Effective homology and discrete Morse theory for the computation of homology of groups

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Maths, Algorithms and Proofs
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- Algebra: $H_i(G)$ are computed by means of *resolutions*.

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Definition

A *resolution* F_* for a group G is an acyclic chain complex of $\mathbb{Z}G$-modules

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} F_{-1} = \mathbb{Z} \longrightarrow 0$$

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Theorem

Let G be a group and F_*, F'_* two free resolutions of G. Then

$$H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*) \cong H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F'_*) \cong H_n(K(G,1)) \quad \textit{for all } n \in \mathbb{N}$$

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Definition

Given a group G, the *homology groups* $H_n(G)$ are defined as $H_n(G) = H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*)$, $n \in \mathbb{N}$, where F_* is any free resolution for G.

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For some particular cases, small (or minimal) resolutions can be directly constructed.

For instance, let $G = C_m$ with generator t. The resolution F_*

$$\cdots \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

produces

$$H_i(G) = \begin{cases} \mathbb{Z} & \text{if } \; i = 0 \\ \mathbb{Z}/m\mathbb{Z} & \text{if } \; i \text{ is odd} \\ 0 & \text{if } \; i \text{ is even}, i > 0 \end{cases}$$

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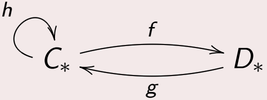
A *reduction* ρ between two chain complexes C_* and D_* (denoted by $\rho : C_* \Rrightarrow D_*$) is a triple $\rho = (f, g, h)$



satisfying the following relations:

1) $fg = \mathrm{Id}_{D_*}$;
2) $d_C h + h d_C = \mathrm{Id}_{C_*} - gf$;
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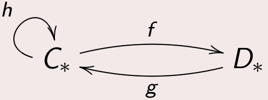

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2) $d_C h + h d_C = \mathrm{Id}_{C_*} - gf$;
3) $fh = 0$; $hg = 0$; $hh = 0$.

If $C_* \Rrightarrow D_*$, then $C_* \cong D_* \oplus A_*$, with A_* acyclic, which implies that $H_n(C_*) \cong H_n(D_*)$ for all n.

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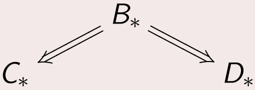
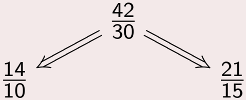
A *(strong chain) equivalence* ε between C_* and D_*, $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$, is a triple $\varepsilon = (B_*, \rho, \rho')$ where B_* is a chain complex, $\rho : B_* \Rrightarrow C_*$ and $\rho' : B_* \Rrightarrow D_*$.





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$$C_* \Lleftarrow B_* \Rrightarrow D_* \qquad\qquad \frac{14}{10} \Lleftarrow \frac{42}{30} \Rrightarrow \frac{21}{15}$$

Definition

An *object with effective homology* is a quadruple $(X, C_*(X), HC_*, \varepsilon)$ where HC_* is an effective chain complex and $\varepsilon : C_*(X) \lll\!\!\Longrightarrow\!\!\ggg HC_*$.

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This implies that $H_n(X) \cong H_n(HC_*)$ for all n.

Algorithm computing the effective homology of a group

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Given G a group, F_* a (small) free $\mathbb{Z}G$-resolution with a *contracting homotopy* $h_n : F_n \to F_{n+1}$.

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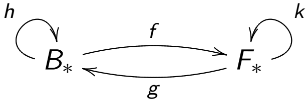
It is well known that there exists a morphism of chain complexes of $\mathbb{Z}G$-modules $f : B_* \to F_*$ which is a homotopy equivalence. An algorithm has been designed constructing the explicit expressions of f and the corresponding maps g, h and k



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Applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ we obtain an equivalence of chain complexes (of $\mathbb{Z}$-modules):

$$\overset{h}{\circlearrowright}\ \mathbb{Z} \otimes_{\mathbb{Z}G} B_* \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{Z} \otimes_{\mathbb{Z}G} F_*\ \overset{k}{\circlearrowleft}$$

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In order to obtain a strong chain equivalence we make use of the mapping cylinder construction.

$$\mathbb{Z} \otimes_{\mathbb{Z}G} B_* \overset{\rho'}{\Lleftarrow} \text{Cylinder}(f)_* \overset{\rho}{\Rrightarrow} \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$$

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Finally we observe that the left chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} B_*$ is equal to $C_*(K(G,1))$. Moreover, if the initial resolution F_* is of finite type (and small), then the right chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} F_* \equiv E_*$ is effective.

Algorithm computing the effective homology of a group

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Algorithm

Input: a group G and a free resolution F_* of finite type with contracting homotopy.
Output: the effective homology of $K(G,1)$, that is, a (strong chain) equivalence $C_*(K(G,1)) \Longleftarrow\!\!\Longrightarrow E_*$ where E_* is an effective chain complex.

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- Implemented in Common Lisp, enhancing the Kenzo system (developed by F. Sergeraert and some coworkers).
- It allows to compute homology of groups and, what is more important, to use the space $K(G,1)$ in other constructions allowing new computations.

Applications and examples

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Computations with $K(C_p, n)'s$

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```
> (setf C7 (cyclicGroup 7))
[K1 Abelian-Group]
> (setf KC71 (K-G-1 C7))
[K2 Abelian-Simplicial-Group]
> (efhm KC71)
[K50 Homotopy-Equivalence K2 <= K40 => K31]
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```
> (setf KC72 (classifying-space KC71))
[K51 Abelian-Simplicial-Group]
> (efhm KC72)
[K190 Homotopy-Equivalence K51 <= K180 => K176]
> (homology KC72 3 6)
Homology in dimension 3 :
---done---
Homology in dimension 4 :
Component Z/7Z
---done---
Homology in dimension 5 :
---done---
```

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Homology of 2-types

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Let $G = C_3$, $A = \mathbb{Z}/3\mathbb{Z}$ with trivial G-action, and $[f] \in H^3(G, A) = \mathbb{Z}/3\mathbb{Z}$ a non-trivial cohomology class.

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```
> (setf K-C3-1 (K-Cm-n 3 1))
[K261 Abelian-Simplicial-Group]
> (setf chml-clss (chml-clss K-C3-1 3))
[K308 Cohomology-Class on K288 of degree 3]
> (setf tau (zp-whitehead 3 K-C3-1 chml-clss))
[K323 Fibration K261 -> K309]
> (setf x (fibration-total tau))
[K329 Kan-Simplicial-Set]
> (efhm x)
[K541 Homotopy-Equivalence K329 <= K531 => K527]
> (homology x 5)
Homology in dimension 5 :
Component Z/3Z
---done---
```

Applications and examples

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Central extensions

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$$E = \langle x, y, z \mid x^p = y^p = z^{p^{n-2}} = [x, z] = [y, z] = 1; [x, y] = z^{p^{n-3}} \rangle$$

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can be seen as a central extension of the groups
$A = \langle z \mid z^{p^{n-2}} = 1 \rangle \cong C_{p^{n-2}}$
and $G = \langle x, y \mid x^p = y^p = [x, y] = 1 \rangle \cong C_p \oplus C_p$.

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```
(progn
     (setf p 3 n 4) (setf A (cyclicGroup (expt  p (- n 2))))
     (setf G (gr-crts-prdc (cyclicGroup p) (cyclicGroup p)))
     (setf cocycle #'(lambda (crpr1 crpr2)
                    (with-grcrpr (x1 y1) crpr1
                      (with-grcrpr (x2 y2) crpr2
                        (mod (* y1 x2 (1- p) (expt p (- n 3))) (expt p (- n 2)))))))
     (setf E (gr-cntr-extn A G cocycle)))
[K663 Group]
```

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```
> (setf KE1 (K-G-1 E))
[K776 Simplicial-Group]
> (efhm KE1)
[K884 Homotopy-Equivalence K776 <= K870 => K866]
> (homology KE1 3)
Homology in dimension 3 :
Component Z/9Z
Component Z/3Z
Component Z/3Z
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---done---
```

Discrete Morse theory

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Definition

Let $C_* = (C_p, d_p)_{p\in\mathbb{Z}}$ a free chain complex with distinguished $\mathbb{Z}$-basis $\beta_p \subset C_p$. A *discrete vector field* V on C_* is a collection of pairs $V = \{(\sigma_i;\tau_i)\}_{i\in I}$ satisfying the conditions:

- Every σ_i is some element of β_p, in which case $\tau_i \in \beta_{p+1}$. The degree p depends on i and in general is not constant.
- Every component σ_i is a *regular face* of the corresponding τ_i.
- Each generator (*cell*) of C_* appears at most one time in V.

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- Each generator (*cell*) of C_* appears at most one time in V.

Definition

A V-path of degree p and length m is a sequence $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0 \leq k < m}$ satisfying:

- Every pair $((\sigma_{i_k}, \tau_{i_k}))$ is a component of V and τ_{i_k} is a p-cell.
- For every $0 < k < m$, the component σ_{i_k} is a face of $\tau_{i_{k-1}}$, non necessarily regular, but different from $\sigma_{i_{k-1}}$.

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A discrete vector field V is *admissible* if for every $p \in \mathbb{Z}$, a function $\lambda_p : \beta_p \to \mathbb{N}$ is provided satisfying the following property: every V-path starting from $\sigma \in \beta_p$ has a length bounded by $\lambda_p(\sigma)$.

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A cell σ which does not appear in a discrete vector field V is called a *critical cell*.

Theorem

Let $C_* = (C_p, d_p)_{p \in \mathbb{Z}}$ be a free chain complex and $V = \{(\sigma_i; \tau_i)\}_{i \in I}$ be an admissible discrete vector field on C_*. Then the vector field V defines a canonical reduction $\rho = (f, g, h) : (C_p, d_p) \Rightarrow\!\!\!\!\Rightarrow (C_p^c, d_p')$ where $C_p^c = \mathbb{Z}[\beta_p^c]$ is the free $\mathbb{Z}$-module generated by the critical p-cells.

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1. *The critical cells are the unique 0-simplex $[\,]$, the $(2k)$-simplex $[p-1, 1, \dots, p-1, 1]$ for each $k \geq 1$ and the $(2k+1)$-simplex $[1, p-1, 1, \dots, p-1, 1]$ for each $k \geq 0$.*
2. *The source cells are the k-simplices $[a_1, \dots, a_{k-2r}, p-1, 1, \dots, p-1, 1]$ with $a_{k-2r} > 1$ and $0 \leq r < k/2$.*
3. *The target cells are the $(k+1)$-simplices $[a_1, \dots, a_{k-2r}, 1, p-1, 1, \dots, p-1, 1]$ with $a_{k-2r} < p-1$ and $0 \leq r < k/2$.*
4. *The pairing [source cell $\leftrightarrow$ target cell] associates to the source cell $[a_1, \dots, a_{k-2r}, p-1, 1, \dots, p-1, 1]$ the target cell $[a_1, \dots, a_{k-2r} - 1, 1, p-1, 1, \dots, p-1, 1]$.*

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This gives us a reduction $C_*(K(C_p, 1)) \Rrightarrow \mathbb{Z}[\beta^c]$, where $\mathbb{Z}[\beta^c]$ is the free chain complex generated by the critical cells, which is an effective (and small) chain complex.

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$$C_*(F \times_\tau B) \Longleftarrow\!\!\Rightarrow EFF'_*$$

where EFF'_* is an effective chain complex.

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- A central extension E satisfies $K(E,1) \cong K(A,1) \times_\tau K(G,1)$. Again the discrete vector field for the Eilenberg-Zilber reduction improves the effective homology of $K(E,1)$.

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- With respect to computation time, our first experiments have not been as good as expected. Work should be done trying to improve it.