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Effective Homotopy of fibrations

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Abstract The well-known effective homology method provides algorithms computing homology groups of spaces. The main idea consists in keeping systematically a deep and subtle connection between the *homology* of any object and the object itself. Now applying similar ideas to the computation of *homotopy groups*, we aim to develop a new *effective homotopy* theory which allows one to determine homotopy groups of spaces. In this work we introduce the notion of *a solution for the homotopical problem of a simplicial set*, which will be the main definition of our theory, and present an algorithm computing the effective homotopy of a fibration. We also illustrate with examples some applications of our results.

Keywords Constructive Algebraic Topology · Homotopy groups · Fibrations · Spectral sequences

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1 Introduction

The homology groups of topological spaces can be difficult to reach, for example when loop spaces or classifying spaces are involved. In particular, knowing the homology groups of a topological group or space does not imply that the homology groups of its classifying space or loop space can also be determined. In the same way, given a fibration $F \hookrightarrow E \to B$, there does not exist a general algorithm computing the homology groups of E from the homology groups of B and F.

The methods of *effective homology* (introduced in [14] and explained in depth in [12] and [13]) solve the previous problem and give their user *algorithms* for computing for example the homology groups of the total space of a fibration, of an arbitrarily iterated loop space (Adams' problem), of a classifying space, etc. One of the main ideas in this method is the notion of *a solution for the homological problem* of a chain complex [15], which consists of four algorithms describing in a *constructive* way the homology groups of a space, which is said to have *effective homology*. These algorithms produce in particular the homology groups of the chain complex but they give also some additional information which is necessary if we want to use the space inside more complicated constructors. The effective homology method has been concretely implemented in the Kenzo system [6], a Common Lisp program developed by Francis Sergeraert and some coworkers which has made it possible to compute some complicated homology groups so far unreachable.

The computation of homotopy groups is even harder than homology and is in fact one of the most challenging problems in the field of Algebraic Topology. In 1953 Serre obtained a general *finiteness result* [16] which states that, if X is a simply connected space such that the homology groups $H_n(X;\mathbb{Z})$ are of *finite type*, then the homotopy groups $\pi_n(X)$ are also Abelian groups of finite type. In 1957, Edgar Brown published in [4] a *theoretical* algorithm for the computation of these groups, based on the Postnikov tower and making use of finite approximations of infinite simplicial sets, transforming in this way the finiteness results of Serre into a computability result. Nevertheless, Edgar Brown himself quoted in his paper that his algorithm has no practical use, even with the most powerful computer you can imagine: it is a consequence of the hyper-exponential complexity of the algorithm designed by Brown. Other theoretical methods have been also designed trying to determine homotopy groups of spaces, but up to our knowledge there does not exist a real implementation in a computer of a general algorithm producing the homotopy groups of a space.

Inspired by the fundamental ideas of the effective homology method, we try now to develop an *effective homotopy* theory, which would allow the computation of *homotopy* groups of spaces. The most important notion will be that of *a solution for the homotopical problem* of a simplicial set. As in the case of homology, we will start with some spaces whose effective homotopy can be directly determined, and then different constructors of Algebraic Topology should produce new spaces with effective homotopy. As a first work in this research, we have developed some results allowing in particular to determine homotopy groups of fibrations when the fiber and the base spaces are objects with effective homotopy.

The paper is organized as follows. Section 2 begins with some elementary definitions and ideas about simplicial sets and homotopy groups. In Section 3 we present the notion of *a solution for the homotopical problem of a simplicial set*, which will be the main ingredient of our theory. As a first important result, an algorithm determining a solution for the homotopical problem of a Kan fibration is constructed in Section 4. Section 5 contains some examples of applications of the previous algorithm. The paper ends with a section of conclusions and further work.

2 Preliminaries

In this section we introduce some elementary ideas about simplicial sets, which can be considered a useful combinatorial model for topological spaces. More concretely, given a Kan simplicial set K with a base point $\star \in K_0$, an algebraic definition of the homotopy groups of K can be given such that they are isomorphic to the homotopy groups of the corresponding topological space by means of the *realization functor*. All the definitions and results of this section (and details about the connection of simplicial sets and topological spaces) can be found in [7].

Definition 1 A *simplicial set* K is a simplicial object over the category of sets, that is to say, K consists of

- a set K_n for each integer $n \geq 0$;
- for every pair of integers (i, n) such that $0 \leq i \leq n$, *face* and *degeneracy* maps $\partial_i : K_n \to K_{n-1}$ and $\eta_i : K_n \to K_{n+1}$ satisfying the *simplicial identities*:

$$\begin{aligned}
\partial_i \partial_j &= \partial_{j-1} \partial_i && \text{if } i < j \\
\eta_i \eta_j &= \eta_{j+1} \eta_i && \text{if } i \leq j \\
\partial_i \eta_j &= \eta_{j-1} \partial_i && \text{if } i < j \\
\partial_i \eta_j &= \text{Id} && \text{if } i = j, j+1 \\
\partial_i \eta_j &= \eta_j \partial_{i-1} && \text{if } i > j+1
\end{aligned}$$

The elements of K_n are called the *n-simplices* of K.

Definition 2 A simplicial set K is said to satisfy the *extension condition* if for every collection of $n+1$ n-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ which satisfy the compatibility condition $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j, i \neq k$, and $j \neq k$, there exists an $(n+1)$-simplex $x \in K_{n+1}$ such that $\partial_i x = x_i$ for every $i \neq k$. A simplicial set which satisfies the extension condition is called a *Kan simplicial set*.

Definition 3 Let K be a simplicial set. Two n-simplices x and y of K are said to be *homotopic*, written $x \sim y$, if $\partial_i x = \partial_i y$ for $0 \le i \le n$, and there exists an $(n+1)$-simplex z such that $\partial_n z = x, \partial_{n+1} z = y$, and $\partial_i z = \eta_{n-1}\partial_i x = \eta_{n-1}\partial_i y$ for $0 \le i < n$.

If K is a Kan simplicial set, then $\sim$ is an equivalence relation on the set of n-simplices of K for every $n \ge 0$.

Let $\star \in K_0$ be a 0-simplex of K, called a *base point*; we also denote by $\star$ the degeneracies $\eta_{n-1} \dots \eta_0 \star \in K_n$ for every n. We define $S_n(K)$ as the set of all $x \in K_n$ such that $\partial_i x = \star$ for every $0 \le i \le n$, which is said to be the set of *n-spheres* of K.

Definition 4 Given a Kan simplicial set K and a base point $\star \in K_0$, we define

$$\pi_n(K, \star) \equiv \pi_n(K) = S_n(K)/(\sim)$$

The set $\pi_n(K, \star)$ admits a group structure for $n \ge 1$ and it is Abelian for $n \ge 2$. It is called the *n-homotopy group* of K.

The Kan simplicial sets K which will be considered in this paper will be *connected* simplicial sets, that is to say, such that $\pi_0(K)$ has only one homotopy class, $\pi_0(K) = \{\star\}$.

Definition 5 A Kan simplicial set K is said to be *minimal* if, for every pair of n-simplices $x, y \in K_n$ and $0 \le k \le n$ such that $\partial_i x = \partial_i y$ for all $0 \le i \le n$ with $i \ne k$, then $\partial_k x = \partial_k y$.

It can be proved that a Kan simplicial set K is minimal if and only if $x \sim y$ implies $x = y$, so that each homotopy class has only one element.

Definition 6 Let $p : E \to B$ be a simplicial map. The map p is a *Kan fibration* if for every collection of $n+1$ n-simplices $x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}$ of E which satisfy the compatibility condition $\partial_i x_j = \partial_{j-1} x_i, i < j, i \ne k, j \ne k$, and for every $(n+1)$-simplex y of B such that $\partial_i y = p(x_i), i \ne k$, there exists an $(n+1)$-simplex x of E such that $\partial_i x = x_i, i \ne k$, and $p(x) = y$. E is called the *total complex* and B is the *base complex*. If Φ denotes the simplicial set generated by a vertex of B (usually the base point $\star$), then $F = p^{-1}(\Phi)$ is called the *fiber* over Φ.

We finish this section by introducing two interesting examples of Kan simplicial sets which will appear in our work.

Definition 7 Let π be a group. An Eilenberg-MacLane space $K(\pi, n)$ is a minimal Kan complex K such that $\pi_n(K) \cong \pi$ and $\pi_i(K) = 0$ for $i \ne n$.

Given an Abelian group π and $n \ge 1$, there exist several models which define Eilenberg-MacLane spaces $K(\pi, n)$, but all of them are isomorphic (and are simplicial Abelian groups, which means that each component $K(\pi, n)_i$ is an Abelian group and the face and degeneracy operators are compatible with the group operation). If π is not Abelian, then one can only construct $K(\pi, n)$ for $n = 1$. See [7] for details.

Definition 8 The *standard simplex* Δ is a simplicial set built as follows. An n-simplex of Δ is any $(n+1)$-tuple $(a_0, \ldots, a_n)$ of integers such that $0 \leq a_0 \leq \cdots \leq a_n$, and the face and degeneracy operators are defined as

$$\partial_i(a_0, \ldots, a_n) = (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$$
$$\eta_i(a_0, \ldots, a_n) = (a_0, \ldots, a_i, a_i, a_{i+1} \ldots, a_n)$$

3 The homotopical problem of a Kan simplicial set

Although in Section 2 we have introduced a combinatorial definition for the homotopy groups of a Kan simplicial set K, in general this *formal* notion is not sufficient to produce an algorithm *computing* these groups. In some *good* situations, some theoretical reasoning could lead to claim that a homotopy group $\pi_n(K)$ is equal to some given group, for instance $\pi_6(K) = \mathbb{Z}_{12}$. The result $\pi_6(K)$ "=" $\mathbb{Z}_{12}$ is in fact a shorthand for the more precise statement: there exists an isomorphism between the group $\pi_6(K)$ and the well-known group $\mathbb{Z}_{12}$, to be considered as a preferred representative of an isomorphism class. But it is exceptional this proof is *constructive*, we mean such an isomorphism is rarely made explicit. If we then intend to determine some unknown homology or homotopy group H after calculations involving the intermediate group $\pi_6(K)$ (and perhaps other intermediate groups), if the result about the computed $\pi_6(K)$ is not constructive, then often the alleged algorithm $\pi_6(K) \to H$ in fact fails: maybe for example some necessary differential in a spectral sequence is in fact out of scope, or some extension problem is unsolvable with the available information. The effective homotopy method tries to avoid this problem by defining in a precise way what a *constructive* solution for the problem of computing the homotopy groups of a space is.

Let K be a simplicial set satisfying the Kan extension condition (Definition 2). Let us observe that the existence of the $(n+1)$-simplex $x \in K$ for each collection of n-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ satisfying the compatibility condition does not imply it is always possible to obtain it. We say that the Kan simplicial set K is *constructive* if the desired x is given explicitely by an algorithm.

Definition 9 A *constructive Kan simplicial set* is a simplicial set K together with an algorithm σ_K, which given an integer n, an index k, and a list of $n+1$ n-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ such that $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j, i \neq k$, and $j \neq k$, *returns* an $(n+1)$-simplex $x \in K_{n+1}$ with $\partial_i x = x_i$ for every $i \neq k$.

Although the word *constructive* is sometimes omitted, all Kan simplicial sets which appear in this paper are supposed to be constructive Kan simplicial sets. Furthermore, since we aim to work with the groups $\pi_*(K)$ in a constructive way, we only consider Kan simplicial sets whose homotopy groups $\pi_*(K)$ are Abelian groups of finite type. One could also consider simplicial sets K where $\pi_1(K)$ is not Abelian ($\pi_i(K)$ is always Abelian for $i \geq 2$), but we have decided to focus on the Abelian situation because, as we will see later, this is the case of our examples of applications (more concretely, we will usually work with simply connected simplicial sets, that is, such that $\pi_1(K) = 0$).

Definition 10 A *solution for the homotopical problem* (SHmtP) posed by a constructive Kan simplicial set K is a graded 4-tuple $(\pi_n, f_n, g_n, h_n)_{n\geq 1}$ where:

- The component π_n is a standard presentation of a finitely generated Abelian group (that is to say, each π_n is a direct sum of several copies of the infinite cyclic group $\mathbb{Z}$ and some finite cyclic groups $\mathbb{Z}_{p_i^n}$, $\pi_n = \mathbb{Z}^{\alpha_n} \oplus \mathbb{Z}_{p_1^n}^{\beta_1^n} \oplus \cdots \oplus \mathbb{Z}_{p_r^n}^{\beta_r^n}$. The component π_n is therefore a well-known group which is given and where computations can be done). As we will see later, this group will be isomorphic to the desired homotopy group $\pi_n(K) = S_n(K)/(\sim)$.
- The component g_n is an algorithm $g_n : \pi_n \to S_n(K)$ giving for every "abstract" homotopy class $a \in \pi_n$ a sphere $x = g_n(a) \in S_n(K)$ representing this homotopy class.
- The component f_n is an algorithm $f_n : S_n(K) \to \pi_n$ computing for every sphere $x \in S_n(K)$ "its" homotopy class $a = f_n(x) \in \pi_n$. This algorithm f_n must satisfy the following properties. First of all, the composition $f_n g_n$ must be the identity of π_n. Moreover, given $z \in K_{n+1}$ such that $\partial_i z = \star$ for all $0 \leq i \leq n$, then $f_n(\partial_{n+1} z) = 0$; in other words, $f_n(x) = 0$ for all $x \in S_n(K)$ with $x \sim \star$. Furthermore, f must be a "group" morphism, in the following sense: given $x, y \in S_n(K)$ and $w =$"$x + y$" representative of the homotopy class $[x] + [y]$ (computed by using the Kan property of K as explained in [7]), then $f_n(w) = f_n(x) + f_n(y)$.
- The component h_n is an algorithm $h_n : \ker f_n \to K_{n+1}$ satisfying $\partial_i h_n = \star$ for all $0 \leq i \leq n$ and $\partial_{n+1} h_n = \mathrm{Id}_{\ker f_n}$. This algorithm produces a *certificate* for a sphere $x \in S_n(K)$ claimed having a null homotopy class by the algorithm f_n.

If K is a Kan simpicial set and a solution for its homotopical problem is given, we say the K is an *object with effective homotopy*. The interesting point of this definition is the fact that, if K has effective homotopy, one can easily construct an algorithm computing the homotopy groups $\pi_n(K)$.

Proposition 1 *Let K be a constructive Kan simplicial set, $(\pi_n, f_n, g_n, h_n)_{n\geq 1}$ a solution for the homotopical problem of K. Then, for each $n \geq 1$, the homotopy group $\pi_n(K) = S_n(K)/(\sim)$ is isomorphic to the given group π_n.*

Proof The isomorphism is given by a map $\phi : \pi_n(K) = S_n(K)/(\sim) \to \pi_n$ defined as $\phi[x] = f_n(x)$ and its inverse $\psi : \pi_n \to \pi_n(K) = S_n(K)/(\sim)$ constructed as the composition of g_n with the projection to the corresponding quotient. Both maps are well-defined morphisms of groups and provide an explicit isomorphism between the "formal" group $\pi_n(K) = S_n(K)/(\sim)$ and the group π_n.

In this way, a solution for the homotopical problem of a simplicial set allows one to determine its homotopy groups. More concretely, from the given 4-tuple

$(\pi_n, f_n, g_n, h_n)_{n\geq 1}$ we can obtain a standard presentation of the groups (that is, the component π_n) and also the generators, which could be useful when these groups are involved in other constructions. Now the problem is: how can we determine a solution for the homotopical problem of a given Kan simplicial set K?

In some simple situations, the effective homotopy of a space can be determined in an easy way. This is the case, for example, of Eilenberg-MacLane spaces $K(\pi, n)$ for finitely generated Abelian groups π or the standard simplex Δ. It is not difficult to observe that both of them are constructive Kan simplicial sets (see [7]).

Proposition 2 *Let π be a finitely generated Abelian group $\pi = \mathbb{Z}^\alpha \oplus \mathbb{Z}_{p_1}^{\beta_1} \oplus \cdots \oplus \mathbb{Z}_{p_r}^{\beta_r}$, and $K = K(\pi, n)$. Then one can define a tuple $(\pi_n, f_n, g_n, h_n)_{n\geq 1}$ which provides a solution for the homotopical problem of K.*

Proof By the definition of the space $K(\pi, n)$, one has $\pi_n(K(\pi, n)) \cong \pi$ and $\pi_i(K(\pi, n)) = 0$ for every $i \neq n$. Moreover, one can observe that in fact $S_n(K(\pi, n)) = K(\pi, n)_n \cong \pi$ and $S_i(K(\pi, n)) = \{\star\} \cong 0$ for $i \neq n$. Then it is clear that the components π_n, f_n, g_n and h_n can be given in the following way:

- $\pi_n = \pi = \mathbb{Z}^\alpha \oplus \mathbb{Z}_{p_1}^{\beta_1} \oplus \cdots \oplus \mathbb{Z}_{p_r}^{\beta_r}$ and $\pi_i = 0$ for each $i \neq n$.
- $g_i : \pi_i \to S_i(K(\pi, n))$ is the identity morphism in dimension n and the null map for $i \neq n$.
- $f_i : S_i(K(\pi, n)) \to \pi_i$ is again the identity if $i = n$ and null if $i \neq n$.
- $\ker f_i = \{\star\}$ for every i and therefore h_i is always null.

Proposition 3 *Let $K = \Delta$ be the standard simplex introduced in Definition 8. Then one can define a tuple $(\pi_n, f_n, g_n, h_n)_{n\geq 1}$ which provides a solution for the homotopical problem of K.*

Proof The standard simplex Δ is known to be contractible, and in fact $S_n(\Delta) = \{\star\}$ for all n. Again the definition of the solution for the homotopical problem follows in a straightforward manner.

Unfortunately the spaces for which a solution for the homotopical problem can be constructed in a direct way are not common, and more powerful techniques are necessary to show the interest of our definition. The main idea of the effective homotopy method should be the following: given some Kan simplicial sets $K_1, \ldots, K_n$, a topological constructor Φ produces a new simplicial set K. If solutions for the homotopical problems of the spaces $K_1, \ldots, K_n$ are known, then one should be able to build a solution for the homotopical problem of K, and this construction would allow us to compute the homotopy groups $\pi_*(K)$. The following section presents an outstanding example of this type of situation: given two constructive Kan simplicial sets F and B with effective homotopy and a constructive fibration $F \hookrightarrow E \to B$, an algorithm is obtained producing a solution for the homotopical problem of the total space E.

4 Solution for the homotopical problem of a Kan fibration

Let $p : E \to B$ be a Kan fibration (Definition 6), where the extension condition is again *constructive*.

Definition 11 A Kan fibration $p : E \to B$ is said to be a *constructive Kan fibration* if the following *algorithm* σ_p is provided: given a dimension n, an index k, a list of $n+1$ n-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ of E which satisfy the compatibility condition $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j, i \neq k$ and $j \neq k$, and an $(n+1)$-simplex y of B such that $\partial_i y = p(x_i)$ for $i \neq k$, then σ_p *returns* an $(n+1)$-simplex x of E such that $\partial_i x = x_i$ for $i \neq k$ and $p(x) = y$.

One can observe that, if $p : E \to B$ is a constructive Kan fibration and B is a constructive Kan simplicial set, then E and F are constructive Kan simplicial sets. The proof for the non constructive case is included in [7] and can be automatically suited to the constructive framework.

We suppose now that the groups $\pi_*(F)$ and $\pi_*(B)$ are known: is it then possible to determine the homotopy groups of the total space, $\pi_*(E)$? The answer is negative in general. However, the effective homotopy problem allows one to solve this problem: if instead of *only* knowing $\pi_*(F)$ and $\pi_*(B)$, both F and B are provided with a solution for their homotopical problems (that is, we also have the algorithms f_n, g_n and h_n), then one can also determine a solution for the homotopical problem of E, which in particular makes it possible to determine its homotopy groups.

Theorem 1 *An algorithm can be written down:*

- ***Input:***
  - *A constructive Kan fibration $p : E \to B$ where B is a constructive Kan complex (which implies F and E are also constructive Kan simplicial sets), and F or B are simply connected.*
  - *Respective $SHmtP_F$ and $SHmtP_B$ for the simplicial sets F and B.*
- ***Output:*** *A $SHmtP_E$ for the Kan simplicial set E.*

Proof The proof is a bit tedious; it follows from successive applications of the Kan properties of F, B, E and p and the solutions for the homotopical problems of B and F. First of all we include here a general sketch summarizing the main steps and then we detail some of them; some parts will be skipped here, but the complete proof can be found in [10].

Beginning with the summary, let us say that the proof starts with the long exact sequence of homotopy [7]:

$$\cdots \xrightarrow{p_*} \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \xrightarrow{\text{inc}_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \xrightarrow{\text{inc}_*} \cdots$$

From this one can deduce a short exact sequence

$$0 \longrightarrow \text{Coker}[\pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F)] \xrightarrow{i} \pi_n(E) \xrightarrow{j} \text{Ker}[\pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F)] \longrightarrow 0$$

which implies the desired group $\pi_n(E)$ can be expressed as an extension of $\mathrm{Ker} \equiv \mathrm{Ker}[\pi_n(B) \stackrel{\partial}{\to} \pi_{n-1}(F)]$ by $\mathrm{Coker} = \mathrm{Coker}[\pi_{n+1}(B) \stackrel{\partial}{\to} \pi_n(F)]$, $\pi_n(E) \cong \mathrm{Coker} \times_\chi \mathrm{Ker}$, for a cohomology class $\chi \in H^2(\mathrm{Ker}, \mathrm{Coker})$ classifying the extension. This cohomology class is in principle unknown, but can be determined if the short exact sequence is *constructive* [15]. The most important part of the proof consists in proving that this is the case: the short exact sequence can be made constructive; in other words, we are going to define a set-theoretic section $\sigma : \mathrm{Ker} \to \pi_n(E)$ and a set-theoretic retraction $\rho : \pi_n(E) \to \mathrm{Coker}$ such that $\rho i = \mathrm{Id}_{\mathrm{Coker}}$, $i\rho + \sigma j = \mathrm{Id}_{\pi_n(E)}$ and $j\sigma = \mathrm{Id}_{\mathrm{Ker}}$; both maps will be defined by means of a suitable game of successive applications of the Kan properties of B, F and the fibration p, and the solutions for the homotopical problems of B and F. From the maps σ and ρ we will give a *constructive* definition of the cohomology class which will allow us to *compute* the homotopy group $\pi_n(E)$. Furthermore the algorithms f_n and g_n will be immediately deduced from i, j, σ and ρ. We will end the proof with the computation of h_n.

Once we have given an overview of the main ideas of the proof, let us begin detailing some parts of it.

As already said, the Kan fibration $p : E \to B$ produces a long exact sequence of homotopy [7]:

$$\cdots \xrightarrow{p_*} \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \xrightarrow{\mathrm{inc}_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \xrightarrow{\mathrm{inc}_*} \cdots$$

where the maps p_* and inc_* are the morphisms between the corresponding homotopy groups induced respectively by the fibration $p : E \to B$ and the inclusion $F \hookrightarrow E$, and $\partial : \pi_*(B) \to \pi_{*-1}(F)$ is the *connection morphism* (see [7] for the definition of this map).

Let us emphasize that the groups $\pi_*(B)$ and $\pi_*(F)$ with the corresponding generators can be computed thanks to the SHmtP for both simplicial sets B and F. The *connection morphism* $\partial : \pi_*(B) \to \pi_{*-1}(F)$ can also be *constructively* defined thanks to the Kan property of the fibration p and the solutions for the homotopical problems of B and F (see [10] for details). Since $\pi_*(B)$ and $\pi_*(F)$ are Abelian groups of finite type, it is possible to determine by means of elementary operations the groups $\mathrm{Ker} \equiv \mathrm{Ker}[\pi_n(B) \stackrel{\partial}{\to} \pi_{n-1}(F)]$ and $\mathrm{Coker} \equiv \mathrm{Coker}[\pi_{n+1}(B) \stackrel{\partial}{\to} \pi_n(F)] = \pi_n(F)/\operatorname{Im}\partial$. We obtain then a short exact sequence

$$0 \longrightarrow \mathrm{Coker}[\pi_{n+1}(B) \stackrel{\partial}{\to} \pi_n(F)] \xrightarrow{i} \pi_n(E) \xrightarrow{j} \mathrm{Ker}[\pi_n(B) \stackrel{\partial}{\to} \pi_{n-1}(F)] \longrightarrow 0$$

which implies $\pi_n(E) \cong \mathrm{Coker} \times_\chi \mathrm{Ker}$, for a cohomology class $\chi \in H^2(\mathrm{Ker}, \mathrm{Coker})$ classifying the extension. Although the cohomology class $\chi \in H^2(\mathrm{Ker}, \mathrm{Coker})$ is in principle not known, it can be determined if the short exact sequence is made *constructive* [15], that is to say, if one is able to define a set-theoretic section $\sigma : \mathrm{Ker} \to \pi_n(E)$ and a set-theoretic retraction $\rho : \pi_n(E) \to \mathrm{Coker}$ such that $\rho i = \mathrm{Id}_{\mathrm{Coker}}$, $i\rho + \sigma j = \mathrm{Id}_{\pi_n(E)}$ and $j\sigma = \mathrm{Id}_{\mathrm{Ker}}$. Let us focus now therefore on the construction of the desired σ and ρ.

In order to represent the elements of $\pi_n(E)$, instead of homotopy classes inside an unknown group, we will use n-spheres in $S_n(E)$ *representing* these homotopy classes.

Let us observe that the map i of the short exact sequence is obtained by following the diagram chasing path:

$$\text{Coker} \longrightarrow \pi_n(F) \xrightarrow{g_n} S_n(F) \overset{\text{inc}}{\hookrightarrow} S_n(E)$$

where the map g_n is the algorithm given in SHmtP_F, and the map $\text{Coker} \to \pi_n(F)$ consists in choosing a representative of an element of the cokernel (which can be elementary done since $\pi_n(F)$ is an Abelian group of finite type). The morphism $i : \text{Coker} \to \pi_n(E)$ is therefore implemented as a map $i : \text{Coker} \to S_n(E)$.

The map j of the diagram is implemented as a map $j : S_n(E) \to \text{Ker}$, simply obtained from the path:

$$S_n(E) \xrightarrow{p} S_n(B) \xrightarrow{f_n} \text{Ker} \subseteq \pi_n(B)$$

The map f_n is in fact defined as $f_n : S_n(B) \to \pi_n(B)$, but one can observe that given an element of $S_n(B)$ which is in the image of p, then its homotopy class is necessarily in Ker.

Let us define now the desired section $\sigma : \text{Ker} \to \pi_n(E)$, which will be implemented as a map $\sigma : \text{Ker} \to S_n(E)$. Let $\beta \in \text{Ker} \subseteq \pi_n(B)$, we choose a representative $b \in \beta$ produced by g_n in SHmtP of B. Since $\partial[b] = [\star] \in \pi_{n-1}(F)$, and following the definition of the connection morphism ∂ [7], the algorithm σ_p produces $x \in E_n$ with $\partial_i x = \star$ for $1 \leq i \leq n$ and $p(x) = b$, and then h_{n-1} of SHmtP_F provides us $z \in F_n$ such that $\partial_i z = \star$ for $0 \leq i < n$ and $\partial_n z = \partial_0 x$.

Let us consider now the $n+1$ n-simplices $z, \star, \ldots, \star, -, x$ of E_n which clearly satisfy the necessary compatibility conditions $\partial_i x_j = \partial_{j-1} x_i$. The algorithm σ_E (which is available since E is a constructive Kan simplicial set) returns an $(n+1)$-simplex $y \in E$ such that $\partial_0 y = z$, $\partial_i y = \star$ for $1 \leq i \leq n-1$, and $\partial_{n+1} y = x$. Then the n-simplex $\partial_n y$ is an n-sphere of E. Moreover, one can prove $p(\partial_n y) \sim b$ in B. Therefore, given $\beta \in \text{Ker} \subseteq \pi_n(B)$ we define $\sigma(\beta) = \partial_n y \in S_n(E)$. The map is well defined since the selection of a representative b for the homotopy class β is uniquely done by the algorithm g_n of SHmtP_B and it can be proved that $j\sigma(\beta) = \beta$ (see [10]). In this way σ satisfies the desired property $j\sigma = \text{Id}_{\text{Ker}}$.

The construction of a retraction $\rho : \pi_n(E) \to \text{Coker}$ that we implement as a map $\rho : S_n(E) \to \text{Coker}$ follows the same ideas but is a bit more complicated. The details can be found in [10].

Once that a section $\sigma : \text{Ker} \to \pi_n(E)$ and a retraction $\rho : \pi_n(E) \to \text{Coker}$ satisfying $\rho i = \text{Id}_{\text{Coker}}$, $i\rho + \sigma j = \text{Id}_{\pi_n(E)}$ and $j\sigma = \text{Id}_{\text{Ker}}$ have been defined, one can construct the 2-cocycle $\chi \in H^2(\text{Ker}, \text{Coker})$ which classifies the extension. Given $\alpha, \beta \in \text{Ker}$ two homotopy classes, $\chi(\alpha, \beta)$ is defined as:

$$\chi(\alpha, \beta) = \rho(\sigma(\alpha) + \sigma(\beta) - \sigma(\alpha + \beta))$$

One can observe that χ satisfies the necessary properties of a 2-cocycle. The standard extension theory proves then that $\pi_n(E) \cong \text{Coker} \times_\chi \text{Ker}$ and an elementary calculation can produce some explicit isomorphism $\pi_n \leftrightarrow \text{Coker} \times_\chi \text{Ker}$ for some finitely generated Abelian group π_n (that is to say, π_n is a well-known group $\pi_n = \mathbb{Z}^\alpha \oplus \mathbb{Z}_{p_1}^{\beta_1} \oplus \cdots \oplus \mathbb{Z}_{p_r}^{\beta_r}$). The condition of B or F being simply connected is necessary here in order to avoid the possibility of $\pi_1(E)$ being a non-Abelian extension of two (non-null) Abelian groups $\pi_1(B)$ and $\text{Coker} \subseteq \pi_1(F)$. In this way the first element π_n of the 4-tuple in dimension n of the solution for the homotopical problem of E has been reached.

We need also the three components f_n, g_n and h_n to achieve the construction of SHmtP_E. We define f_n and g_n firstly with respect to the model $\text{Coker} \times_\chi \text{Ker}$ of $\pi_n(E)$. It is easy to justify $g_n(\alpha, \beta) = i(\alpha) + \sigma(\beta) \in S_n(E)$ if $\alpha \in \text{Coker}$ and $\beta \in \text{Ker}$. In the same way, $f_n(x) = (\rho(x), j(x))$ is the unique possible definition of f_n. In this way, one has the desired identity $f_n g_n = \text{Id}_{\text{Coker} \times_\chi \text{Ker}}$ and it is not difficult to prove that f_n satisfies the two additional conditions that we have required. The definitions of $g_n : \text{Coker} \times_\chi \text{Ker} \to S_n(E)$ and $f_n : S_n(E) \to \text{Coker} \times_\chi \text{Ker}$ can then be converted into correspondances with π_n thanks to an arbitrary group isomophism $\text{Coker} \times_\chi \text{Ker} \cong \pi_n$, so that the required properties are still satisfied.

Constructing the map h_n is a little more complicated, a small *game* with the different Kan extension properties and the solutions for the homotopical problems of B and F is again necessary. Let $e \in S_n(E)$ such that $e \in \text{Ker}\, f_n$, that is to say, $\rho(e) = 0 \in \text{Coker} = \pi_n(F)/\operatorname{Im}\partial$ and $j(e) = 0 \in \text{Ker} \subseteq \pi_n(B)$. We begin with the second property $j(e) = 0$, which implies $p(x) \sim \star$ in B. Following the definition of ρ, there exists $b \in B_{n+1}$ with $\partial_1 b = p(e)$ and $\partial_i b = \star$ for all $i \neq 1$, and $x \in E_{n+1}$ such that $\partial_0 x \in S_n(F)$, $\partial_1 x = e$ and $\partial_i x = \star$ for all $2 \leq i \leq n+1$. The map ρ was defined then as $\rho(e) = [f_n(\partial_0 x)] \in \pi_n(F)/\operatorname{Im}\partial$. Now, since $\rho(e) = 0$, one has $f_n(\partial_0 x) \in \operatorname{Im}\partial$ and taking into account that both $\pi_{n+1}(B)$ and $\pi_n(F)$ are finite type Abelian groups one can find $\beta \in \pi_{n+1}(B)$ such that $\partial(\beta) = f_n(\partial_0 x)$ in $\pi_n(F)$.

Let $v \in S_{n+1}(B)$ be a representative of β given by the algorithm g_n of SHmtP_B, the definition of ∂ considers $w \in E_{n+1}$ such that $p(w) = v$, $\partial_0 w \in S_n(F)$ and $\partial_i w = \star$ for $1 \leq i \leq n+1$, and then $\partial(\beta)$ is defined as $\partial(\beta) = f_n(\partial_0 w)$. Since one knows that $\partial(\beta) = f_n(\partial_0 x)$, we deduce $f_n(\partial_0 w) = f_n(\partial_0 x)$ and then $f_n(\partial_0 w - \partial_0 x) = 0$. Then the algorithm h_n for F returns $a \in F_{n+1}$ with $\partial_i a = \star$ for $0 \leq i \leq n$, $\partial_{n+1} a = \partial w - \partial x$. Using the Kan property of F it is not difficult to obtain then $t \in E_{n+1}$ with $\partial_i t = \star$ if $0 \leq i \leq n-1$, $\partial_n t = \partial_0 w$ and $\partial_{n+1} t = \partial_0 x$. The $n+2$ $(n+1)$-simplices of E $t, -, \star, \ldots, \star, w, x$ satisfy the compatibility condition and therefore there exists $y \in E_{n+2}$ with $\partial_0 y = t$, $\partial_i t = \star$ for $2 \leq i \leq n$, $\partial_{n+1} t = w$ and $\partial_{n+2} t = x$. Then the unknown face $\partial_1 y$ satisfies $\partial_0 \partial_1 y = \partial_0 t = \star$, $\partial_i \partial_1 y = \star$ for all $1 \leq i \leq n-1$, $\partial_n \partial_1 y = \partial_1 w = \star$ and $\partial_{n+1} \partial_1 y = \partial_1 x = e$. Therefore we can define $h_n(e) = \partial_1 y$ which satisfies the desired properties, and a solution for the homotopical problem of E has been finally obtained. In this way the proof has been finished. □

As we will see in the following section, this result makes it possible to determine, for instance, the homotopy groups of fibrations where the base and the fiber spaces are Eilenberg-MacLane spaces. Moreover, it has been an important ingredient in the development of a constructive version of the Bousfield-Kan spectral sequence.

5 Examples and applications

5.1 Fibrations of Eilenberg-MacLane spaces

As already said in Section 3, given a finitely generated Abelian group π and an integer $n \geq 1$, one can construct in an elementary way a solution for the homotopical problem of the space $K(\pi, n)$. If we have now two Eilenberg-MacLane spaces $F = K(\pi, n)$ and $B = K(\pi', m)$ where both π and π' are finitely generated groups, and a constructive Kan fibration $F \hookrightarrow E \to B$, our Theorem 1 allows one to construct a solution for the homotopical problem of the total space E.

We observe that this does not give new information when $n \neq m$, since in that case the long exact sequence of homotopy of the fibration provides directly the homotopy groups of E, which are $\pi_n(E) \cong \pi$, $\pi_m(E) \cong \pi'$ and $\pi_i(E) = 0$ for $i \neq n, m$. A particular case is obtained when $m = 1$ and $n = 2$, then E is called a 2-*type* [5] and it is well known that it corresponds to a cohomology class $[f]$ in $H^3(\pi', \pi)$.

The interest of our application can be seen when $n = m \neq 1$. Then the long exact sequence of homotopy produces

$$0 \longrightarrow \pi \longrightarrow \pi_n(E) \longrightarrow \pi' \longrightarrow 0$$

which implies $\pi_n(E)$ is an extension of π' by π, but several extensions could be possible and in general $\pi_n(E)$ can not always be determined. Let us observe that the condition $n = m \neq 1$ implies $F = K(\pi, n)$ and $B = K(\pi', m)$ are simply connected. Then, thanks to our algorithm, one can determine a solution for the homotopical problem of E, compute $\pi_n(E)$ with its generators, and then use this group inside other constructions.

Our results can also be applied when the fiber and the base space of a fibration are direct sums (or Cartesian products) of Eilenberg-MacLane spaces. For example, let us consider a fibration

$$K(\pi_1, n_1) \oplus K(\pi_2, n_2) \hookrightarrow E \longrightarrow K(\pi'_1, m_1) \oplus K(\pi'_2, m_2)$$

where B or F are simply connected, that is to say, $n_1, n_2 \neq 1$ or $m_1, m_2 \neq 1$.

Several non-null groups appear now in the long exact sequence of homotopy. Depending on n_1, n_2, m_1 and m_2 it can happen that the information of the long exact sequence is not sufficient to determine $\pi_*(E)$. But both the fiber and the base spaces are objects with effective homotopy (a solution for their homotopical problem can be easily determined from the solutions for the homotopical problems of the different Eilenberg-MacLane spaces); if the fibration satisfies the Kan property, one can determine a solution for the homotopical problem of E, and in particular its homotopy groups.

Once that all these total spaces E are objects with effective homotopy, they can be used as fiber or base spaces of different Kan fibrations producing new spaces with effective homotopy.

5.2 Bousfield-Kan spectral sequence

The Bousfield-Kan spectral sequence first appeared in [3]. It presents the Adams spectral sequence [1] in the setting of combinatorial topology and makes its algebraic properties more accessible. The Adams spectral sequence and its satellite spectral sequences were designed to compute homotopy groups, in particular stable and unstable sphere homotopy groups. They did allow topologists to compute some homotopy groups, but no constructive version of this spectral sequence is yet available; in other words no routine translation work allows a programmer to implement this spectral sequence on a theoretical or concrete machine to produce an algorithm computing homotopy groups.

A spectral sequence [8] is a family of "pages" $E^r = (E^r_{p,q}, d^r_{p,q})$ of differential bigraded modules, each page being made of the homology groups of the preceding one. In many cases, only the first levels are given, and then some extra information is necessary to determine the successive differential maps. This implies that a spectral sequence can only be *computed* in some simple situations and in general it is not an *algorithm*.

In a previous work [9] we used the effective *homology* method [13] to produce algorithms computing the first two levels of the Bousfield-Kan spectral sequence of a simplicial set, but the higher levels of the spectral sequence were not determined. Our new effective *homotopy* technique makes it possible now to develop general algorithms computing the different components of *all levels* of the Bousfield-Kan spectral sequence.

Given a simplicial set X, the Bousfield-Kan spectral sequence is defined by means of a tower of fibrations

$$\begin{tikzcd}
\cdots \arrow[r, "f_4"] & Y_3 \arrow[r, "f_3"] & Y_2 \arrow[r, "f_2"] & Y_1 \arrow[r, "f_1"] & Y_0 \arrow[r, "f_0"] & \star \\
 & F_3 \arrow[u, "i"] & F_2 \arrow[u, "i"] & F_1 \arrow[u, "i"] & &
\end{tikzcd}$$

and under good conditions this spectral sequence converges to the homotopy groups $\pi_*(X)$ (see [2] for the complete definition and details about the Bousfield-Kan spectral sequence). More concretely, the long exact sequence of homotopy of a fibration [7] provides us the following diagram:

$$\begin{tikzcd}
{} \arrow[d, dotted, "f"] & & {} \arrow[d, dotted, "f"] & \\
\pi_{q-p+1}(Y_p) \arrow[d, "f"] & & \pi_{q-p}(Y_{p+1}) \arrow[r, dotted, "\partial"] \arrow[d, dotted, "f"] & \pi_{q-p-1}(F_{p+2}) \\
\pi_{q-p+1}(Y_{p-1}) \arrow[r, "\partial"] \arrow[d, dotted, "f"] & \pi_{q-p}(F_p) \arrow[r, "i"] & \pi_{q-p}(Y_p) \arrow[d, "f"] & \\
\pi_{q-p+1}(Y_{p-2}) \arrow[d, dotted, "f"] & & \pi_{q-p}(Y_{p-1}) \arrow[r, "\partial"] \arrow[d, dotted, "f"] & \pi_{q-p-1}(F_p) \\
{} & & {} &
\end{tikzcd}$$

where F_n is the fiber of f_n, $\partial : \pi_*(Y_{n-1}) \to \pi_{*-1}(F_n)$ is the connection morphism and $i : \pi_*(F_n) \to \pi_*(Y_n)$ is induced by the inclusion $F_n \hookrightarrow Y_n$. The groups $E^r_{p,q}$ of the spectral sequence are defined then as

$$\begin{aligned}
E^r_{p,q} &= \frac{i^{-1}(\operatorname{Im} f^{r-1})}{\partial(\operatorname{Ker} f^{r-1})} && \text{for } q \geq p \\
E^r_{p,q} &= 0 && \text{otherwise}
\end{aligned}$$

It is clear that, if the homotopy groups $\pi_*(Y_n)$ and $\pi_*(F_n)$ are finitely generated Abelian groups and they are explicitly known (with the corresponding generators) for all n, then the groups $E^r_{p,q}$ are computable because the involved maps f, i and ∂ can be expressed as finite integer matrices. In this way, as we want to develop an algorithm computing the spectral sequence associated with the tower of fibrations, we will first try to construct algorithms which determine the homotopy groups of the simplicial sets Y_n and of the fiber spaces F_n.

The first space in the tower is $Y_0 = RX$, the simplicial Abelian group

$$RX = \frac{R[X]}{R[\star]}$$

where $R[X]$ denotes the simplicial $\mathbb{Z}$-module freely generated by the simplices of X, and $R[\star]$ is the simplicial submodule generated by the base point $\star$ and its degeneracies. It is well-known that, given X a pointed simplicial set, there exists a canonical isomorphism

$$\pi_*(RX) \cong \widetilde{H}_*(X; \mathbb{Z})$$

where $\widetilde{H}_*(X; \mathbb{Z})$ denotes the reduced homology groups of X with coefficients in $\mathbb{Z}$. Let us observe then that, if X is a finite simplicial set (as for instance one of the spheres S^n), its homology groups $\widetilde{H}_*(X; \mathbb{Z})$ (with generators) can be elementarily computed and therefore it is not difficult to construct a solution for the homotopical problem of RX. In a more general framework, if the simplicial set X has *effective homology*, that is to say, there exists a solution for the *homological* problem of X, thanks to the isomorphism $\pi_*(RX) \cong \widetilde{H}_*(X;\mathbb{Z})$ it is also possible to determine a solution for the *homotopical* problem of the simplicial Abelian group RX.

If we apply the constructor R to the simplicial set RX, we obtain a new simplicial Abelian group

$$R^2X = R(RX) = \frac{R[R[X]/R[\star]]}{R[R[\star]]}$$

which satisfies $\pi_*(R^2X) \cong \widetilde{H}_*(RX)$. Since RX is an infinite simplicial set, in principle it is not easy to compute its (reduced) homology groups. However, in [9] we developed an algorithm which determines the effective homology of RX from the effective homology of a simplicial set X, supposing that X is 1-reduced (that is to say, X has only one simplex in dimension 0 and has not non-degenerate simplices in dimension 1. In particular, it implies that X is simply connected). Therefore, if X is a 1-reduced simplicial set with effective homology (or, in particular, if it is finite), then RX has effective homology and therefore R^2X is an object with effective homotopy. And the result can be iterated obtaining that R^nX has effective homotopy for every $n \geq 1$.

The first fiber in the Bousfield-Kan tower of fibrations is $F_1 = \Omega(R^2X \cap \operatorname{Ker}\eta^0)$, where Ω is the loop space constructor and $\eta^0 : R^2X \to RX$ is the first *codegeneracy map* [2]. Its homotopy groups are

$$\pi_*(F_1) = \pi_*(\Omega(R^2X \cap \operatorname{Ker}\eta^0)) \cong \pi_{*+1}(R^2X \cap \operatorname{Ker}\eta^0) \cong \pi_{*+1}(R^2X) \cap \operatorname{Ker}\eta^0$$

The relation $\pi_*(\Omega K) \cong \pi_{*+1}(K)$ is well-known for every Kan simplicial set K. In the particular situation of K being a simplicial Abelian group (as is the case of $R^2X \cap \operatorname{Ker}\eta^0$) the isomorphism can be made explicit by means of discrete vector fields [11]. The second equation $\pi_{*+1}(R^2X \cap \operatorname{Ker}\eta^0) \cong \pi_{*+1}(R^2X) \cap \operatorname{Ker}\eta^0$ is a direct consequence of η^0 being a simplicial Abelian group morphism. Using now that, given a 1-reduced simplicial set X with effective homology, RX and R^2X have effective homotopy, one can determine their homotopy groups $\pi_*(RX)$ and $\pi_*(R^2X)$ (with generators). The kernel of the maps $\eta^0 : \pi_*(R^2X) \to \pi_*(RX)$ can be elementarily computed and therefore a solution for the homotopical problem of $\pi_*(F_1)$ can be given.

On the other hand, by using again discrete vector fields [11] we have proved that the fibrations in the Bousfield-Kan tower are constructive Kan fibrations. In this way, from our Theorem 1 (let us observe that the base $Y_0 = RX$ and the fiber $F_1 = \Omega(R^2X \cap \operatorname{Ker}\eta^0)$ are 1-reduced, and therefore simply connected) one can deduce that the first total space Y_1 has effective homotopy. A similar reasoning proves that all the fibers F_n in the tower have effective homotopy, and iterating the process over the fibrations we obtain that all the spaces Y_n are objects with effective homotopy. Once we can determine the homotopy groups of F_n and Y_n we can compute the different components of the Bousfield-Kan spectral sequence, producing our desired general algorithms. The details of this process will be explained in a subsequent paper.

6 Conclusions and further work

Following the ideas of the effective homology method, in this paper we have started to develop a new *effective homotopy* theory, which can make it possible to compute homotopy groups of interesting spaces. We have beginning by introducing the definition of *a solution for the homotopical problem* of a simplicial set and showing some examples where this can be determined in a direct way. The main result of the paper is an algorithm computing the effective homotopy of the total space of a fibration from solutions for the homotopical problems of the base and the fiber. As a direct application of our algorithm, we can compute the homotopy groups of fibrations of Eilenberg-MacLane spaces. Furthermore, it has been an important ingredient in the development of an algorithm computing all levels of the Bousfield-Kan spectral sequence.

The theorem computing the effective homotopy of the total space of a fibration could be enhanced by obtaining similar algorithms producing the effective homotopy of the base space (respectively the fiber space) from the effective homotopies of the total space and the fiber (resp. the base). Furthermore, other constructions (loop spaces, classifying spaces, suspensions, etc) in Algebraic Topology should be studied, as already done in the effective homology framework (see [13]). In other words, given a Kan simplicial set and a solution for its homotopical problem, algorithms should be designed computing the effective homotopy of its loop space, suspension, etc.

With respect to implementation, this should be done in Common Lisp as new modules for the Kenzo system, where the effective homology method has been implemented and which is already capable of computing some homotopy groups of spaces.

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