O-minimal structures

Michel Coste, IRMAR (Rennes)

MAP 2010, Logroño

Tutorial 1

Motivating example: semialgebraic and subanalytic sets

The *semialgebraic* subsets of $\mathbb{R}^n$ form the smallest class $\mathcal{SA}_n$ of subsets of $\mathbb{R}^n$ such that:

1. If $P \in \mathbb{R}[X_1, \ldots, X_n]$, then $\{x \in \mathbb{R}^n \ ; \ P(x) = 0\} \in \mathcal{SA}_n$ and $\{x \in \mathbb{R}^n \ ; \ P(x) > 0\} \in \mathcal{SA}_n$.
2. If $A \in \mathcal{SA}_n$ and $B \in \mathcal{SA}_n$, then $A \cup B$, $A \cap B$ and $\mathbb{R}^n \setminus A$ are in $\mathcal{SA}_n$.

Semianalytic sets : locally described by analytic functions (as semialgebraic sets by polynomials). *Subanalytic* sets : locally projection of relatively compact semianalytic sets.

Features of semialgebraic geometry

- Semialgebraic sets are stable under many constructions : projection (Tarski-Seidenberg), closure, taking connected components,...
- Tame topology : no pathological behavior, semialgebraic sets are triangulable,...
- Finiteness, uniform bounds: finitely many topological types in a semialgebraic family,...

The same holds for globally subanalytic sets (= subanalytic in a compactification of $\mathbb{R}^n$)

O-minimal structures

O-minimal strutures: axiomatic generalization of the preceding examples (semialgebraic and globally subanalytic), retaining their nice features. They include other interesting examples:

- with the exponential function,
- more generally, pfaffian functions,
- with some functions associated with non convergent power series.

The entire sine function cannot be in a o-minimal structure (infinitely many zeroes); only its restriction to a compact segment.

Real closed fields

- Semialgebraic geometry works over *real closed fields*, which retain all algebraic properties of $\mathbb{R}$.
- Real closed field R : ordered field such that every $P \in R[X]$ satisfies the Intermediate Value Theorem :

$$a < b \text{ and } P(a)P(b) < 0 \Rightarrow \exists c \in (a,b)\ P(c) = 0 \ .$$

- Examples: the field of real algebraic numbers, the field of real Puiseux series $\bigcup_{p \in \mathbb{N}^*} \mathbb{R}((x^{1/p}))$.
- We shall introduce o-minimal structures in the framework of real closed fields. Real closed fields will appear related to "ideal points".
- Interval in a real closed field R: (a,b) where $a < b$ in $R \cup \{-\infty, +\infty\}$. Products of intervals generate the topology of R^n.

Structure

Definition

A *structure expanding the real closed field R* is a collection $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$, where each $\mathcal{S}_n$ is a set of subsets of the affine space R^n, satisfying the following axioms:

1. All semialgebraic subsets of R^n are in $\mathcal{S}_n$.
2. For every n, $\mathcal{S}_n$ is a Boolean subalgebra of the powerset of R^n.
3. If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$, then $A \times B \in \mathcal{S}_{m+n}$.
4. If $p : R^{n+1} \to R^n$ is the projection on the first n coordinates and $A \in \mathcal{S}_{n+1}$, then $p(A) \in \mathcal{S}_n$.

O-minimal structure

Definition

The structure $\mathcal{S}$ is said to be *o-minimal* if, moreover, it satisfies:

5. The elements of $\mathcal{S}_1$ are precisely the finite unions of points and intervals.

The elements of $\mathcal{S}_n$ are called the *definable subsets of* R^n. A map $f : A \to B$ between definable sets is called *definable* if its graph is definable.

Exercise : an ordered field carrying an o-minimal structure is real closed.

Elementary facts

- The image of a definable set by a definable map is definable. The composite of definable maps is definable. Definable functions $A \to R$ form an R-algebra.
- The closure and the interior of a definable subset $A \subset R^n$ are definable.

$$\mathrm{clos}(A) = R^n \setminus \left(p_{n+1,n}\left(R^{n+1} \setminus p_{2n+1,n+1}(B)\right)\right) ,$$

where B is

$$(R^n \times R \times A) \cap \{(x, \varepsilon, y) \in R^n \times R \times R^n \mid \sum_{i=1}^{n}(x_i - y_i)^2 < \varepsilon^2\} .$$

Language of an o-minimal structure

- First-order formula ($x = (x_1, \ldots, x_n)$):
  1. $P(x) = 0$, $P(x) > 0$ for $P \in R[X_1, \ldots, X_n]$, $x \in A$ for $A \subset R^n$ definable,
  2. Φ and Ψ, Φ or Ψ, not Φ, $\Phi \Rightarrow \Psi$.
  3. $\exists x \in A\ \Phi(y, x)$, $\forall x \in A\ \Phi(y, x)$ where $A \subset R^n$ is definable subset of R^n.
- If $\Phi(x)$ is a first-order formula, then $\{x \in R^n \mid \Phi(x)\}$ is definable.
- Exercise: $x \mapsto \mathrm{dist}(x, A) = \inf\{\|y - x\| \mid y \in A\}$, for $A \subset R^n$ non empty definable, is a continuous definable function on R^n.
- $\{(x, y) \in \mathbb{R}^2 \mid \exists n \in \mathbb{N}\ y = nx\}$ is not definable.

Monotonicity Theorem

Theorem

Let $f : (a, b) \to R$ be a definable function. There exists a finite subdivision $a = a_0 < a_1 < \ldots < a_k = b$ such that, on each interval (a_i, a_{i+1}), f is continuous and either constant or strictly monotone.

If there is no subinterval on which f is constant, there is a subinterval on which f is injective, hence there is a subinterval on which f is strictly monotone, hence there is a subinterval on which f is continuous strictly monotone.

Remark : one can ask f to be C^k on each (a_i, a_{i+1}).
Exercise: every continuous definable $f : (a, b) \to R$ has left and right derivatives in $R \cup \{-\infty, +\infty\}$ at every point.

CDCD (Cylindrical Definable Cell Decomposition)

Definition

A cdcd of R^n is a finite partition of R^n into definable *cells*, given:

$n = 1$: by a finite subdivision $a_1 < \ldots < a_\ell$ of R.
Cells : the singletons $\{a_i\}$, $0 < i \leq \ell$, and the intervals (a_i, a_{i+1}), $0 \leq i \leq \ell$ ($a_0 = -\infty$, $a_{\ell+1} = +\infty$).

$n > 1$: by a cdcd of R^{n-1} and continuous definable functions $\zeta_{D,1} < \ldots < \zeta_{D,\ell(D)} : D \to R$ for each cell $D \subset R^{n-1}$.
Cells : the *graphs* of the $\zeta_{D,i}$, and the *bands* $(\zeta_{D,i}, \zeta_{D,i+1})$, $0 \leq i \leq \ell(D)$ cut in $D \times R$ by these graphs ($\zeta_{D,0} = -\infty$, $\zeta_{D,\ell(D)+1} = +\infty$).

Dimension of cells

- Definition by induction:
  - dim(point) $= 0$, dim(interval)$=1$,
  - dim(graph in $D \times R$) $=$ dim(D),
  - dim(band in $D \times R$) $=$ dim(D)$+1$.
- A cell C of a cdcd is definably homeomorphic to $R^{\dim(C)}$.
- A cell C of a cdcd of R^n is open in R^n iff $\dim(C) = n$.

Adapted cdcd

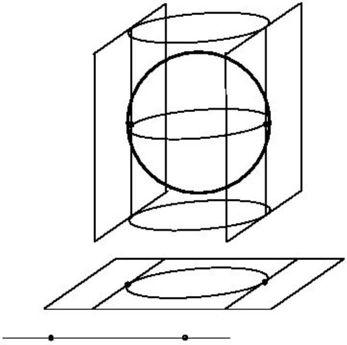


Figure: cdcd adapted to the sphere

A cdcd is said to be *adapted* to a definable set A if A is a union of cells.

Theorem (Cell Decomposition CDCD_n)

Let $A_1, \ldots, A_k$ be definable subsets of R^n. There is a cdcd of R^n adapted to $A_1, \ldots, A_k$.

Uniform Finiteness and Piecewise Continuity

The Cell Decomposition Theorem is proved by induction on the dimension n, together with the following results:

Theorem (Uniform Finiteness UF_n)

Let A be a definable subset of R^n such that, for every $x \in R^{n-1}$, the set $A_x = \{y \in R \mid (x, y) \in A\}$ is finite. Then there exists $k \in \mathbb{N}$ such that $\sharp A_x \leq k$ for every $x \in R^{n-1}$.

Theorem (Piecewise Continuity PC_n)

Let A be a definable subset of R^n and $f : A \to R$ a definable function. There is a cdcd of R^n adapted to A such that, for every cell C contained in A, $f|_C$ is continuous.

First UF_n, then CDCD_n and finally PC_n.

Tutorial 2

In Tutorial 1 : o-minimal structures, cylindrical definable cell decomposition.
Today: tameness properties following from this decomposition (connected components, dimension, ...) and description of the topology of definable sets in finite terms

Definable choice, curve selection

Theorem (Definable Choice)

$A \subset R^m \times R^n$ definable, $p : R^m \times R^n \to R^m$ the projection. There is a definable function $f : p(A) \to R^n$ such that, for every $x \in p(A)$, $(x, f(x))$ belongs to A.

Theorem (Curve Selection Lemma)

$A \subset R^n$ definable, $b \in \mathrm{clos}(A)$. There is a continuous definable map $\gamma : [0,1) \to R^n$ such that $\gamma(0) = b$ and $\gamma((0,1)) \subset A$.

Apply definable choice to
$X = \{(t,x) \in R \times R^n \; ; \; x \in A \text{ and } \|x - b\| < t\}$, and then monotonicity.

Definable compactness

Theorem

Let $A \subset R^n$ definable. TFAE:

1. *A is closed and bounded.*
2. *Every definable continuous map $(0,1) \to A$ extends by continuity to a map $[0,1) \to A$.*
3. *For every definable continuous function $f : A \to R$, $f(A)$ is closed and bounded.*

Such a A is called *definably compact*. This is an intrinsic notion (invariant by definable homeomorphism). Being *locally closed* ($=$ locally definably compact) is also intrinsic.

Definable connectednes

Definition

A definable set A is said to be *definably connected* if, for all disjoint definable open subsets U and V of A such that $A = U \cup V$, one has $A = U$ or $A = V$.
It is said to be *definably arcwise connected* if, for all points a and b in A, there is a definable continuous map $\gamma : [0, 1] \to A$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

Cells of a cdcd are definably arcwise connected.

Exercise: definable + definably connected $\Rightarrow$ definably arcwise connected.

Connected components

Theorem

Let A be a definable subset of R^n. There is a partition of A into finitely many definable subsets $A_1, \dots, A_k$ such that each A_i is nonempty, open and closed in A, and definably arcwise connected. Such a partition is unique. The $A_1, \dots, A_k$ are called the definable connected components *of A.*

Theorem (Uniform Finiteness revisited)

Let A be a definable subset of $R^m \times R^n$. There is $\beta \in \mathbb{N}$ such that, for every $x \in R^m$ the number of definable connected components of $A_x = \{y \in R^n \mid (x, y) \in A\}$ is not greater than β.

Dimension

The dimension of cells coincide with the following intrinsic notion:

Definition

The dimension of a definable set A is the sup of d such that there exists a injective definable map from R^d to A.

If $\mathcal{C}$ is a cdcd adapted to A, then $\dim(A) = \max_{C \in \mathcal{C}} \dim(C)$.

Theorem

Let A be a definable subset of $R^m \times R^n$. For $d \in \mathbb{N} \cup \{-\infty\}$, set $X_d = \{x \in R^m \mid \dim(A_x) = d\}$. Then X_d is a definable subset of R^m, and $\dim(A \cap (X_d \times R^n)) = \dim(X_d) + d$.

Dimension (2)

Theorem

Let A be a nonempty definable subset of R^n. Then $\dim(\mathrm{clos}(A) \setminus A) < \dim(A)$*.*

Dimension is invariant by definable bijection (not necessarily continuous). Another such invariant is *Euler characteristic*: for $A \subset R^n$ definable, set $\chi(A) = \sum_{C \subset A} (-1)^{dim(C)}$, where the sum is taken over the cells of an adapted cdcd contained in A. This does not depend on the cdcd.

Exercise: there is a definable bijection $A \to B$ iff $\dim(A) = \dim(B)$ and $\chi(A) = \chi(B)$.

Extension by continuity

Adjacency between cells in different cylinders of a cdcd is not clear.
Problem : extend by continuity $\zeta : D \to R$ to $\mathrm{clos}(D)$.

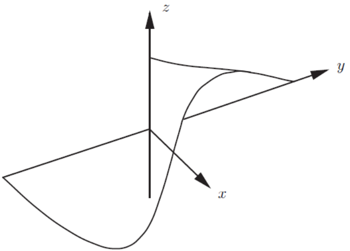


Figure: $\Gamma \subset R^3$ near the origin

Ex: ζ defined by $\zeta(x, y) = 2xy/(x^2 + y^2)$ on $x > 0$ does not extend continuously to $(0, 0)$. The closure Γ of the graph of ζ has dimension 2, but the projection $\Gamma \to R^2$ is not finite-to-one.

What if we tilt it a little?

Good change of coordinates

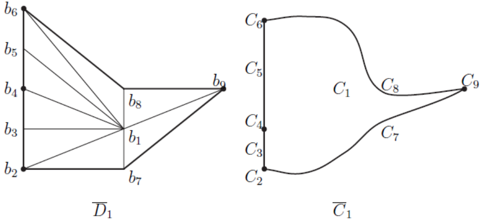
Given: $F \subset R^n$ definable, closed and bounded, such that the restriction to F of the projection $p : R^n \to R^{n-1}$ is finite-to-one.
$X \subset p(F)$ definable, such that every $x' \in \mathrm{clos}(X)$ has a basis of neighborhoods U such that $U \cap X$ is definably connected.
Then every continuous definable function $\zeta : X \to R$ with graph contained in F extends continuously to $\mathrm{clos}(X)$.

Lemma

$G \subset R^q \times R^n$ definable such that, for every t in R^q, the dimension of $G_t \subset R^n$ is $< n$. Then there is a polynomial automorphism u of R^n such that the restriction to G of the projection $R^q \times R^n \to R^q \times R^{n-1}$ is finite-to-one.

If $q = 0$, the change of coordinates u can be taken linear.

Triangulation of definable sets



Using induction on n and the "good coordinates" lemma for $q = 0$, one obtains:

Theorem

Let A be a closed and bounded definable subset of R^n and B_i, $i = 1, \ldots, k$, definable subsets of A. Then there exist a finite simplicial complex K with vertices in $\mathbb{Q}^n$ and a definable homeomorphism $\Phi : |K|_R \to A$ such that each B_i is a union of images by Φ of open simplices of K.

Triangulation of definable functions

Still using induction on n and the "good coordinates" lemma (now with $q = 1$) for the graph of a function, we obtain:

Theorem

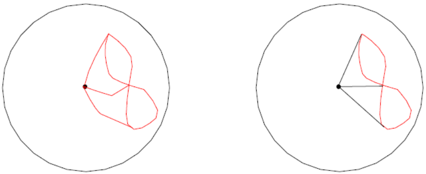
Let $X \subset R^n$ be definable, closed and bounded, $f : X \to R$ continuous definable. Then there exist a finite simplicial complex K in R^{n+1} and a definable homeomorphism $\rho : |K|_R \to X$ such that $f \circ \rho$ is an affine function on each simplex of K.
Given $B_1, \ldots, B_k \subset X$ definable, we may choose the triangulation so that each B_i is a union of images of open simplices of K.

Beware! No triangulation of definable continuous maps $X \to Y$ if $\dim Y > 1$. Ex.: show that $f : [0,1]^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x, xy)$ cannot be triangulated.

Local conic structure

Theorem

Let $A \subset R^n$ be a closed definable set, $a \in A$. There is $r > 0$ such that there exists a definable homeomorphism h from the cone with vertex a and base $S(a,r) \cap A$ onto $\overline{B}(a,r) \cap A$, satisfying $h|_{S(a,r)\cap A} = \mathrm{Id}$ and $\|h(x) - a\| = \|x - a\|$ for all x in the cone.



Easily obtained by triangulating the function $x \mapsto \|x - a\|$.

Tutorial 3

- First two lectures: o-minimal structures, Cylindrical Definable Cell Decomposition, tameness properties, triangulation.
- Today: uniformity in definable families $X \subset R^m \times R^n$, using "generic fibers" X_α at "ideal points" α of the parameter space.

Ultrafilters

Ideal point α of R^m = ultrafilter of the Boolean algebra $\mathcal{S}_m$.

1. $R^m \in \alpha$
2. $A \cap B \in \alpha$ if and only if $A \in \alpha$ and $B \in \alpha$
3. $\emptyset \notin \alpha$
4. $A \cup B \in \alpha$ if and only if $A \in \alpha$ or $B \in \alpha$

Notation: $\widetilde{R^m}$ for the set of ideal points. Note $R^m \subset \widetilde{R}^m$.
Examples:

- in $\widetilde{R}$: $\{A \in \mathcal{S}_1 \mid \exists a \in R \ (a, +\infty) \subset A\}$; this ideal point may be called $+\infty$.
- in $\widetilde{R^2}$: ultrafilter generated by all curvilinear triangles $\{(x, y) \in R^2 \mid 0 < x < a, \ 0 < y < f(x)\}$ where $a > 0$ and $f : (0, a) \to (0, +\infty)$ definable.

Topologies on $\widetilde{R^m}$

Notation: for $A \in \mathcal{S}_m$, set $\widetilde{A} = \{\alpha \in \widetilde{R^m} \mid A \in \alpha\}$.

Stone space of the Boolean algebra. Compact, Hausdorff, totally disconnected. Clopens: all $\widetilde{A}$ for $A \in \mathcal{S}_m$.

Stone space of the lattice of definable opens. Compact, not Hausdorff (spectral space). Compact opens: all $\widetilde{U}$ for $U \subset R^m$ open definable.

In the semialgebraic case, ideal points = prime cones of $R[X_1, \ldots, X_m]$ = couples (prime $\mathfrak{p}$, ordering of $k(\mathfrak{p})$). The real spectrum of $R[X_1, \ldots, X_m]$ is $\widetilde{R^m}$ with the second topology.

Residual field at an ideal point

- For $\alpha \in \widetilde{R^m}$, the elements of $\kappa(\alpha)$ are definable functions $f : A \to R$ where $A \in \alpha$, modulo identification of f and g when they coincide on some $B \in \alpha$ (germs of definable functions along α). Denote by $f(\alpha) \in \kappa(\alpha)$ the class of f.
- $\kappa(\alpha)$ is a real closed field.
- If $\alpha = t \in R^m$, then $\kappa(\alpha) = R$.
- In the semialgebraic case, $\kappa(+\infty)$ is the field of algebraic Puiseux series in $1/x$. More generally, if $\alpha = (\mathfrak{p}, \leq)$, then $\kappa(\alpha)$ is the real closure of $k(\mathfrak{p})$ for $\leq$.

Fiber of a definable family at an ideal point

- $X \subset R^m \times R^n$ definable = definable family of subsets of R^n parametrized by R^m. If $A \subset R^m$, set $X|_A = X \cap (A \times R^n)$.
- If $t \in R^m$, $X_t = \{x \in R^m \mid (t, x) \in X\}$.
- For $\alpha \in \widetilde{R^m}$, define the fiber X_α to be the set of $f(\alpha) \in \kappa(\alpha)^m$ such that there exists $A \in \alpha$ with $(t, f(t)) \in X$ for all $t \in A$.
- Set $\mathcal{S}_n(\alpha)$ to be the set of fibers X_α for all definable families $X \subset R^m \times R^n$.

Residual o-minimal structure

Theorem

The collection $(S_n(\alpha))_{n\in\mathbb{N}}$ is an o-minimal structure expanding the real closed field $\kappa(\alpha)$.

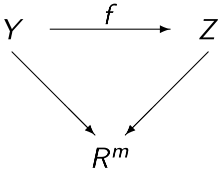
Main points of the proof:

- stability by projection: definable choice implies that the projection of the fiber at α is the fiber at α of the projection.
- elements of $\mathcal{S}_1(\alpha)$ are unions of points and intervals in $\kappa(\alpha)$: CDCD in $R^m \times R$.

The extension of a definable subset $A \subset R^n$ to $\kappa(\alpha)$ is $A_{\kappa(\alpha)} = (R^m \times A)_\alpha$.
From a model-theoretic point of view, $\kappa(\alpha)$ is an elementary extension of the o-minimal structure on R.

What means validity at an ideal point?

- In algebraic geometry, the properties of the generic fiber hold for almost all fibers.
- Here : a property (expressible by a first-order formula) holds for X_α iff there exists $A \in \alpha$ such that it holds for all X_t with $t \in A$.
- For instance (property = being the graph of a function): definable functions $Y_\alpha \to Z_\alpha$ are precisely the fibers at α of definable families of functions, i.e. definable functions $f : Y \to Z$ commuting with projection to R^m :



Fiberwise and global properties

If $X_\alpha \subset \kappa(\alpha)^n$, we know that there exists $A \in \alpha$ such that $X_t \subset R^n$ is closed for every $t \in A$. Actually, someting better holds:

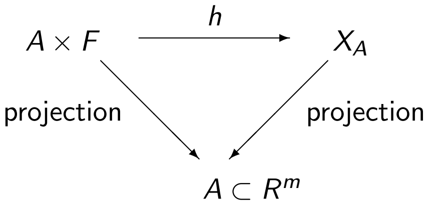
Theorem

Let $X \subset Y$ be definable families. The fiber X_α is closed in Y_α if and only if there exists $A \in \alpha$ such that $X|_A$ is closed in $Y|_A$.
Let $f : X \to Y$ be a definable family of maps. Then f_α is continuous if and only if there exists $A \in \alpha$ such that $f|_A : X|_A \to Y|_A$ is continuous.

Exercise: Suppose $f : X \to Y$ is a definable family such that f_t is continuous for each $t \in R^m$. Then there exists a finite definable partition $R^m = \bigcup C_i$ such that $f|_{C_i}$ is continuous for each i. (Hint: use compactness of $\widetilde{R^m}$).

Trivialisation

Let $A \subset R^m$ definable. The definable family $X \subset R^m \times R^n$ is said to be *definably trivial over* A if there exist a definable set F and a definable homeomorphism $h : A \times F \to X|_A$ such that the folllowing diagram commutes:



The trivialization h is said to be compatible with a definable subset $Y \subset X$ if there is a definable subset G of F such that $h(A \times G) = Y|_A$.

Hardt's theorem

Theorem (Hardt's Theorem for Definable Families)

Let $X \subset R^m \times R^n$ be a definable family, $Y_1, \ldots, Y_\ell$ definable subsets of X. There exists a finite partition of R^m into definable sets $C_1, \ldots, C_k$ such that X is definably trivial over each C_i and, moreover, the trivializations over each C_i are compatible with $Y_1, \ldots, Y_\ell$.

Proof:

- Can assume X_t closed and bounded for every t.
- For every $\alpha \in \widetilde{R^m}$, triangulate X_α: $h_\alpha : |K|_{\kappa(\alpha)} \to X_\alpha$.
- This gives a trivialisation $h : A \times |K|_R \to X|_A$ for some $A \in \alpha$.
- Hardt's theorem follows using the compactness of $\widetilde{R^m}$.

Finiteness of topological types in definable families

- Example: given n and d, there are finitely many topological types of $V \subset \mathbb{R}^n$ where V is an algebraic subset described by equations of degrees $\leq d$.
- There is also a "Hardt's theorem" for families of definable functions (with values in R), using the triangulation of functions.
- Example of consequence: finiteness of topological types of fewnomials. For n and p fixed, there is a finite number of topological types of fewnomials $\mathbb{R}^n \to \mathbb{R}$ with $\leq p$ monomials (whatever are the degrees). These fewnomials can be put in a definable family for the o-minimal structure $\mathbb{R}_{\exp}$.
- There are semialgebraic families of maps $\mathbb{R}^2 \to \mathbb{R}^2$ with infinitely many topological types.

Tutorial 4

Yesterday: how to obtain triviality results for definable families (Hardt's theorem) almost for free from triangulation theorems, using ideal points of the parameter space.
Today: relations between complexity and families, some effective bounds including a constant depending on the family. Problem of effectiveness of uniform bounds. Finally, a few words concerning effectiveness issues in proving that a given structure is o-minimal.

Complexity and families

Say a semialgebraic set in R^n has complexity $\leq (c, d)$ if it can be described by a boolean combination of sign conditions on $\leq c$ polynomials. of degrees $\leq d$. Then:

- Given a semialgebraic family, there is (c, d) such that every semaialgebraic set in the family has complexity $\leq (c, d)$.
- Semialgebraic subsets of R^n of complexity $\leq (c, d)$ form a semialgebraic family.

So uniform bounds in term of complexity = uniform bounds in families.

A metric uniform bound

Theorem

Let $X \subset \mathbb{R}^m \times \mathbb{R}^2$ be a family of curves, definable for some o-minimal structure. Then there is a constant $c(X)$ depending only on the family X such that, for every disk $D(a,r) \subset \mathbb{R}^2$ and every $t \in \mathbb{R}^m$, one has $\mathrm{length}(X_t \cap D(a,r)) \leq c(X)\, r$.

Proof: Uniform finiteness for the definable family of $X_t \cap D(a,r) \cap L$ (L a line) parameterized by t, a, r and L + Cauchy-Crofton formula which computes the length of a curve by integrating over the Grassmannian of lines its number of intersections with a line.

Combinatorial complexity (S. Basu)

If $X_1, \ldots, X_s$ are definable subsets of R^m, denote by $\mathcal{C}(X_1, \ldots, X_s)$ the collection of all non empty intersections of the form $Y_1 \cap \ldots \cap Y_s$ where $Y_i = X_i$ or $Y_i = R^m \setminus X_i$.

Theorem

Let $X \subset R^m \times R^n$ be a definable family for some o-minimal structure. Then there is a constant $c(X)$ depending only on the family X such that, for every positive integer s and every $t_1, \ldots, t_s$ in R^m, we have

$$\sum_{Y \in \mathcal{C}(X_{t_1}, \ldots, X_{t_s})} \beta_i(Y) \le c(X)\, s^{m-i}$$

S. Basu, Proc. London Math. Soc. (2010) 100: 405-428.

h-cobordism

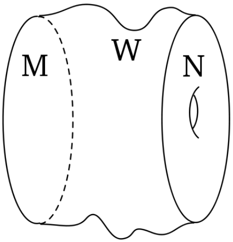


Figure: Cobordism between M and N

- A *cobordism* between M and N is a compact manifold W with boundary equal to the disjoint union of M and N.
- It is called *h-cobordism* if both M and N are deformation retract of W.

Theorem (Smale, 1961)

Let (W, M, N) be a simply connected h-cobordism, $\dim W \geq 6$. *Then $W \simeq M \times [0,1]$.*

Semialgebraic h-cobordism with complexity

Theorem

There exists $\Psi(m,n,p,q) \in \mathbb{N}^2$ *such that for every simply connected semialgebraic h-cobordism* (W,M,N) *in* $\mathbb{R}^n$ *with* $\dim(W) = m \geq 6$, *of complexity* $\leq (p,q)$, *there exists a semialgebraic homeomorphism* $h : W \to M \times [0,1]$ *of complexity* $\leq \Psi(m,n,p,q)$.

Sketch of proof:

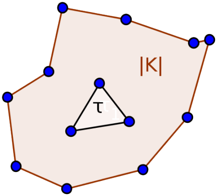
- Put all semialgebraic triples (W,M,N) of subsets of $\mathbb{R}^n$ of complexity $\leq (p,q)$ in a semialgebraic family.
- Apply Hardt's trivialisation theorem and retain the pieces of the parameter space over which the assumptions of the PL-h cobordism theorem are fulfilled.
- Over each piece, apply PL-h cobordism theorem to one fiber.

Non-effectiveness of the bound

Theorem

$\Psi(m,n,p,q)$ *cannot be recursive.*

Proof. Reduction of the problem of recognizing PL balls to h-cobordism: K a simplicial complex such that $|K|$ is a PL m-ball. There is a subdivision K' of K simplicially isomorphic to a subdivision of the m-simplex; for $m \geq 6$, $\sharp K'$ cannot be bounded by a recursive function of $\sharp K$.



K. Demdah, to appear in Ann. Inst. Fourier

Proving o-minimality

- Usual tools for proving that a structure (presented by a family of functions) is o-minimal are quantifier elimination and model completeness.
- Quantifier elimination: every definable set may be described by a quantifier free formula (combination of $f > 0$, $f = 0$). Example: semialgebraic sets, Tarski-Seidenberg theorem. Then one has to check that such a set has finitely many connected components.
- Model-completeness: every definable set may be described by an existential formula (projection of a set defined by a combination of $f > 0$, $f = 0$). Example: subanalytic sets, theorem of the complement (A. Gabrielov). It is then sufficient to check that sets defined by a combination of $f > 0$, $f = 0$ have finitely many connected components (Khovanskii, ...).

Algorithmic issues

- The case of quantifier elimination for real closed fields is well known.
- Effective (algorithmic) aspects of the theorem of the complement for a specified structure? Gabrielov, Wilkie, Macintyre, Speisseger, Rolin...