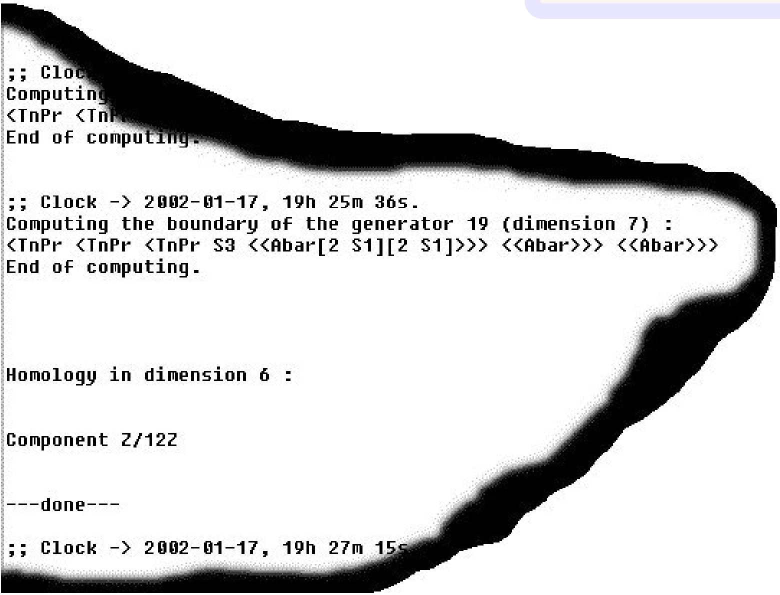

Constructive Homology Classes and Constructive Triangulations

Dedicated to *Mirian Andrès*

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Mathematics Algorithms and Proofs
Logroño, Spain, 8-12 November, 2010

Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective, algorithm, machine object, . . .**
Violet	=	**Problem, difficulty, obstacle, disadvantage, . . .**
Green	=	**Solution, essential point, mathematicians, . . .**
Dark Orange	=	**Fuzzy objects.**
Pale grey	=	**Hyper-Fuzzy objects.**

Plan.

1. **Constructive Homological Algebra.**

2. **Triangulations and fundamental cycles.**

3. **Complex projective spaces.**

4. **Connection $P^n\mathbb{C} \longleftrightarrow P^\infty\mathbb{C}$.**

5. **Kenzo program + Constructive Homological Algebra**

   $\Rightarrow$ Constructive Triangulation of $P^n\mathbb{C}$.

1/5. Constructive Homological Algebra.

General style of Homological Algebra:

First step in the classification of angiosperms:

Number of cotyledons = 1 or 2.

$n = 1 \Rightarrow$ Monocotyledons ($\sim$ 60.000 species).

$n = 2 \Rightarrow$ Dicotyledons ($\sim$ 200.000 species)

First step in the classification of topological spaces:

$$(\forall X \in \underline{\text{Top}}) \Rightarrow [(\forall d \in \mathbb{N}) \Rightarrow H_d(X) \in \underline{\text{AbGroup}}].$$

Only partial classification !!!

Main problem:

Let $\Phi : \underline{\mathbf{Top}} \times \underline{\mathbf{Top}} \to \underline{\mathbf{Top}}$ be a constructor.

Example: $\Phi(X, Y) := X \times Y$.

Homological version of this constructor ??

$$\Phi_H : (H_*(X), H_*(Y)) \xrightarrow{???} \boxed{\overset{???}{H_*(\Phi(X,Y))}}$$

Sometimes possible, for example

for the product constructor (Künneth formulas).

In general not !!

Example:

The loop space constructor $\Omega : X \mapsto \Omega X := \mathcal{C}(S^1, X)$

Example²:

$$X = S^2 \vee S^4 \qquad Y = P^2\mathbb{C}$$

$$H_*(X) = H_*(Y) = (\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, 0, 0, \ldots)$$

$$H_*(\Omega X) = (\mathbb{Z},\ \mathbb{Z},\ \mathbb{Z},\ \mathbb{Z}^2,\ \mathbb{Z}^3,\ \mathbb{Z}^4,\ \mathbb{Z}^6,\ \mathbb{Z}^9,\ \mathbb{Z}^{13},\ \ldots)$$

$$H_*(\Omega Y) = (\mathbb{Z},\ \mathbb{Z},\ 0,\ 0,\ \mathbb{Z},\ \mathbb{Z},\ 0,\ 0,\ \mathbb{Z},\ \ldots)$$

Corollary: $\nexists$ algorithm $\Omega_H : H_*(X) \mapsto H_*(\Omega X)$.

Analysis of the problem.

Ordinary homological algebra is not constructive.

$H_4(X)$ "=" $\mathbb{Z}$ means:

$$\exists \text{ isomorphism } H_4(X) \xleftrightarrow{\cong} \mathbb{Z} \ ;$$

But most often $\exists$ is not constructive.

Reorganizing Homological Algebra

to make these $\exists$'s constructive

$\Rightarrow$ Constructive Homological Algebra

$\Rightarrow$ Algorithms:

$$\Phi_{CH} : (CH_*(X), CH_*(Y)) \mapsto CH_*(\Phi(X,Y)).$$

$\Rightarrow$ (JR) Efficient solution of Adams' problem for loop spaces.

2/5. Triangulations and fundamental cycles.

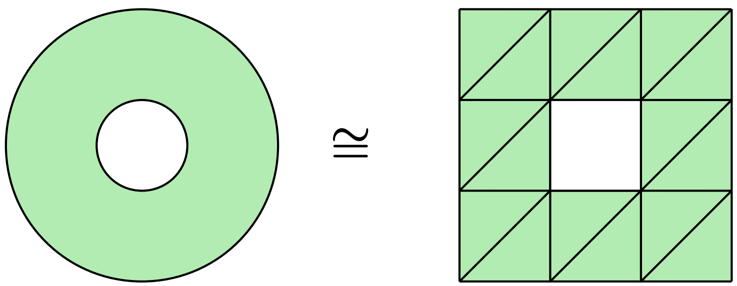
Amazing spin-off of Constructive Homological Algebra:

Using constructive isomorphisms

to produce difficult triangulations.

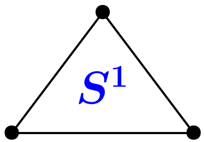
Notions of triangulation.

Triangulation as a simplicial complex of $S^1 \times I$.



Triangulations of S^1 as simplicial:

complex



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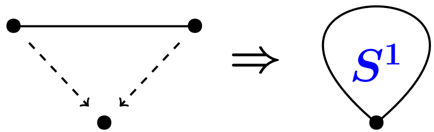
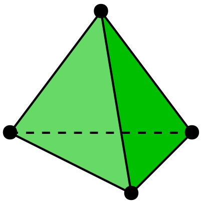


Triangulations of S^2 as simplicial:

complex

S^2



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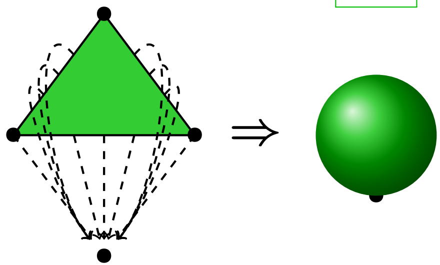


S^2

Fundamental Homological Theorem for closed manifolds:

$M =$ closed n-manifold $\Rightarrow$ M is triangulable.

We assume M orientable.

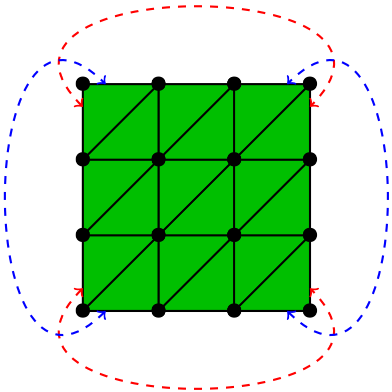
Let $\mathcal{T}$ be some triangulation

and T_n the corresponding collection of n-simplices.

Then $H_n(M) = \mathbb{Z}$

and a cycle representing a generator of H_n is $z = \sum_{\tau \in T_n} \varepsilon_\tau \tau$.

Example for $M =$ 2-torus:

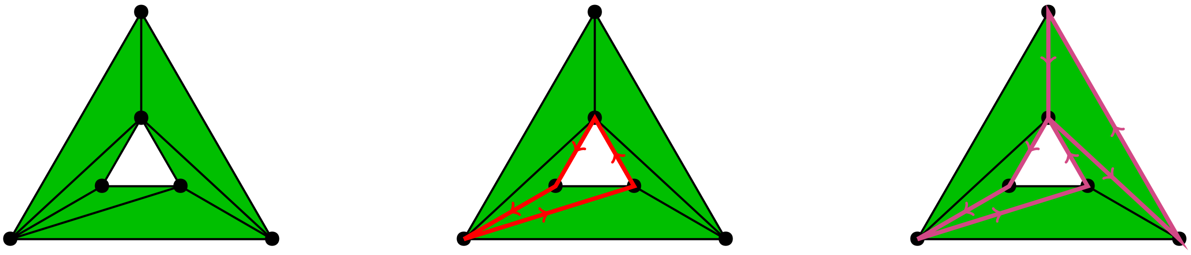


In a context of Constructive Homological Algebra,

the result can sometimes be reversed.

Toy example with $S^1 \times I \overset{H}{\sim} S^1$.

$$H_*(S^1 \times I) = H_*(S^1) = (\mathbb{Z}, \mathbb{Z}, 0, 0, 0, \ldots)$$



$S^1 \times I$ | **Good generator of $H_1(S^1 \times I)$** | **Bad generator of $H_1(S^1 \times I)$**

3/5. Complex projective spaces.

Using this method to construct a triangulation of $P^n\mathbb{C}$.

$$S^{2n+1} = \text{unit sphere}(\mathbb{C}^{n+1})$$

$$P^n\mathbb{C} := S^{2n+1}/S^1$$

$$S^1 \subset S^3 \subset S^5 \subset \cdots \subset S^\infty$$

$$* \subset P^1\mathbb{C} \subset P^2\mathbb{C} \subset P^3\mathbb{C} \subset \cdots \subset P^\infty\mathbb{C}$$

Universal fibration:

$$K(\mathbb{Z},1) = S^1 \hookrightarrow S^\infty \rightarrowtail P^\infty\mathbb{C}$$

$$\Rightarrow P^\infty\mathbb{C} = K(\mathbb{Z},2)$$

$K(\mathbb{Z},2)$ = "catalog" space =

collection of all the possible configurations

of elements $z \in H^2(-,\mathbb{Z})$

Standard simplicial model for $K(\mathbb{Z},2)$

due to Eilenberg-MacLane.

$K(\mathbb{Z},2)$ = Monster: $K(\mathbb{Z},2)_n \sim \mathbb{Z}^{\frac{n(n-1)}{2}}$

But the methods of Constructive Algebraic Topology

can handle this monster.

4/5. Connection $P^n C \longleftrightarrow P^\infty\mathbb{C}$.

$P^\infty\mathbb{C} = \lim_{n\to\infty} P^n\mathbb{C}$ has a good homological translation:

$$\begin{aligned}
H_*(P^\infty\mathbb{C}) &= (\overset{0}{\mathbb{Z}}, 0, \overset{2}{\mathbb{Z}}, 0, \overset{4}{\mathbb{Z}}, 0, \overset{6}{\mathbb{Z}}, 0, \overset{8}{\mathbb{Z}}, 0, \overset{10}{\mathbb{Z}}, 0, \ldots)\\
H_*(P^1\mathbb{C}) &= (\mathbb{Z}, 0, \mathbb{Z}, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots)\\
H_*(P^2\mathbb{C}) &= (\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, 0, 0, 0, 0, 0, 0, \ldots)\\
H_*(P^3\mathbb{C}) &= (\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, 0, 0, 0, 0, \ldots)\\
\cdots &= \cdots
\end{aligned}$$

Also the inclusion $P^n\mathbb{C} \hookrightarrow P^\infty\mathbb{C}$

induces an inclusion $H_*(P^n\mathbb{C}) \hookrightarrow H_*(P^\infty\mathbb{C})$.

So that a generator g_{2n} of $H_{2n}(P^\infty\mathbb{C})$

corresponds to a generator g_{2n} of $H_{2n}(P^n\mathbb{C})$

which could correspond to a triangulation of $P^n\mathbb{C}$.

5/5. Kenzo calculations.

1. kz2 $:= K(\mathbb{Z}, 2)$

2. "Local" calculations are possible.

3. The effective homology is computable:

$$[C_*(K(\mathbb{Z},2)) = \mathtt{K86}] \Lleftarrow\!\!\Lleftarrow \mathtt{K216} \Rrightarrow\!\!\Rrightarrow \mathtt{K212}$$

4. G4 = generator of $H_4(\mathtt{K212}) = \mathbb{Z}$.

5. GP4 = generator of $H_4(\mathtt{K86}) = H_4(K(\mathbb{Z},2)) = \mathbb{Z}$.

6. P2C? = finite simplicial subset of $K(\mathbb{Z},2)$ generated by GP4.

Kenzo calculations (continuation):

5. `GP4` $=$ **generator of** $H_4(\texttt{K86}) = H_4(K(\mathbb{Z},2)) = \mathbb{Z}$.

6. `P2C?` $=$ **finite simplicial subset of** $K(\mathbb{Z},2)$

generated by `GP4`.

Next question: $\texttt{P2C?} \overset{???}{=} P^2\mathbb{C}$

Proposition: $\texttt{P2C?} = P^2\mathbb{C} \Leftarrow$ **the inclusion P2C?** $\hookrightarrow K(\mathbb{Z},2)$

induces isomorphisms:

$$H_k(\texttt{P2C?}) \xrightarrow{\boxed{\cong\ ??}} H_k(K(\mathbb{Z},2))$$

for $k \leq 4$.

Proof: Hurewicz-Whitehead Theorem.

$\texttt{P2C?} = P^2\mathbb{C} \quad \Leftrightarrow \quad H_k(\texttt{P2C?}) \xrightarrow{\boxed{\cong\ ??}} H_k(K(\mathbb{Z},2))$

Cone constructor:

$$\texttt{P2C?} \overset{\text{inclusion}}{\hookrightarrow} K(\mathbb{Z},2)$$

$$C_*(\texttt{P2C?}) \overset{\text{inclusion}}{\hookrightarrow} C_*(K(\mathbb{Z},2))$$

$$\text{Cone(inclusion)} := C_*(\texttt{P2C?})^{[+1]} \oplus_{\text{inclusion}} C_*(K(\mathbb{Z},2))$$

Proposition: $H_k(\texttt{P2C?}) \xrightarrow{\boxed{\cong\ ??}} H_k(K(\mathbb{Z},2))$ for $k \leq 4$

$\Leftrightarrow$

$H_k(\text{Cone(inclusion)}) = 0$ for $k \leq 5$

Proof: Elementary homological algebra.

Kenzo calculations (continuation):

5. `GP4` = **generator** of $\boldsymbol{H_4}(\texttt{K86}) = \boldsymbol{H_4}(\boldsymbol{K}(\mathbb{Z}, 2)) = \mathbb{Z}$.

6. `P2C?` = **finite simplicial subset** of $\boldsymbol{K}(\mathbb{Z}, 2)$ **generated by** `GP4`.

7. **Construction of Cone** $\left\{ \boldsymbol{C_*}(\texttt{P2C?}) \xhookrightarrow{\text{inclusion}} \boldsymbol{C_*}(\boldsymbol{K}(\mathbb{Z}, \mathbf{2})) \right\}$

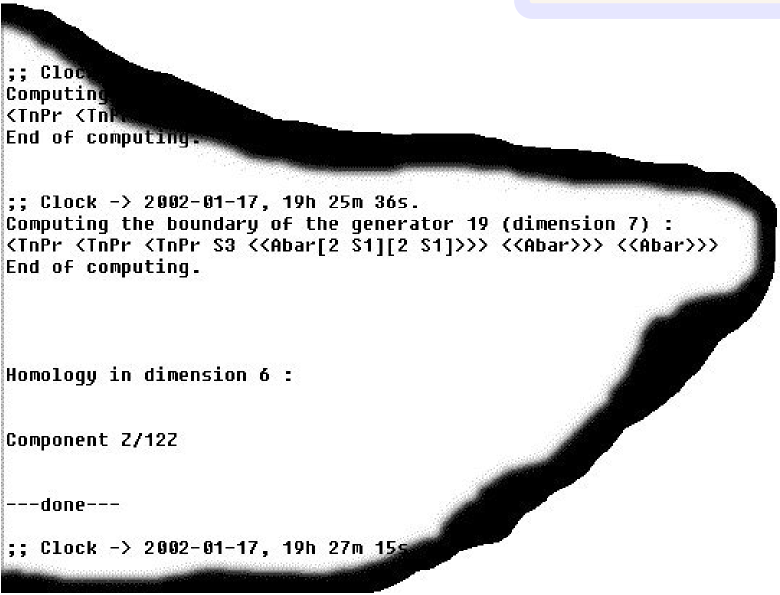
8. **Calculation of** $\boldsymbol{H_k}(\textbf{Cone}\,\{\cdots\})$ **for** $\boldsymbol{k} \leq \mathbf{6}$.

9. $\boldsymbol{H_k}(\textbf{Cone}) = \mathbf{0}$ **for** $\boldsymbol{k} \leq \mathbf{5} \Rightarrow$ `P2C?` $= \boldsymbol{P}^2\mathbb{C}$.

   $\Rightarrow$ **a triangulation of** $\boldsymbol{P}^2\mathbb{C}$ **as** `P2C?` **is obtained.**

10. **The same for higher dimensions.**

The END



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