalent to a product of Eilenberg-MacLane spaces of the form

$$R^pX \simeq \prod_{n\geq 0} K(\pi_n(R^pX), n) = \prod_{n\geq 2} K(\pi_{p-1,n}, n)$$

which implies that

$$H_q(R^pX) \cong H_q\left(\prod_{n\geq 2} K(\pi_{p-1,n}, n)\right)$$

If we apply in a recursive way the Eilenberg-Zilber Theorem [10], which claims that the homology groups of the Cartesian product of two simplicial sets are isomorphic to the homology groups of the tensor product of the associated chain complexes, we obtain an isomorphism

$$\pi_{p,q} \cong H_q(R^pX) \cong H_q\left(\bigotimes_{n\geq 2} C_*(K(\pi_{p-1,n}, n))\right)$$

Then, we can make use of the Künneth formula [8], which relates the homology groups of the tensor product of two chain complexes with the homologies of the components. In our case, this formula gives a decomposition of the group $\pi_{p,q} \cong H_q(R^pX)$ in terms of $H_i(K(\pi_{p-1,n}, n))$. Taking into account that, given a group π and a positive integer m, one has $H_m(K(\pi, m)) = \pi$ and $H_{m+1}(K(\pi, m)) = 0$ (a property of Eilenberg-MacLane spaces which can easily be proved making use of the Hurewicz Theorem [13]), for instance for $q = 6$ we obtain:

$$\begin{aligned}\pi_{p,6} \cong H_6(R^pX) \cong \pi_{p-1,6} \oplus H_6(K(\pi_{p-1,2}, 2)) \oplus H_6(K(\pi_{p-1,3}, 3))\\ \oplus H_6(K(\pi_{p-1,4}, 4)) \oplus (\pi_{p-1,2} \otimes \pi_{p-1,4}) \oplus (\pi_{p-1,2} * \pi_{p-1,3})\end{aligned}$$

where $*$ denotes the *torsion product* of two groups (see [8] for details).

If we iterate the process for the groups

$$\pi_{p-1,n} = \pi_n(R^pX) \cong H_n(R^{p-1}X) \cong H_n\left(\prod_{m\geq 2} K(\pi_{p-2,m}, m)\right)$$

and we do the same for $\pi_{p-2,m}$ and so on, we obtain an expression for the group $\pi_{p,q} \cong H_q(R^p X)$ based on the initial groups $\pi_{0,i} \cong H_i(X) \equiv H_i$. As an example, we show in the following lines the decomposition of the group $\pi_{3,7} \cong H_7(R^3 X)$.

$$\begin{aligned}
H_7(R^3X) \cong\ & H_7(K(H_2,2) \times K(H_3,3) \times K(H_4,4) \times K([H_4(K(H_2,2))]^2,4) \\
& \times K(H_5,5) \times K([H_5(K(H_2,2))]^2,5) \times K([H_5(K(H_3,3))]^2,5) \\
& \times K([H_2 \otimes H_3]^2,5) \times K(H_6,6) \times \cdots) \\
\cong\ & H_7 \oplus [H_7(K(H_2,2))]^3 \oplus [H_7(K(H_3,3))]^3 \oplus [H_7(K(H_4,4))]^3 \\
& \oplus [H_7(K(H_4(K(H_2,2)),4))]^3 \oplus [H_7(K(H_5,5))]^3 \\
& \oplus [H_7(K(H_5(K(H_2,2)),5))]^3 \oplus [H_7(K(H_5(K(H_3,3)),5))]^3 \\
& \oplus [H_7(K(H_2 \otimes H_3,5))]^3 \oplus [H_4(K(H_2,2)) \otimes H_3]^3 \\
& \oplus [H_3 \otimes H_4]^3 \oplus [H_3 \otimes H_4(K(H_2,2))]^3 \oplus [H_2 \otimes H_5(K(H_3,3))]^3 \\
& \oplus [H_2 \otimes H_5]^3 \oplus [H_2 \otimes H_5(K(H_2,2))]^3 \oplus [H_2 \otimes H_5(K(H_3,3))]^3 \\
& \oplus [H_2 \otimes (H_2 \otimes H_3)]^3 \oplus [H_2 * H_4]^3 \oplus [H_2 * H_4(K(H_2,2))]^3
\end{aligned}$$

For higher p and q this decomposition becomes more complicated, but in all cases the group $\pi_{p,q} \cong H_q(R^p X)$ is isomorphic to a direct sum of different components (which may occur several times) obtained recursively from the initial groups $H_i \equiv H_i(X)$.

We can formalize this construction defining the notion of *genealogy*. The set of genealogies Gen is a totally ordered graded set defined as the disjoint union of a family of totally ordered graded sets $\{GG_n\}_{n\in\mathbb{N}}$, which are built in a recursive way. The starting point is

$$GG_0 = \{H_i\}_{i\geq 2}$$

where each H_i is a symbol. The *degree* of H_i is i, and the *ordering* in GG_0 is defined by $H_i < H_j$ if $i < j$.

Let us suppose now that we have built GG_m for $m < n$. We consider the set GGG_{n-1} given by the disjoint union

$$GGG_{n-1} = \coprod_{m<n} GG_m$$

which becomes totally ordered simply by making use of the ordering of each GG_m and considering that if $m_1 < m_2$ then each element in GG_{m_1} is less than every element in GG_{m_2}.

We define then GG_n as the set of expressions G of the form

$$G = [(d_1\ G_1)c_1(d_2\ G_2)c_2 \cdots c_{k-1}(d_k\ G_k)]$$

where

- k is a positive integer ($k \geq 1$);
- $G_j \in GGG_{n-1}$ for all $1 \leq j \leq k$;
- $G_{j-1} \leq G_j$ for $1 < j \leq k$;
- $G_k \in GG_{n-1}$;
- if each G_j has degree $\tilde{d}_j$, then $d_j \geq \tilde{d}_j$ and $d_j \neq \tilde{d}_j + 1$. Moreover, if $k = 1$, then $d_j \neq \tilde{d}_j$;
- each c_j is a *connector* (a symbol) in the set $\{\otimes, *\}$.

The degree of G is defined as

$$d = d_1 + \cdots + d_k + n_*$$

where n_* is the number of connectors $*$ which appear in G.

The ordering of GG_n is the lexicographical ordering obtained by considering in a successive way, for instance,

$$k, G_1, \ldots, G_k, d_1, \ldots, d_k, c_1, \ldots, c_{k-1}$$

Once we have built recursively GG_n for all $n \in \mathbb{N}$, we define the *set of genealogies*

$$Gen = \coprod_{n \in \mathbb{N}} GG_n = \bigcup_{n \in \mathbb{N}} GGG_n$$

It is worth remarking that given $q \geq 2$ it is possible to construct (in a recursive way, starting with the initial symbols H_i) the set Gen_q of genealogies of degree q, which is a finite set. For instance, for $q = 4$ one has two genealogies $G_4^1 = H_4$ and $G_4^2 = [(4\ H_2)]$. For degree 5 we obtain $G_5^1 = H_5$, $G_5^2 = [(5\ H_2)]$, $G_5^3 = [(5\ H_3)]$, and $G_5^4 = [(2\ H_2) \otimes (3\ H_3)]$. For $q = 6, 7$, and 8 there exist 8, 19, and 45 genealogies, respectively.

Let us suppose now that each symbol H_i represents the group

$$H_i \equiv H_i(X) \cong \pi_i(RX), \quad i \geq 2$$

Then one can associate to every genealogy $G \in Gen_q$ of degree $q \geq 2$ an Eilenberg-MacLane space $K(G)$ built as follows:

- if $G = H_q$, then $K(G) = K(H_q, q)$;
- if $G = [(d_1\ G_1)c_1(d_2\ G_2)c_2 \cdots c_{k-1}(d_k\ G_k)]$, then

$$K(G) = K(H_{d_1}(K(G_1))c_1(\cdots c_{k-1}H_{d_k}(K(G_k))\cdots), q)$$

where each c_j represents a tensor product operator $\otimes$ or a torsion product operator $*$. We define then the group π_G associated with G by

$$\pi_G = H_q(K(G))$$

which is given in fact by

- if $G = H_q$, then $\pi_G = H_q = H_q(X)$;
- if $G = [(d_1\ G_1)c_1(d_2\ G_2)c_2 \cdots c_{k-1}(d_k\ G_k)]$, then

$$\pi_G = H_{d_1}(K(G_1))c_1(\cdots c_{k-1} H_{d_k}(K(G_k)) \cdots)$$

One can then immediately observe that $\pi_{p,q}$ can be described as

$$\pi_{p,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n_G(p)}$$

where the function n_G counts the number of times the group π_G appears as a component of $\pi_{p,q} \cong H_q(R^p X)$. Furthermore, it is not difficult to prove in a recursive way that the function $n_G(p)$, for G a genealogy of degree q, is in fact a polynomial in p of degree $< q/2$, a bound which only depends on the ordinate q.

The first “stage” in the normalization process replaces $\pi_{p,q}$ by

$$\pi^1_{p,q} \equiv \pi_{p,q} \cap \operatorname{Ker} \eta^{p-1}, \quad p \geq 1$$

The usual considerations for the decompositions in direct sums deduced from the cosimplicial identity $\eta^{p-1}\partial^p = \mathrm{Id}$ give

$$\pi^1_{p,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n_G(p) - n_G(p-1)}$$

That is to say, a stage of normalization replaces the exponent n_G of π_G in the expression of $\pi_{p,q}$ by its *discrete derivative*, $n_G(p) - n_G(p-1)$.

Iterating the process, the group $\pi^s_{p,q} \equiv \pi_q(R^{p+1}X) \cap \operatorname{Ker} \eta^{p-s} \cap \cdots \cap \operatorname{Ker} \eta^{p-1}$ can be seen as

$$\pi^s_{p,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n^s_G(p)}$$

where the exponent $n^s_G(p)$ is obtained recursively from $n^{s-1}_G(p)$ as the discrete derivative

$$n^s_G(p) = n^{s-1}_G(p) - n^{s-1}_G(p-1)$$

Finally, the normalized group $\pi'_{p,q}$ is given by

$$\pi'_{p,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n^p_G(p)}$$

Since $n_G(p)$ is a polynomial in the variable p, its degree is then decreased by one unit at each stage of normalization. The number of stages of normalization which one must apply to $\pi_{p,q}$ in order to obtain $E^1_{p,q} = \pi'_q(R^{p+1}X) \equiv \pi'_{p,q}$ is exactly p; since the possible degree for n_G is bounded by $q/2$, it follows that $\pi'_{p,q} = 0$ for $q < 2p+2$.

This important property we have proved implies in particular that the bigraded module $E^1 = \{E^1_{p,q}\}_{p,q\in\mathbb{Z}}$ is tapered (that is, $E^1_{p,q} = 0$ if $q < 2p$), which guarantees the convergence of the spectral sequence: given $p, q \in \mathbb{Z}$ then $d^r_{p,q} = 0$ whenever $r > q - 2p - 3$, and if $r > p$ then $d^r_{p-r,q-r+1} = 0$, which implies that $E^\infty_{p,q} = E^r_{p,q}$ for $r > \max\{p, q - 2p - 3\}$.

The convergence of the Bousfield-Kan spectral sequence is in fact well-known, and was proved by other means, for instance in [5] and [6]. Our proof is more elementary but can be useful: it allows one to compute the first level of the spectral sequence, as it will be explained in the next section.

4 Computing the level E^1

As already said in the previous sections, the first level of the Bousfield-Kan spectral sequence associated with a (1-reduced) simplicial set X is given by the normalized homotopy groups:

$$E^1_{p,q} = \pi'_q(\mathcal{R}X^p) = \pi_q(R^{p+1}X) \cap \operatorname{Ker}\eta^0 \cap \cdots \cap \operatorname{Ker}\eta^{p-1}$$

and the first differential map $d^1_{p,q} : E^1_{p,q} \to E^1_{p+1,q}$ is the morphism induced by the alternate sum $\delta^{p+1}_q = \sum_{j=0}^{p+1}(-1)^j \partial^j$.

The group $\pi'_q(\mathcal{R}X^p) = \pi'_q(R^{p+1}X) \equiv \pi'_{p,q}$ can be seen as a direct sum of (groups associated with) genealogies

$$\pi'_{p,q} \cong \bigoplus_{G\in Gen_q} (\pi_G)^{n^p_G(p)}$$

We have also already mentioned that it is possible to construct all the genealogies G of a given degree q, and one can also determine the corresponding enumeration functions $n_G(p)$, and then the iterated derivative $n^p_G(p)$. Furthermore, if the initial groups $H_*(X)$ are known (and they are finitely generated), then the Betti number and the torsion coefficients of each genealogy G (which is constructed by means of tensor and torsion products of finitely generated groups) can also be determined, and in this way we can *compute* the normalized groups $\pi'_q(R^{p+1}X) = E^1_{p,q}$ for every pair (p, q): once our description is understood, the value of $E^1_{p,q}$ is a direct consequence of Cartan's computation for $H_*(K(\pi, k); \mathbb{Z})$.

For instance, let us consider the case where X is the simplicial set $\Omega(S^3) \cup_2 e^3$, obtained from the loop space $\Omega(S^3)$ by attaching a 3-disk by a map $\gamma : S^2 \to \Omega(S^3)$ of degree 2 (several simplicial models can be defined for the loop space constructor; the one used by the Kenzo system is the Kan's simplicial loop functor G, see [13]). For $i \leq 8$, the initial homology groups $H_i \equiv H_i(X)$ are the following: $H_2 = \mathbb{Z}_2$, $H_4 = H_6 = H_8 = \mathbb{Z}$, and $H_i = 0$ if i is odd. A little calculation with the genealogies and the corresponding enumeration functions makes it possible then to compute the normalized groups $\pi'_q(R^{p+1}X) = E^1_{p,q}$ for $q \leq 8$, which satisfy $\pi'_q(R^{p+1}X) = 0$ whenever $q < 2p + 2$.

$$
\begin{array}{l|ccccc}
q & & & & & \\
\mathbb{Z} & \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z} & \mathbb{Z}_8 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^9 & \mathbb{Z}_4 \oplus \mathbb{Z}_2^4 & 0 & \qquad \boxed{r = 1} \\
0 & \mathbb{Z}_2 & \mathbb{Z}_2^4 & 0 & 0 & \\
\mathbb{Z} & \mathbb{Z}_2^3 & \mathbb{Z}_2^2 & 0 & 0 & \\
0 & \mathbb{Z}_2 & 0 & 0 & 0 & \\
\mathbb{Z} & \mathbb{Z}_4 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
\mathbb{Z}_2 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
\hline
0 & 0 & 0 & 0 & 0 & p
\end{array}
$$

Nevertheless, let us observe that in order to compute the groups of the second level of the spectral sequence, $E^2_{p,q} \cong \operatorname{Ker} d^1_{p,q} / \operatorname{Im} d^1_{p-1,q}$, the previous information is not sufficient: the horizontal differential $d^1_{p,q}$ for the groups $(R^{p+1}X)_q$ can be easily described, but defining the induced differential $d^1_{p,q} : E^1_{p,q} \to E^1_{p+1,q}$ requires a *complete description* of the relationship between the homology group $E^1_{p,q} = \pi_q(R^{p+1}X) \cong H_q(R^pX)$ and the original Abelian simplicial group $R^{p+1}X$; this is exactly the role of the *effective* homology, to be studied now.

5 Effective homology of RX

In the following we present the general ideas of the effective homology method, deeply explained in [17] and [18].

Definition 11 A *reduction* ρ between two chain complexes $C_* = (C_n, d_{C_n})_{n \in \mathbb{N}}$ and $D_* = (D_n, d_{D_n})_{n \in \mathbb{N}}$ (denoted $\rho : C_* \Rightarrow\!\!\!\Rightarrow D_*$) is a triple (f, g, h) where: (a) the components f and g are chain complex morphisms $f : C_* \to D_*$ and $g : D_* \to C_*$; (b) the component h is a homotopy operator $h : C_* \to C_{*+1}$ (a graded group homomorphism of degree +1); (c) the following relations are satisfied:(1) $fg = \operatorname{Id}_D$; (2) $d_C h + h d_C = \operatorname{Id}_C - gf$;(3) $fh = 0$; (4) $hg = 0$; (5) $hh = 0$.

Remark 1 These relations express that C_* is the direct sum of D_* and a contractible (acyclic) complex. This decomposition is simply $C_* = \operatorname{Ker} f \oplus \operatorname{Im} g$, with $\operatorname{Im} g \cong D_*$ and $H_*(\operatorname{Ker} f) = 0$. In particular, this implies that the graded homology groups $H_*(C_*)$ and $H_*(D_*)$ are canonically isomorphic.

Definition 12 A (*strong chain*) *equivalence* ε between two chain complexes C_* and D_*, denoted by $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$, is a triple (B_*, ρ_1, ρ_2) where B_* is a chain complex, and ρ_1 and ρ_2 are reductions from B_* over C_* and D_*, respectively.

$$\begin{tikzcd} & B_* \arrow[dl, Rrightarrow, "\rho_1"'] \arrow[dr, Rrightarrow, "\rho_2"] & \\ C_* & & D_* \end{tikzcd}$$

Remark 2 We must use the notion of *effective* chain complex: it is essentially a free chain complex C_* where each group C_n is finitely generated, and a provided algorithm returns a (distinguished) $\mathbb{Z}$-basis in each degree n; in particular, its homology groups are elementarily computable (for details, see [17]).

Definition 13 An *object with effective homology* X is a quadruple $(X, C_*(X), HC_*, \varepsilon)$ where $C_*(X)$ is a chain complex canonically associated with X (which allows us to study the homological nature of X), HC_* is an effective chain complex, and ε is an equivalence $\varepsilon : C_*(X) \Lleftarrow\!\!\Rrightarrow HC_*$.

Remark 3 It is important to understand that in general the HC_* component of an object with effective homology is *not* made of the homology groups of X; this component HC_* is a free $\mathbb{Z}$-chain complex of finite type, in general with a non-null differential, whose homology groups $H_*(HC_*)$ can be determined by means of an elementary algorithm. The equivalence ε allows one to *prove* the isomorphism $H_*(X) := H_*(C_*(X)) \cong H_*(HC_*)$, so that one can compute the homology groups of the initial space X.

In this way, the notion of object with effective homology makes it possible to compute homology groups of complicated spaces by means of homology groups of effective complexes. This method is based on the following idea: given some topological spaces $X_1, \ldots, X_n$, a topological constructor Φ produces a new topological space X. If effective homology versions of the spaces $X_1, \ldots, X_n$ are known, then one should be able to build an effective homology version of the space X, and this version would allow us to compute the homology groups of X.

A typical example of this kind of situation is the loop space constructor. Given a 1-reduced simplicial set X with effective homology, it is possible to determine the effective homology of the loop space $\Omega(X)$, which in particular allows one to compute the homology groups $H_*(\Omega(X))$. Moreover, if X is m-reduced, this process may be iterated m times, producing an effective homology version of $\Omega^k(X)$, for $k \leq m$. This provides a solution for the Adams' problem [1], unsolved by means of traditional methods. In 1956 Frank Adams designed an algorithm computing the homology groups of the first loop space of a 1-reduced simplicial set X. Adams then asked for some analogous solution for the iterated loop space $\Omega^n(X)$. Eighteen years later, Hans Baues [3] gave a solution for the second loop space $\Omega^2(X)$; it depends on an ingenious possible geometrical model for the second loop space; but again it is not possible to extend this model to the third loop space $\Omega^3(X) \ldots$. The problem is in fact in the non-constructive nature of Adams' solution for the first loop space. As previously explained, the effective homology method solves the problem. Effective homology versions are also known for classifying spaces or total spaces of fibrations, see [18] for details.

One can also try to use this technique in order to determine a *constructive* version of the Bousfield-Kan spectral sequence, and the first step to this aim is the determination of an effective homology version of the constructor R. Given X a 1-reduced simplicial set with effective homology, our problem consists in obtaining the effective homology of the simplicial Abelian group RX, that is to say, an equivalence of the form $C_*(RX) \Longleftrightarrow HR_*$, where HR_* must be an effective chain complex. The rest of this section includes a brief explanation of the construction of an algorithm computing this effective homology, which has been obtained as the composition of two equivalences.

From now on in this section all the chain complexes associated with simplicial sets are normalized, that is to say, only the non-degenerate simplices are considered as generators. In particular, $\widetilde{C}_*(X)$ will denote the reduced normalized chain complex associated with a simplicial set X.

Let us suppose that X is a 1-reduced simplicial set with effective homology; an equivalence

$$\begin{tikzcd} & DX_* \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ C_*(X) & & HX_* \end{tikzcd}$$

is given, where HX_* is an effective chain complex.

The main ingredient in the computation of the effective homology of the free simplicial Abelian group RX is the Dold-Kan correspondence [13] between the categories $\mathcal{A}$ of simplicial Abelian groups and $\mathcal{C}$ of (positive) chain complexes, given by the functors $\Gamma : \mathcal{C} \to \mathcal{A}$ and $N_* : \mathcal{A} \to \mathcal{C}$ which satisfy $\Gamma \circ N_* \equiv \mathrm{Id}_{\mathcal{A}}$ and $N_* \circ \Gamma \equiv \mathrm{Id}_{\mathcal{C}}$.

Let us begin the description of our construction for the effective homology of RX, which consists in the composition of two equivalences. We start by briefly explaining the left equivalence, based on the following two results:

Proposition 2 *Given a simplicial set X, there exists an explicit isomorphism*

$$RX \cong \Gamma(\widetilde{C}_*(X))$$

Proposition 3 *Let C_* and D_* be chain complexes and $\rho : C_* \Rightarrow D_*$ a reduction between them. Then one can construct a new reduction*

$$\Gamma(\rho) : C_*(\Gamma(C_*)) \Rightarrow C_*(\Gamma(D_*))$$

The proofs of both propositions are not complicated and can be found in [15].

We consider now the given effective homology of X, that is, the equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$. It is not difficult to construct a new equivalence

$$\begin{tikzcd}[column sep=small] & \widetilde{DX_*} \arrow[dl, Rightarrow, two heads] \arrow[dr, Rightarrow, two heads] & \\ \widetilde{C}_*(X) & & \widetilde{HX_*} \end{tikzcd}$$

where $\widetilde{C}_*(X)$ is the reduced (normalized) chain complex associated with X, $\widetilde{DX_*}$ and $\widetilde{HX_*}$ are chain complexes easily deduced from DX_* and HX_*, respectively, and $\widetilde{HX_*}$ is effective and null in dimensions 0 and 1. Composing then the results of Propositions 2 and 3 we obtain the following equivalence:

$$\begin{tikzcd}[column sep=small] & C_*(\Gamma(\widetilde{DX_*})) \arrow[dl, Rightarrow, two heads] \arrow[dr, Rightarrow, two heads] & \\ C_*(RX) \cong C_*(\Gamma(\widetilde{C}_*(X))) & & C_*(\Gamma(\widetilde{HX_*})) \end{tikzcd}$$

This will be the left equivalence μ_L in the effective homology of the simplicial Abelian group RX. We have obtained in this way an algorithm which has the following input and output.

Algorithm 1 *Input:*
- a 1-reduced pointed simplicial set X,
- an equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$, where HX_* is an effective chain complex.

Output: an equivalence $\mu_L : C_*(RX) \Lleftarrow C_*(\Gamma(\widetilde{DX_*})) \Rrightarrow C_*(\Gamma(\widetilde{HX_*}))$, where $\widetilde{DX_*}$ and $\widetilde{HX_*}$ are obtained from DX_* and HX_*, respectively, $\widetilde{HX_*}$ is effective and $\widetilde{HX}_0 = \widetilde{HX}_1 = 0$.

In order to determine the effective homology of RX, a second (right) equivalence $\mu_R : C_*(\Gamma(\widetilde{HX_*})) \Lleftarrow\!\!\Rrightarrow HR_*$ is necessary, HR_* being an effective chain complex. In other words, we need to compute the effective homology of the simplicial Abelian group $\Gamma(\widetilde{HX_*})$. More generally, we will determine the effective homology of $\Gamma(E_*)$ for a *general* effective chain complex E_* which is null in degrees 0 and 1.

As explained in [8], an effective chain complex E_* is isomorphic to a direct sum:

$$E_* \cong \bigoplus_k C_*^k$$

where each C_*^k is *elementary*, that is to say, there exists $m = m(k) \in \mathbb{N}$ such that $C_n^k = 0$ for $n \neq m, m+1$, $C_m^k \cong \mathbb{Z}$, and $d_{m+1} : C_{m+1}^k \to C_m^k$ is monomorphic (which implies that $C_{m+1}^k \cong \mathbb{Z}$ or $C_{m+1}^k \cong 0$).

Since the functor Γ has a good behavior with respect to the direct sum of chain complexes, when applying it to the chain complex E_* we obtain

$$\Gamma(E_*) \cong \Gamma\left(\bigoplus_k C_*^k\right) \cong \bigoplus_k \Gamma(C_*^k)$$

In general the infinite direct sum of a family of simplicial Abelian groups does not coincide with the corresponding Cartesian product, but one must bear in mind that in this case we have special properties. Since each group E_n is a sum of a finite number of components $C_n^{k_i^n}$, one has

$$\Gamma(E_*) \cong \bigoplus_k \Gamma(C_*^k) \cong \prod_k \Gamma(C_*^k)$$

The effective homology of a finite Cartesian product can be computed when the effective homologies of the components are known (see [18] for details). This method cannot always be generalized to an infinite Cartesian product, but in this case the result holds since the product is finite in each degree. In this way, in order to compute the effective homology of $\Gamma(E_*)$, we need to determine the effective homology of $\Gamma(C_*)$, for C_* an elementary chain complex. Two different cases must be considered.

First of all, let C_* be an elementary chain complex such that $C_n = 0$ for all $n \neq m$ and $C_m = \mathbb{Z}$ for some $m \geq 2$, denoted by $C_* = C_*(\mathbb{Z}, m)$. Then

$$\Gamma(C_*) = \Gamma(C_*(\mathbb{Z}, m))$$

and this is in fact one of the possible models for the Eilenberg-MacLane space $K(\mathbb{Z}, m)$ (see [13]). On the other hand, the space $K(\mathbb{Z}, m)$ is known to be an object with effective homology for every $m \geq 1$, and therefore we can suppose that an equivalence $C_*(K(\mathbb{Z}, m)) \Longleftarrow\!\!\!\Rightarrow HK_*^m$ is available, HK_*^m being an effective chain complex.

In the second case to be considered, the elementary chain complex C_* is of the form

$$0 \longleftarrow 0 \longleftarrow \cdots \longleftarrow 0 \longleftarrow \mathbb{Z} \overset{d_{m+1}}{\longleftarrow} \mathbb{Z} \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

where the only non-null differential map $d_{m+1} : \mathbb{Z} \to \mathbb{Z}$ is given by $d_{m+1}(1) = t$ for some $t \in \mathbb{Z}$.

One can easily observe that this chain complex can be expressed as the Cone of the morphism

$$f : C_*(\mathbb{Z}, m+1) \longrightarrow C_*(\mathbb{Z}, m+1)$$

defined by $f(1) = d_{m+1}(1) = t$, and the following *effective short exact sequence* [18] is obtained:

$$0 \longrightarrow \mathrm{Desusp}_*(C_*(\mathbb{Z}, m+1)) \underset{r}{\overset{i}{\rightleftarrows}} \mathrm{Cone}(f)_* \underset{s}{\overset{j}{\rightleftarrows}} C_*(\mathbb{Z}, m+1) \longrightarrow 0$$

where Desusp_* is the *Desuspension* constructor; in this case one can observe that $\mathrm{Desusp}_*(C_*(\mathbb{Z}, m+1))$ is equal to $C_*(\mathbb{Z}, m)$.

Now, what happens when we apply the functor Γ to a short exact sequence? The following proposition provides the answer.

Proposition 4 *Given an effective short exact sequence of chain complexes*

$$0 \longrightarrow A_* \underset{r}{\overset{i}{\rightleftarrows}} B_* \underset{s}{\overset{j}{\rightleftarrows}} C_* \longrightarrow 0$$

there exists an explicit isomorphism between the simplicial Abelian group $\Gamma(B_*)$ and a twisted Cartesian product $\Gamma(A_*) \times_\tau \Gamma(C_*)$.

Proof The arrows i and j are compatible with the differentials, but of course the section s and the retraction r in general are not. These section and retraction produce a connection morphism $\partial := r d_B s$, which *is* compatible with the differentials if you do not forget the degree shift and the Koszul rule about signs; this connection morphism naturally induces the twisting function τ defining the principal fibered structure.

In our particular case, we have the fibration

$$\Gamma(C_*(\mathbb{Z}, m)) \hookrightarrow \Gamma(\mathrm{Cone}(f)_*) \to \Gamma(C_*(\mathbb{Z}, m+1))$$

where the total space $\Gamma(\mathrm{Cone}(f)_*) \cong \Gamma(C_*)$ can be expressed as a twisted product $\Gamma(C_*(\mathbb{Z}, m)) \times_\tau \Gamma(C_*(\mathbb{Z}, m+1))$. Recalling now that $\Gamma(C_*(\mathbb{Z}, m)) = K(\mathbb{Z}, m)$ and $\Gamma(C_*(\mathbb{Z}, m+1)) = K(\mathbb{Z}, m+1)$, one has

$$\Gamma(C_*) \cong K(\mathbb{Z}, m) \times_\tau K(\mathbb{Z}, m+1)$$

Provided that $K(\mathbb{Z}, m)$ and $K(\mathbb{Z}, m+1)$ are objects with effective homology (and $K(\mathbb{Z}, m+1)$ is 1-reduced since $m \geq 2$), the effective homology of the total space of the fibration $\Gamma(C_*)$ can be computed (see [18]). The effective homology of $\Gamma(C_*)$ has then been determined for the two different types of elementary chain complexes.

In this way one can obtain the effective homology of $\Gamma(E_*)$, which is a Cartesian product of several factors $\Gamma(C_*)$ for C_* elementary, so that we obtain a second algorithm with the following input and output.

Algorithm 2 *Input:* an effective chain complex E_* such that $E_0 = E_1 = 0$.
Output: an equivalence $C_*(\Gamma(E_*)) \Lleftarrow D\Gamma E_* \Rrightarrow H\Gamma E_*$, where $H\Gamma E_*$ is an effective chain complex.

This algorithm can be applied in particular to the effective chain complex $\widetilde{HX}_*$ deduced from HX_* (which satisfies $\widetilde{HX}_0 = \widetilde{HX}_1 = 0$), producing the looked-for right equivalence μ_R for the effective homology of RX

$$\begin{array}{ccc} & \widetilde{DR}_* & \\ \swarrow\!\!\swarrow & & \searrow\!\!\searrow \\ C_*(\Gamma(\widetilde{HX}_*)) & & HR_* \end{array}$$

Once we have developed algorithms for the left and right equivalences, the last step for the construction of the effective homology of RX consists simply in composing μ_L and μ_R.

Algorithm 3 *Input:*
- a 1-reduced pointed simplicial set X,
- an equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$, where HX_* is an effective chain complex.

Output: an equivalence $C_*(RX) \Lleftarrow DR_* \Rrightarrow HR_*$, where HR_* is effective.

This algorithm can be considered as the main result of this paper. The homology groups $H_*(R^{p+1}X)$ leading to the $E^1_{p,q}$ groups of the Bousfield-Kan spectral sequence can be obtained through Cartan's algorithm [7], but installing the horizontal differential $d^1_{p,q}$ between them requires much more: this is just the role of Algorithm 3, as explained in the next section.

6 Computing the level E^2

Let us recall again that the level E^1 of the Bousfield-Kan spectral sequence is given by the normalized groups

$$E^1_{p,q} = \pi'_{p,q} = \pi_q(R^{p+1}X) \cap \text{Ker}\,\eta^0 \cap \cdots \cap \text{Ker}\,\eta^{p-1}$$

We consider now the isomorphism $\pi_*(RX) \cong \widetilde{H}_*(X)$, satisfied by any simplicial set X, which implies $\pi_*(R^{p+1}X) \cong \widetilde{H}_*(R^pX)$ for every $p > 1$. Hence one has

$$E^1_{p,q} \cong \widetilde{H}_q(R^pX) \cap \text{Ker}\,\eta^0 \cap \cdots \cap \text{Ker}\,\eta^{p-1}$$

If X is a 1-reduced simplicial set with effective homology, then our Algorithm 3 provides us the effective homology of the simplicial Abelian group RX. Iterating the process (taking into account that RX is also 1-reduced), it is possible to obtain the effective homology of R^pX for every $p \geq 1$, that is, an equivalence of the form $C_*(R^pX) \Lleftarrow\Rrightarrow HR^p_*$. In this way, if X is an object with effective homology, this implies the groups $\pi_q(R^{p+1}X) \cong \widetilde{H}_q(R^pX) \cong \widetilde{H}_q(HR^p_*)$ (with the corresponding generators) are computable.

The codegeneracy maps η^j are well-defined on these homotopy groups:

$$\pi_q(\eta^j) \equiv \eta^j : \pi_q(R^{p+1}X) \longrightarrow \pi_q(R^pX) \quad 0 \leq j \leq p-1$$

and as far as $\pi_q(R^{p+1}X)$ and $\pi_q(R^pX)$ are groups of finite type, these maps can be expressed as finite integer matrices. Therefore, the kernels $\ker \eta^j$ can be computed by means of elementary operations for each $0 \leq j \leq p-1$, and in this way one can determine the normalized groups $\pi'_{p,q} = E^1_{p,q}$ (with generators).

The differential map $d^1_{p,q} : E^1_{p,q} \to E^1_{p+1,q}$ is induced by $\delta^{p+1}_q = \sum_{j=0}^{p+1}(-1)^j \partial^j$. Then, for a class $[x] \in E^1_{p,q}$ (given by means of the coefficients with respect to the generators of the group) it is possible to compute the image $d^1_{p,q}([x]) = [\delta^{p+1}_q(x)]$

in $E^1_{p+1,q}$. For this computation, it is enough to follow the path provided by Algorithm 3 (A_3):

$$
\begin{tikzcd}
E^1_{p,q} \arrow[r, Mapsto] \arrow[d, "d^1_{p,q}"] & H_q(HR^p_*) \arrow[r, Mapsto] & HR^p_q \arrow[r, Mapsto, "A_3"] & C_q(R^pX) \arrow[r, Mapsto] & R^{p+1}X \arrow[d, "\{\partial^j\}_{0\le j\le p+1}"] \\
E^1_{p+1,q} & H_q(HR^{p+1}_*) \arrow[l, Mapsto] & HR^{p+1}_q \arrow[l, Mapsto] & C_q(R^{p+1}X) \arrow[l, Mapsto, "A_3"'] & R^{p+2}X \arrow[l, Mapsto]
\end{tikzcd}
$$

This implies that, if X is an object with effective homology, the first level of the Bousfield-Kan spectral sequence is computable.

Algorithm 4 *Input:*
- a 1-reduced pointed simplicial set X,
- an equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$, where HX_* is an effective chain complex.

Output:
- the groups $E^1_{p,q} = \pi'_q(R^{p+1}X)$ for each $p, q \in \mathbb{Z}$,
- the differential maps $d^1_{p,q}$ for all $p, q \in \mathbb{Z}$.

Let us remark now that the differential maps $d^1_{p,q} : E^1_{p,q} \to E^1_{p+1,q}$ can also be expressed as finite integer matrices. Therefore it is possible to determine their kernel and their image, and using the Smith Normal Form technique [11] we can easily compute the quotient groups

$$E^2_{p,q} = \frac{\operatorname{Ker} d^1_{p,q}}{\operatorname{Im} d^1_{p-1,q}}$$

We obtain so the algorithm to which this paper is devoted.

Algorithm 5 *Input:*
- a 1-reduced pointed simplicial set X,
- an equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$, where the chain complex HX_* is effective.

Output: the groups $\{E^2_{p,q}\}_{p,q\ge 0}$ of the Bousfield-Kan spectral sequence canonically associated with X.

Although the groups $E^2_{p,q}$ of the Bousfield-Kan spectral sequence do not determine in general all the homotopy groups of the space X, in some particular cases they make it possible to obtain some of them. For instance, for the space $X = \Omega(S^3) \cup_2 e^3$ which we have considered in Sect. 6, we have already seen that for $n = q - p \le 3$ the unique groups $E^1_{p,q}$ which are non-null are $E^1_{0,2} = \mathbb{Z}_2$ and $E^1_{1,4} = \mathbb{Z}_4$. The differential $d^1_{0,4} : E^1_{0,4} = \mathbb{Z} \to E^1_{1,4} = \mathbb{Z}_4$ could be different to zero, so that $E^2_{1,4} \ne E^1_{1,4}$; in any case, the group $E^2_{1,4}$ can be determined with our Algorithm 5. Moreover, there is no higher differential which can start or finish in $E^2_{1,4}$; consequently, this must be the final group of the spectral sequence, $E^\infty_{1,4} = E^2_{1,4}$. Similarly, $E^\infty_{0,2} = E^2_{0,2} = E^1_{0,2} = \mathbb{Z}_2$. It is thus clear that from the groups $E^2_{p,q}$ one can deduce the homotopy groups $\pi_n(X)$

for $n \leq 3$, which are $\pi_0(X) = \pi_1(X) = 0, \pi_2(X) = E^2_{0,2} = E^1_{0,2} = \mathbb{Z}_2$ and $\pi_3(X) = E^2_{1,4} \subseteq \mathbb{Z}_4$, which could be computed once our Algorithm 5 will be implemented. For $n = 4$, all the groups $E^2_{p,q}$ are again the final groups of the spectral sequence, but here extension problems could appear to determine the homotopy group $\pi_4(X)$. When trying to obtain $\pi_5(X)$, some differential maps d^3 could be non-null, so that our Algorithm 5 would not be sufficient. For the case of the 2-sphere $X = S^2$, the level 2 of the spectral sequence allows one to determine the homotopy groups $\pi_n(S^2)$ for $n \leq 5$. See [15] for details.

The implementation of our Algorithms 4 and 5 has not been finished yet. As one of the main ingredients, it will be necessary to develop some programs dealing with the computation of the effective homology of Eilenberg-MacLane spaces $K(\pi, n)'s$, a difficult problem by itself. This suggests that the final computations could be often slow. In some cases, days or even weeks could be necessary to determine the desired groups $E^2_{p,q}$.

7 Conclusions and further work

Given a 1-reduced simplicial set X *with effective homology*, Algorithm 4 produces the entire page $\{E^1_{p,q}, d^1_{p,q}\}$ of the Bousfield-Kan spectral sequence converging to the homotopy groups of X. The key point in this work is the fact that the *effective* homology of the magical—magical but enormous!—simplicial groups $R^{p+1}X$ allow us to compute the differentials $d^1_{p,q}$, while classical Cartanć6s algorithm about Eilenberg-MacLane spaces [7] easily gives the groups $E^1_{p,q}$ but without giving any information about the differentials $d^1_{p,q}$.

The page 1 of the spectral sequence being so computed, differentials included, elementary computations produce the *groups* of the next page, that is, the collection $\{E^2_{p,q}\}$; it is Algorithm 5, the main goal of this paper.

Algorithms 4 and 5 are not yet concretely implemented as computer programs. No doubt at all about the feasibility of such a concrete implementation: these new algorithms have the same general style as numerous other algorithms already implemented in the Kenzo program [9], already producing striking results, in particular around complicated loop spaces. The main component of the pending work consists in writing down a *good* implementation of a *good* algorithm computing the *effective* homology of $K(\mathbb{Z}, n)$, an interesting subject by itself: it is easy to prove the computation of the *effective* homology of a simplicial group $R^{p+1}X$ can be reduced to the same problem for the main Eilenberg-MacLane spaces $K(\mathbb{Z}, n)$, see Sect. 5.

What about the next groups $E^r_{p,q}$ for $r \geq 3$ and the next differentials $d^r_{p,q}$ for $r \geq 2$? Of course the main problem is for these differentials. We think it is possible to extend the work of this paper in this direction. The striking Bousfield- Kan paper [6] gives a very precise description of the higher differentials, in a sense "simplex by simplex", see [6, Sect. 5]. A reader having understood the spirit of this technique called *effective homology* should be reasonably convinced a careful study of [6] can lead to a *complete* knowledge of the Bousfield-Kan spectral sequence of a 1-reduced simplicial set. In fact effective homology consists in *permanently* keeping precise relations between homology groups and the initial spaces, even when the latter are not of finite type; and it seems it is exactly the information to be used to concretely exploit the nice paper [6] of Pete Bousfield and Daniel Kan, in particular its fascinating Sect.5.

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