Cylindrical Algebraic Decomposition in Coq

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What we have seen so far

We have investigated two levels of formalization:

- A formal representation of logical objects:
  - A formalization of the first order theory of discrete real closed fields;
  - A sufficient condition to obtain full quantifier elimination;

    Eliminate $\exists$ in $\exists x, \bigwedge_{i=1}^{n} L_i$
- A formal representation of geometrical objects:
  - A formalization of models of the real closed field structures (as records)
  - A geometrical (constructive) proof that the projection of semi-algebraic sets is a semi-algebraic set

    $\pi_{|x_{k+1}} \{x \in R^{k+1} \mid P(x) = 0 \wedge \bigwedge_{Q \in \mathcal{Q}} Q(x) > 0\}$ is a semi-algebraic set.

What we have seen so far

This in fact covers three different levels:

- The programmer (computer algebra)
- The Coq user (formalizing correctness of computer algebra)
- The Coq logician user (formalizing theorems of logic)

The programmer

These are constructive proofs, we can write programs. For instance:

- `l_coef(p : poly R): R := ...`
- `count_sign_changes(l : seq R): nat := ...`
- `signed_prem(p q : poly R): seq poly R := ...`
-
  ```
  tarski_query (p q : poly R) : nat :=
    count_sign_changes
      (map lcoef (signed_prem p (p^`() * q)))
  ```
-
  ```
  test_sas_empty1 (p q : poly R) : bool :=
     (tarski_query p q^2) + (tarski_query p q) > 0
  ```

The Coq user, interested in real algebraic geometry

These are constructive proofs, we can write program (in Coq):

```
Definition test_sas_empty1 (p q : {poly R}) : bool :=
   (tarski_query p q^2) + (tarski_query p q) > 0
```

But moreover we want to formalize the proof that:

```
Theorem test_sas_empty1_correct : forall p q : {poly R},
   p != 0 ->
   (test_sas_empty1 p q) > 0
      <->
   (exists x, p.[x] = 0 /\ q.[x] > 0).
```

The other Coq user, interested in formal proofs in logic

We want a running quantifier elimination algorithm:

- A program:

```
Fixpoint quantifier_elim :
  formula term -> formula term := ...
```

- And formal proofs that:

```
Lemma q_free_quantifier_elim : forall f,
   q_free (quantifier_elim f).

Lemma quantifier_elim_correct : forall (R_rcf : rcf),
  forall (f : formula term)(ctx : seq (R R_rcf)),
  (holds ctx f) <-> (holds ctx (quantifier_elim f)).
```

Sharing the efforts

- Can the programmer rely on mathematics textbooks?

  Yes, if they are reasonably written (for that purpose).
- Can the Coq user rely on the programmer?

  Yes, if the programmer uses a pure functional language.
- Can the logician Coq user rely on the geometer Coq user?

  Unclear at this stage.

Can the logician Coq user rely on the geometer Coq user?

- Is it “easy” to understand the semi-algebraic object described by the input formula?

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Yes, very easy.

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Yes, very easy.

- Is it “easy” to read the expected, quantifier free formula from the computer algebra programs?

?

Can the logician Coq user rely on the geometer Coq user?

- Is it “easy” to understand the semi-algebraic object described by the input formula?

  Yes, very easy.

- Is it “easy” to read the expected, quantifier free formula from the computer algebra programs?

  ?

- Is it “easy” to prove that this formula is correct with respect to the initial one?

  ?

The Coq programs

- `l_coef(p : poly R): R := ...`
- `count_sign_changes(l : seq R): nat := ...`
- `signed_prem(p q : poly R): seq poly R := ...`
-
  ```
  tarski_query (p q : poly R) : nat :=
    count_sign_changes
      (map lcoef (signed_prem p (p^`() * q)))
  ```
-
  ```
  test_sas_empty1 (p q : poly R) : bool :=
     (tarski_query p q^2) + (tarski_query p q) > 0
  ```

Execution of the program for $\{x \in R | \alpha x^2 + \beta x + \gamma = 0\}$

From programs to formulas

- The expected formula should collect the conditions leading to the desired result along all the successful paths.
- From the code as such, it might well be difficult.
- We need to expose (more) the control over the execution flow.

Concrete polynomials

- Univariate polynomials are represented by lists of coefficients.
- We only manipulate polynomials in normal form:
  - The empty list represents the zero polynomial.
  - The head of the list is the constant coefficient.
  - A non empty list has a head non zero element.

Example of program

- A program computing the leading coefficient:

```
Fixpoint lcoef (p : {poly R}) : R :=
  match p with
  | [::] -> 0
  | c :: q -> if q == 0 then c else lcoef q
  end.
```

- A program testing that the leading coefficient is positive:

```
Definition test (p : {poly R}) : bool :=
   lcoef p > 0.
```

- What is the counterpart at the formula level?

Terms, the parameterized ring signature

```
Inductive term (R : Type) : Type :=
 | Var of nat
 | Const of R
 | NatConst of nat
 | Add of term & term
 | Opp of term
 | Mul of term & term.
```

- An atom is a term compared to zero (after reduction).
- Terms are polynomial expressions in their free variables.

First order theory, again with parameters

```
Inductive formula (R : Type) : Type :=
| Equal of (term R) & (term R)
| Leq of (term R) & (term R)
| Lt of (term R) & (term R)
| trueF : formula R
| falseF : formula R
| Not of formula
| And of formula & formula
| Or of formula & formula
| Implies of formula & formula
| Exists of nat & formula
| Forall of nat & formula.
```

Abstract polynomials

Consider the formula with a single existential quantifier:

$$\exists x, \alpha x^2 + (\beta x + 1) + \alpha\gamma = 0$$

- The atom is a sign condition on the term $\alpha x^2 + \beta x + \gamma$;
- The single quantifier binds the variable x;
- The term in the atom should be understood as a polynomial, element of $R[\alpha, \beta, \gamma][x]$

Abstract polynomials

Consider the formula with a single existential quantifier:

$$\exists x, \alpha x^2 + (\beta + 1)x + \alpha\gamma = 0$$

- The term embedded in such an atom can be seen as an abstract univariate polynomial, with abstract polynomial coefficients.
- An abstract univariate polynomial is represented by lists of terms.
  $[\alpha, \beta + 1, \alpha\gamma]$: `(seq (term R))`
- An abstract coefficient is only a term.

Abstract polynomials

- From a `(t : term)` in an atom, and the name i of the variable bound by the existential, we can extract the abstract univariate polynomial in the variable x_i thanks to the function:

```
Fixpoint abstrX (i : nat) (t : term R) : (seq term) :=
    ...
```

- An abstract univariate polynomial can be interpreted to a univariate polynomial given a context:

```
Fixpoint eval_polyF (e : seq R) (ap : (seq term)) : {
    poly R} :=
  match ap with
  |c :: qf => (eval_polyF e qf)*'X + (eval e c)
  |[::] => 0
end.
```

- We want the diagram to commute.

Inside out

```
Fixpoint lcoef (p : {poly R}) : R :=
  match p with
  | [::] → 0
  | c :: q → if (q == 0) then c else (lcoef q)
  end.
```

Inside out

```
Fixpoint lcoef (p : {poly R}) : R :=
  match p with
  | [::] → 0
  | c :: q → if (q == 0) then c else (lcoef q)
  end.
Definition test (p : {poly R}) : bool := lcoef p > 0
```

Inside out

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Definition test (p : {poly R}) : bool := lcoef p > 0

Fixpoint cps_lcoef                  (p : {poly R}) :      :=




Definition cps_test (p : {poly R}) : bool :=
```

Inside out

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Fixpoint lcoef (p : {poly R}) : R :=
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Fixpoint lcoef (p : {poly R}) : R :=
  match p with
  | [::] → 0
  | c :: q → if (q == 0) then c else (lcoef q)
  end.
Definition test (p : {poly R}) : bool := lcoef p > 0

Fixpoint cps_lcoef (k : R → bool) (p : {poly R}) : bool :=
  match p with
  | [::] →
  | c :: q →
  end.
Definition cps_test (p : {poly R}) : bool :=
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Inside out

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Fixpoint lcoef (p : {poly R}) : R :=
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Definition cps_test (p : {poly R}) : bool :=
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Fixpoint lcoef (p : {poly R}) : R :=
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  end.
Definition test (p : {poly R}) : bool := lcoef p > 0

Fixpoint cps_lcoef (k : R → bool) (p : {poly R}) : bool :=
  match p with
  | [::] → (k 0)
  | c :: q →  cps_lcoef                                          q
  end.
Definition cps_test (p : {poly R}) : bool :=
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Inside out

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Definition cps_test (p : {poly R}) : bool :=
cps_lcoef                                     p
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   end.
Definition cps_test (p : {poly R}) : bool :=
cps_lcoef (fun n ⇒ if n > 0 then true else false) p
```

What happened in this transformation?

Consider the emptiness test for one dimensional basic semi-algebraic sets:

$$\{x \in R \mid P(x) = 0 \wedge \bigwedge_{Q \in \mathcal{Q}} Q(x) > 0\}$$

- It consists in assembling sign tests on polynomial expressions in the coefficients of P and the Qs:

$$c(f(p_1, \dots, p_k, q_{11}, \dots, q_{1,k_1}, \dots, q_{1,k_n}))$$

- These tests are the nodes in the tree of execution.
- For test on such a polynomial expression we can abstract over the control operation by:
  - Programming a CPS version *cps_f* of f
  - Giving the continuation k_c as an argument to f

Continuation passing style

- This is not (meant to be) code obfuscation.
- We have exposed the control operations by the mean of a continuation.
- This version of the code is ready to be translated at the formula level:
    - By turning boolean outputs into formulas outputs
    - By turning polynomials and coefficients into terms
- Remark : we can define a branching formula:
  Definition ifF (condF thenF elseF : formula R) : formula R := ((condF ∧ thenF) ∨ ((condF) ∧ elseF)).

Formula level programs

```
Fixpoint cps_lcoef
(k : R → bool) (p : {poly R}) : bool :=
   match p with
   | [::] → (k 0)
   | c :: q → cps_lcoef (fun l ⇒ if (q == 0) then (k c) else (k l)) q
   end.
```

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   end.
```

```
Fixpoint cps_lcoefF
(k :          →              ) (p   :                 )) :              :=
   match p   with
   | [::] →
   | c :: q → cps_lcoefF                                                q
   end.
```

Formula level programs

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Fixpoint cps_lcoef
(k : R → bool) (p : {poly R}) : bool :=
  match p with
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  | c :: q → cps_lcoef (fun l ⇒ if (q == 0) then (k c) else (k l)) q
  end.

Fixpoint cps_lcoefF
(k :        →               ) (pF :                 )) :                :=
  match pF with
  | [::] →
  | c :: q → cps_lcoefF                                                   q
  end.
```

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Fixpoint cps_lcoef
(k : R → bool) (p : {poly R}) : bool :=
  match p with
  | [::] → (k 0)
  | c :: q → cps_lcoef (fun l ⇒ if (q == 0) then (k c) else (k l)) q
  end.

Fixpoint cps_lcoefF
(k :       →             ) (pF : (seq (term R))) :             :=
  match pF with
  | [::] →
  | c :: q → cps_lcoefF                                          q
  end.
```

Formula level programs

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```
Fixpoint cps_lcoefF
(k : term R →                ) (pF : (seq (term R))) :               :=
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  end.

Fixpoint cps_lcoefF
(k : term R → (formula R)) (pF : (seq (term R))) : (formula R) :=
  match pF with
  | [::] →
  | c :: q → cps_lcoefF                                            q
  end.
```

Formula level programs

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Fixpoint cps_lcoef
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  end.

Fixpoint cps_lcoefF
(k : term R → (formula R)) (pF : (seq (term R))) : (formula R) :=
  match pF with
  | [::] → (k (Const 0))
  | c :: q → cps_lcoefF                                              q
  end.
```

Formula level programs

```
Fixpoint cps_lcoef
(k : R → bool) (p : {poly R}) : bool :=
  match p with
  | [::] → (k 0)
  | c :: q → cps_lcoef (fun l ⇒ if (q == 0) then (k c) else (k l)) q
  end.

Fixpoint cps_lcoefF
(k : term R → (formula R)) (pF : (seq (term R))) : (formula R) :=
  match pF with
  | [::] → (k (Const 0))
  | c :: q → cps_lcoefF (fun l ⇒ ifF (Equal l (Const 0)) (k c) (k l)) q
  end.
```

Formula level programs

```
Definition cps_test (p : {poly R}) : bool :=
cps_lcoef
(fun n ⇒ if n > 0 then true else false)
p
```

Formula level programs

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Definition cps_test (p : {poly R}) : bool :=
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p

Definition cps_testF (p : term R) : formula R :=
cps_lcoefF
(fun n ⇒ ifF (Lt (Const n) (Const 0)) trueF falseF)
p
```

Formula level programs

```
Definition cps_test (p : {poly R}) : bool :=
cps_lcoef
(fun n ⇒ if n > 0 then true else false)
p

Definition cps_testF (p : term R) : formula R :=
cps_lcoefF
(fun n ⇒ ifF (Lt (Const n) (Const 0)) trueF falseF)
p

Definition cps_cps_testF
(k : term R → formula R) (p : term R) : formula R :=
cps_lcoefF
(fun n ⇒ k (Lt (Const n) (Const 0)) p
p
```

What happened in this transformation?

Consider an abstract polynomial `pF : seq (term R))`, extracted from a basic formula:

- The concrete shape of this polynomial depends on the values instantiating the parameters.

`(eval_polyF e pF)` denoted `[pF]_e`

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Consider an abstract polynomial `pF : seq (term R))`, extracted from a basic formula:

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- Any polynomial function `f` has a formula CPS counterpart `fF`.

  `cps_lcoef` and `cps_lcoefF`

What happened in this transformation?

Consider an abstract polynomial `pF : seq (term R))`, extracted from a basic formula:

- The concrete shape of this polynomial depends on the values instantiating the parameters.

  `(eval_polyF e pF)` denoted `[pF]_e`
- Any polynomial function `f` has a formula CPS counterpart `fF`.

  `lcoef` and `cps_lcoefF`
- Any test `c` on such a polynomial expression has a formula CPS counterpart `(fF kc)`.

  `cps_testF`

Correctness as observational equivalence

Now we have commutation:

```
Lemma cps_lcoefFP : forall k pF e, acceptable_cont k ->
  qf_sat e (cps_lcoefF k pF)
  =
  qf_sat e (k (Const (lcoef [pF]_e))).
```

A generic and uniform process

- Program the concrete emptiness test for polynomials in $R[X]$;
- For every elementary program used in the previous phase:
  - Turn the concrete program into a CPS-formula one;
  - State the lemma corresponding to its correctness with respect to the concrete program;
  - Prove this lemma by executing symbolically the code of the concrete program in the proof.

Gluing the programs, and the proofs

- Combine the CPS-formula programs in the same way they are combined in the concrete emptiness test program;
- The quantifier elimination procedure of a single $\exists$ follows.
- Combine the CPS-formula correctness lemmas accordingly.
- The correctness proof follows.

Formalized quantifier elimination for discrete RCF

We have defined in Coq:

- The first order language of ordered fields
- The models of the theory of discrete real closed fields

Formalized quantifier elimination for discrete RCF

We have programed in Coq:

- A reduction of the full first order theory decidability to the elimination of a single ∃;
- An emptiness test for semi-algebraic sets;
- A transformation of this test into a procedure elimination a single ∃ by a syntactic process;

Formalized quantifier elimination for discrete RCF

We have formally proved in Coq:

- The reduction of decidability to quantifier elimination
- The reduction of full quantifier elimination to weak;
- The correctness of the emptiness test;
- The correctness of the weak quantifier elimination.

Formalized quantifier elimination for discrete RCF

We have formally proved in Coq:

- The reduction of decidability to quantifier elimination
- The reduction of full quantifier elimination to weak;
- The correctness of the emptiness test;
- The correctness of the weak quantifier elimination.

Disclaimer:

- This is in fact work in progress with C. Cohen.
- The analogue work on algebraically closed fields is completed.

Decidability, effectiveness

We have not addressed tractable programs so far: the complexity of this algorithm is far from elementary.
It is in fact comparable to the one of the original proof of Tarski.

Decision methods in discrete real closed fields

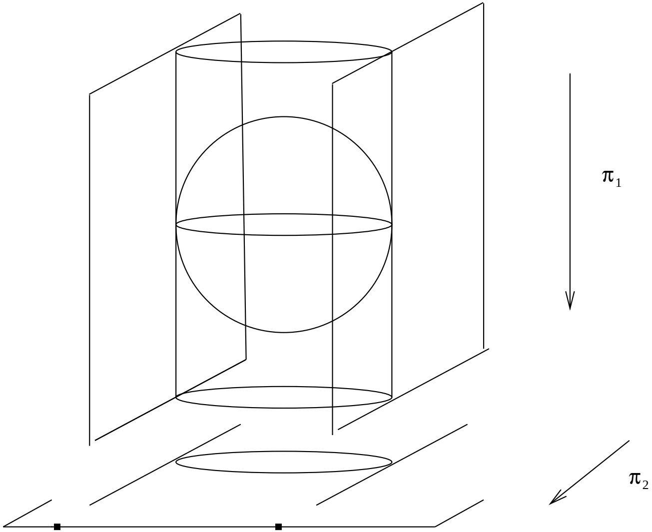
- Real closed field theory is decidable (Tarski, 1948)
- Hörmander method (Hörmander 1983 - Cohen 1969)
- Cylindrical Algebraic Decomposition algorithm (Collins 1975)

CAD in a nutshell: general setting

- Input: A finite family $\mathcal{P} \subset \mathbb{Q}[X_1, \ldots, X_n]$ of polynomials
- Output: A finite partition of $\mathbb{R}^n$ into cylindrical cells over which each element of $\mathcal{P}$ has a constant sign.

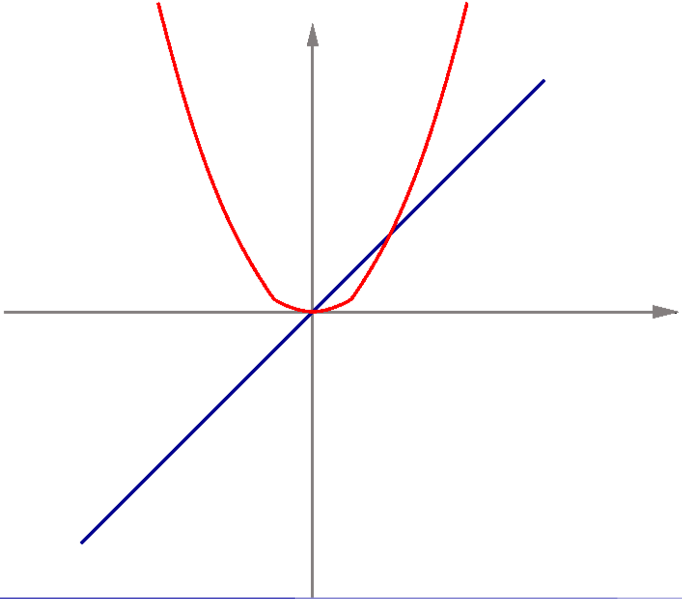
Warning: cylindrical decomposition does not entail decidability
(cf. Michel's tutorial) ...

Example: $X^2 + Y^2 + Z^2 - 1$



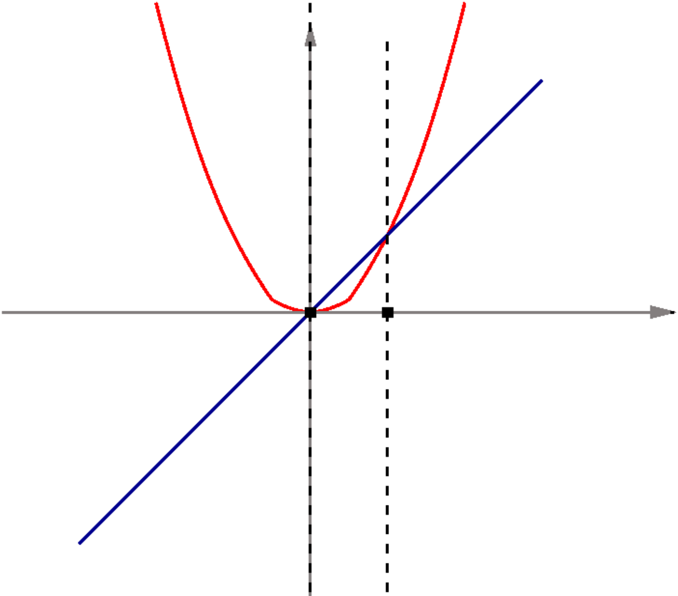
Prove $\exists x \exists y \quad y - x^2 = 0 \wedge x - y = 0$

Extract polynomials: $P_1 = y - x^2$, $P_2 = y - x$



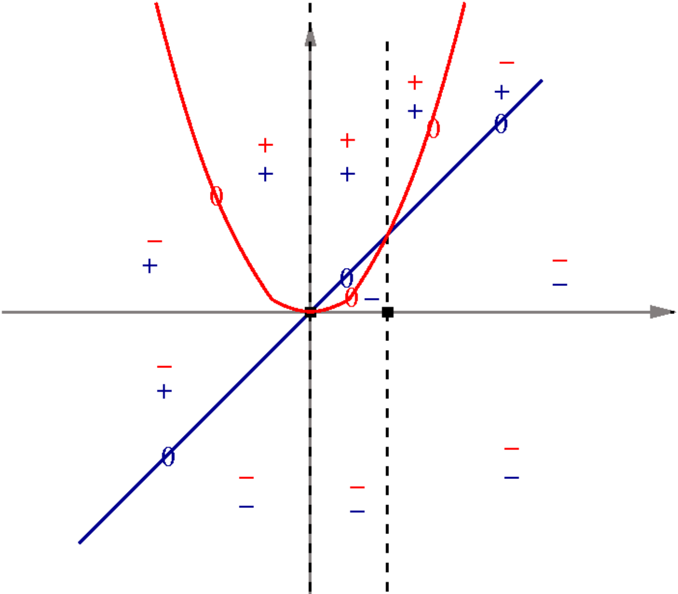
Prove $\exists x \exists y \quad y - x^2 = 0 \wedge x - y = 0$

Projection along y.



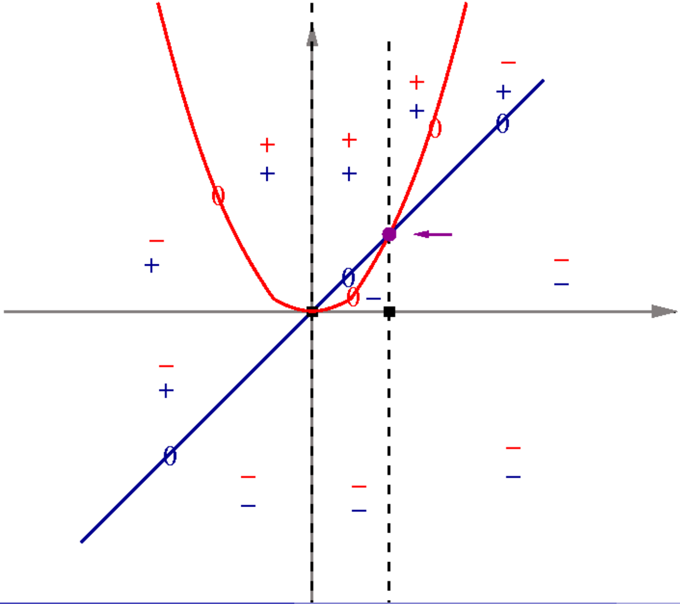
Prove $\exists x \exists y \quad y - x^2 = 0 \wedge x - y = 0$

Lifting phase and sign computation.



Prove $\exists x \exists y \quad y - x^2 = 0 \wedge x - y = 0$

Solution search



CAD in a nutshell

$$\begin{tikzcd}
\mathbb{R}[X_1, \ldots, X_n] & \mathbb{R}[X_1, \ldots, X_{n-1}] \\
\mathcal{P} = P_1, \ldots, P_s \arrow[r, "\text{projection}"] & \mathcal{Q} = Q_1, \ldots, Q_t \\
\text{CAD and signs for } \mathcal{P} & \text{CAD and signs for } \mathcal{Q} \arrow[l, "\text{lifting}"'] \\
\mathbb{R}^n & \mathbb{R}^{n-1}
\end{tikzcd}$$