MAP 2010 - LOGROÑO

November 8-12, 2010

Chain calculus and Krull dimension in distributive lattices

Luis Español

Universidad de La Rioja

luis.espanol@unirioja.es

I. Evolution of Krull dimension of distributive lattices

II. Link dimension of distributive lattices

I always consider bounded distributive lattices L with a bottom element 0 and a top element 1, and lattice morphisms preserving 0 and 1. They form the category $\mathcal{L}$.

Boolean algebras:

$$\mathcal{C}(L) \hookrightarrow L \hookrightarrow L_\neg, \quad \begin{cases} \mathcal{C}(L) & \text{centre of } L \\ L_\neg & \text{freely generated by } L \end{cases}$$

Dimension of distributive lattices

Five definitions (equivalent of course)
in chronological order:

1. *Kdim L*: Krull's notion with chains of prime ideals of L
2. *Sdim L*: Simplicial notion given by Joyal
3. *Bdim L*: Boolean notion given by Español
4. *Edim L*: Elementary notion given by Coquand & Lombardi
5. *Ldim L*: Linked chain dimension given by Español

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1. Classical definition:

$Kdim L \leq n$

if any $(n+1)$-chain of prime ideals in L is degenerated.

2-5. "Constructuive" definitions

Colloque TAC, Amiens 1975.
http://vbm-ehr.pagesperso-orange.fr/ChEh/

Dimension of ordered sets

Given an *ordered set* X, we have the ordered set X_n of all n-chains $x_0 \leq \cdots \leq x_n$ in X, $n \geq 0$ ($n =$ number of $\leq$)

We define *dimension* of ordered sets by the length of its chains:

$dimX \leq n$ if any $(n+1)$-chain is degenerated.

$dimX = n$ if $dimX \leq n$ and there exists a non degenerated n-chain.

In particular, $dimX \leq 0$ if and only if X is a trivial poset.

If we consider the simplicial object $\{X_n\}$, then we can give categorical statements:

- Each X_n is universal for monotone maps
$$(p_0 \leq \cdots \leq p_n) : X_n \to X.$$
- $dimX \leq n$ if and only if
$$\langle s_0, \cdots, s_n \rangle : \coprod_{n+1} X_n \to X_{n+1} \text{ is epi.}$$

Simplicial dimension

We proceed by categorical duality with ordered sets.

Given a distributive lattice L,

- We have a cosimplicial object $\{L_n\}$ of distributive lattices, where L_n is universal for chains of morphisms

$$(p_0 \leq \cdots \leq p_n) : L = L_0 \to L_n.$$

- Definition (1975):

  $SdimL \leq n$

$$\text{if } (s_0, \cdots, s_n) : L_{n+1} \to \prod_{n+1} L_n \text{ is mono.}$$

In the classical (non-constructive) setting we have (prime ideals are morphisms $L \to \{0, 1\}$):

- Prime ideals of L_n are in bijection with n-chains $p_0 \leq \cdots \leq p_n$ of prime ideals of L.

- $SdimL$ is equivalent to *Krull dimension*.

Boolean dimension

In the next formulas the elements x_k are in L.

Definition (1978):

(i) $BdimL \leq 2n$ if for every $x \in L_\neg$, $x = x_0 \vee \bigvee_{i=1}^{n}(x_i \wedge \neg y_i)$.

(ii) $BdimL \leq 2n+1$ if for every $x \in L_\neg$, $x = \bigvee_{i=1}^{n+1}(x_i \wedge \neg y_i)$.

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1. L is discrete if and only if so $L_\neg$ is.
2. This definition can be given in terms of elements of L only.

(Español, talks at Milano, 1987)

(Español, notes 2001, after Coquand & Lombardi)

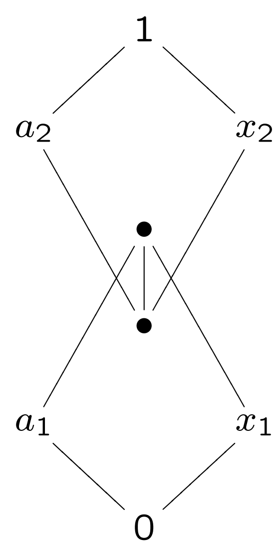
Elementary dimension

Definition: $EdimL \leq n$

if for any $a_1, \ldots, a_{n+1} \in L$ there exists $x_1, \ldots, x_{n+1} \in L$ such that

$$(\bullet\bullet) \begin{cases} a_{n+1} \vee x_{n+1} = 1 \\ a_{i+1} \wedge x_{i+1} \leq a_i \vee x_i, \quad i = 1, \ldots, n \\ a_1 \wedge x_1 = 0 \end{cases}$$

$n = 2$ figure



Link dimension

Definition: $LdimL \leq n$

if for any $(n+1)$-chain $\mathbf{a} : a_1 \leq \cdots \leq a_{n+1}$ in L there exists a $(n+1)$-chain $\mathbf{x} : x_1 \leq \cdots \leq x_{n+1}$ in L such that

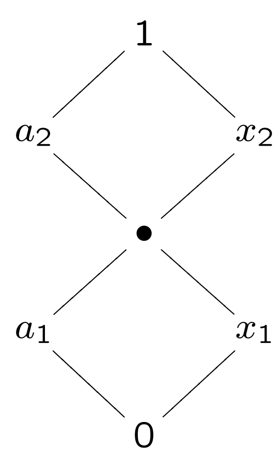
$$(\bullet) \begin{cases} a_{n+1} \vee x_{n+1} = 1 \\ a_{i+1} \wedge x_{i+1} = a_i \vee x_i, i = 1, \ldots, n \\ a_1 \wedge x_1 = 0 \end{cases}$$

$(\bullet)$: $\mathbf{a}, \mathbf{x}$ are *linked*. Then $\mathbf{a}, \mathbf{x}$ are "relatively complemented".

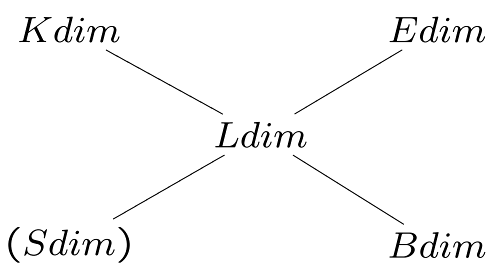
$LdimL \leq n$ if any $(n+1)$-chain is linked

$LdimL \leq 0$ means that L is a Boole algebra

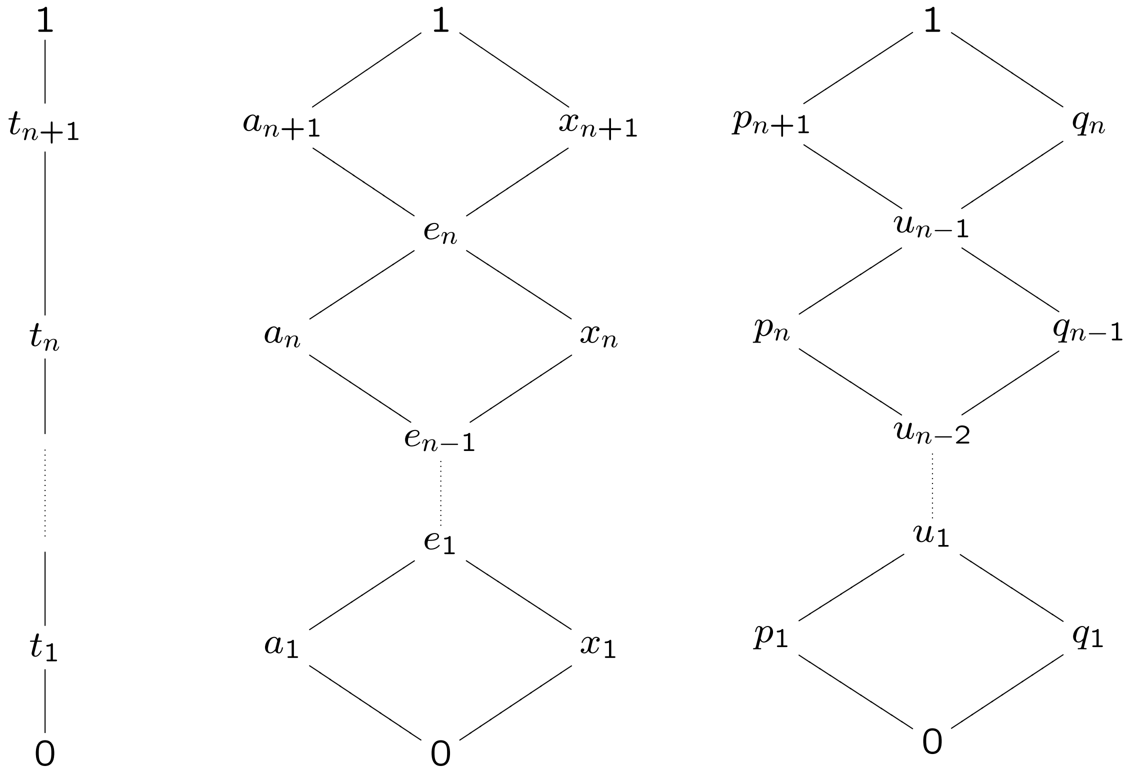
2-link figure



Link dimension in the middle



Linked chains



Ch_{n+1} $\quad$ $\mathbf{a} \diamond_{\mathbf{e}} \mathbf{x}$ in L $\quad$ $Link_{n+1}$

$$Ch_0 = 2 = \{0 < 1\} = Link_0, \;\; Ch_{-1} = \{0 = 1\} = Link_{-1}$$

Then: $\mathbf{a}$, $\mathbf{x}$ are linked with *node chain* $\mathbf{e}$

x_i, a_i are complemented in the interval $[e_{i-1}, e_i]$, ($e_0 = 0$, $e_{i+1} = 1$)

For $\mathbf{a}$, $\mathbf{e}$ given, $\mathbf{x}$ is unique when it exists

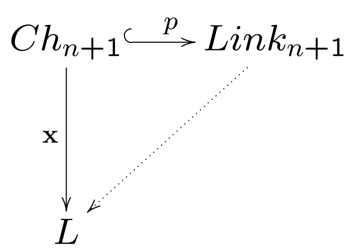
Linked chains as morphisms

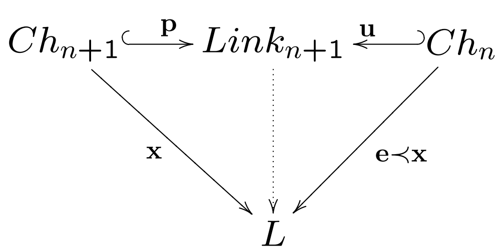
A n-chain in L is a lattice morphisms $\quad \mathbf{x} : Ch_n \to L$

For instance $\mathbf{p} : Ch_n \hookrightarrow Link_n$, $\mathbf{p}(t_i) = p_i$

A n-link in L is a lattice morphisms $\quad \mathbf{x} : Link_n \to L$

$LdimL \leq n$ is an extension property:





$\mathbf{u} \prec \mathbf{p}$

$\mathbf{u} \prec \mathbf{p}$: $\mathbf{u}$ *separates* $\mathbf{p}$

Link dimension and K-dimension

$$LdimL \leq n \qquad \Leftrightarrow \qquad KdimL \leq n$$

We have *cochains*: morphisms $\mathbf{P} : L \to Ch_n$

$\mathbf{P}$ is completely determined by a chain $P_0 \subseteq \cdots \subseteq P_n$ of prime ideals.

$\mathbf{a} \in \mathbf{P}$: $\mathbf{a} : Ch_n \to L$ *belongs* to $\mathbf{P} : L \to Ch_n$ if

$$\mathbf{P} \circ \mathbf{a} = \mathbf{id} : Ch_n \to Ch_n$$

$\mathbf{P}$ is *onto* if and only if the chain of prime ideals is *non-degenerated*.

$\mathbf{a}$ is a *section* if and only if $\mathbf{a}$ is *not linked*.

L. Español: "Finite chain calculus in distributive lattices and elementary Krull dimension". *Contribuciones científicas en honor de Mirian Andrés Gómez*. L. Lambán, A. Romero, J. Rubio (eds.) Ser. de Publ., Univ. de La Rioja, Logroño 2010, pp. 273-285. http://www.unirioja.es/servicios/sp/catalogo/monografias/

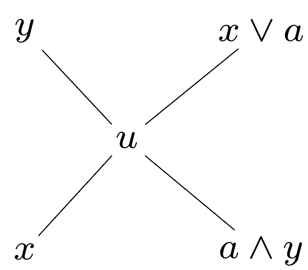
F. W. Anderson, R. L. Blair. "Representations of distributive lattices as lattices of functions". *Math. Ann.* **143** (1961) 187–211.
R. Balbes & Ph. Dwinger, *Distributive lattices*, 1974.

Linked chain calculus

Elementary pieces

To insert into an interval. *Universal* morphism $l : L \to [x, y]$.

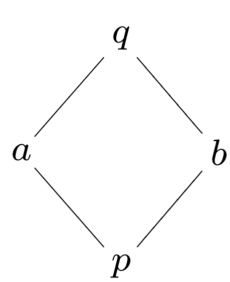
a :



a is *inserted* on $l(a) = u \in [x, y]$

Chain associativity: $u = \begin{cases} (x \vee a) \wedge y \\ x \vee (a \wedge y) \end{cases} \quad (x \leq y)$

The diamond



$q = a \vee b$ $\quad [p, b] \cong [a, q]$ $\quad q + b = p + a$

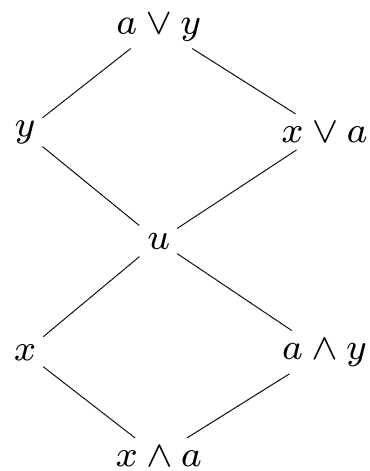
To insert $\quad a + b = p + q$

$p = a \wedge b$ $\quad [p, b] \cong [a, q]$ $\quad p + b = q + a$

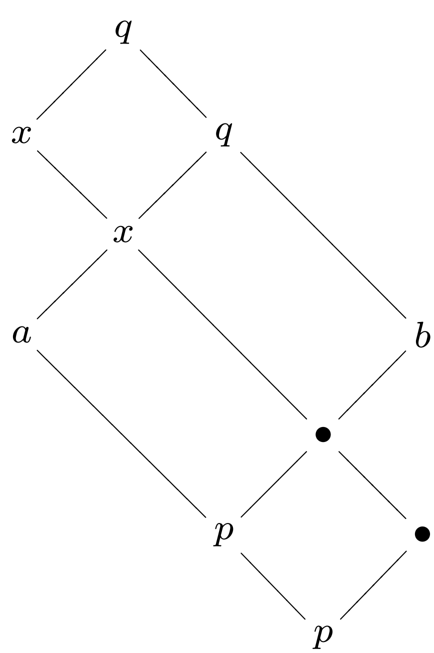
Inserting, diamonds and links

Links in an interval

$u = a$ inserted into $[x, y]$:



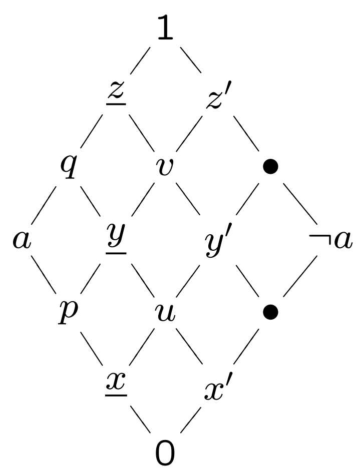
Links with trivial diamonds:



Conditions for linked chains

A n-chain $\mathbf{x}$, $n \geq 1$ is linked when:

(i) $[x_1, x_n] \cap \mathcal{C}(L) \neq \emptyset$



$z' = \neg a$ inserted into $[y, 1]$

$v = \neg a$ inserted into $[y, z]$

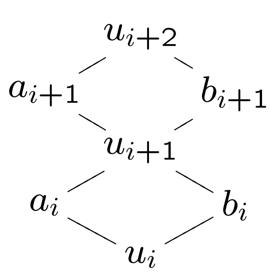
$y' = \neg a$ inserted into $[x, z]$

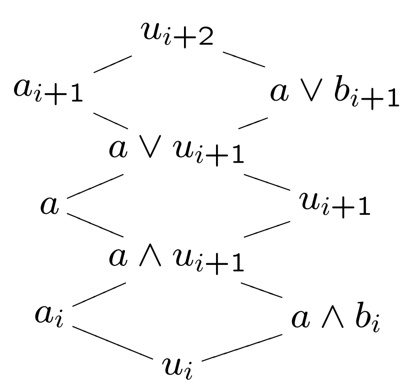
$u = \neg a$ inserted into $[x, y]$

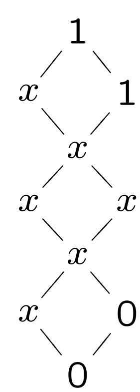
$x' = \neg a$ inserted into $[0, y]$

(i) A k-subchain $a_1 \leq \cdots \leq a_k$, $1 \leq k < n$, is linked in $[0, a_{k+1}]$.

(ii) A subchain is linked.







A n-chain $\mathbf{x}$, $n \geq 1$, is linked when it is degenerated.

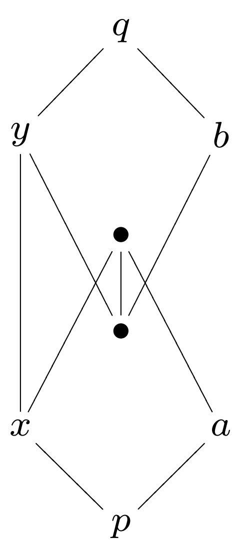
Link dimension and E-dimension

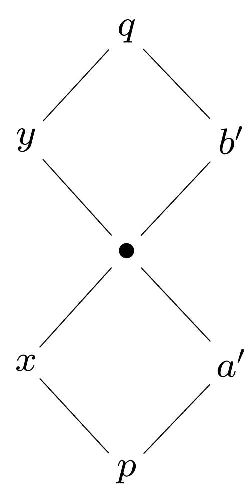
$$LdimL \leq n \qquad \Leftrightarrow \qquad EdimL \leq n$$

($\Rightarrow$) Given $x_1, \ldots, x_{n+1} \in L$, take $\qquad \mathbf{y} : y_i = x_i \wedge \cdots \wedge x_{n+1}$.
If there exist a link $\mathbf{y} \diamond \mathbf{a}$ then $x_1, \ldots, x_{n+1}$ and $\mathbf{a}$ satisfy (●●).

($\Leftarrow$) Given $\mathbf{x}$, a configuration (●●) yields a link (●)

Given $x \leq y$: $\qquad (\bullet\bullet) \begin{cases} y \vee b = q \\ y \wedge b \leq x \vee a \\ x \wedge a = p \end{cases} \qquad (\bullet) \begin{cases} y \vee b' = q \\ y \wedge b' = x \vee a' \\ x \wedge a' = p \end{cases}$





$b' = b$ inserted into $[u, q]$

$u = x \vee a'$

$a' = a$ inserted into $[p, y]$

$$u \text{ satisfies : } \begin{cases} u = a \text{ inserted into } [x, y] \\ y \wedge b \leq u \leq a \vee x \end{cases}$$

Notation for sums of chains

$L_{(n)}$ denotes the distributive lattice of all n-chains in L.

For $a \in L$, $a^n \in L_{(n)}$ is the chain $a \le \cdots \le a$.

$\mathcal{C}(L_{(n)}) \cong \mathcal{C}(L)$.

$$\int_n : L_{(n)} \to L_\neg, \quad \int_n \mathbf{a} = a_1 + \cdots + a_n. \qquad \text{Image} \quad \int_n (L) \subseteq L_\neg$$

$\int_n \mathbf{a} \le a_n, \quad a_1 \le \neg^n \int_n \mathbf{a}$

$\int_{2n} \mathbf{a} = 0 \Leftrightarrow \mathbf{a}$ is *duplicate*, $\mathbf{a} = \mathbf{b}^2 : b_1 \le b_1 \le \cdots b_n \le b_n$

$\int_{2n} \mathbf{a} = x \ne 0 \Leftrightarrow (0, \mathbf{a}_{\widehat{2n}}) \wedge x^{2n}$ is *duplicate*

$$(0, \mathbf{a}_{\hat{i}}) : 0 \le a_1 \le \cdots \le a_{i-1} \le a_{i+1} \le \cdots \le a_n$$

$$0_\mathbf{a} = (0, \mathbf{a}) : 0 \le a_1 \le \cdots \le a_n, \quad 1_\mathbf{a} = (\mathbf{a}, 1) : a_1 \le \cdots \le a_n \le 1$$

$$\mathbf{a}_i : a_1 \le \cdots \le a_i \le a_i \le \cdots \le a_n$$

$$(0, \mathbf{a}_{\hat{i}}, 1) : 0 \le a_1 \le \cdots \le a_{i-1} \le a_{i+1} \le \cdots \le a_n \le 1$$

The lattice of a chain

$$\mathbf{a} \in L_{(n)}: \quad L_{\mathbf{a}} = \prod_{i=0}^{n} [a_i, a_{i+1}] = [0_{\mathbf{a}}, 1_{\mathbf{a}}] \subseteq L_{(n+1)}$$

$$\mathbf{x} \in L_{\mathbf{a}}: \begin{cases} \mathbf{a} \prec \mathbf{x} \\ \mathbf{a} \triangleleft \mathbf{x} : x_1 \leq a_1 \leq x_2 \leq a_2 \leq \cdots \leq x_n \leq a_n \leq x_{n+1} \end{cases}$$

$\mathbf{x} \in \mathcal{C}(L_{\mathbf{a}}) = \prod_{i=0}^{n} \mathcal{C}([a_i, a_{i+1}])$ iff $\mathbf{x}$ is *linked with node* $\mathbf{a}$

Canonical morphism $\quad \ell_{\mathbf{a}} : L \to L_{\mathbf{a}},\ \ell_{\mathbf{a}}(x) = 0_{\mathbf{a}} \vee (1_{\mathbf{a}} \wedge x^{n+1})$

$$\oplus : L_n \times L_m \to L_{n+m}, \begin{cases} \mathbf{a} \oplus b = \ell_{\mathbf{a}}(b) \\ \mathbf{a} \oplus \mathbf{b} = (\mathbf{a} \oplus \mathbf{b}_{\widehat{m}}) \oplus b_m \end{cases}$$

(i) $\mathbf{a} = \mathbf{a}_{\widehat{i}} \oplus a_i = a_1 \oplus \cdots \oplus a_n$

(ii) If $\mathbf{a} \prec \mathbf{x}$ then $\mathbf{a} \oplus \mathbf{x} = \mathbf{a} \triangleleft \mathbf{x}$

(iii) $\int_{n+m} (\mathbf{a} \oplus \mathbf{b}) = \int_n \mathbf{a} + \int_m \mathbf{b}$

Given $a_1, \ldots, a_n \in L$, (iv) $a_1 \oplus \cdots \oplus a_n$ is invariant by permutations

(v) $\int_n (a_1 \oplus \cdots \oplus a_n) = \sum_{i=1}^{n} a_i$

Lattice theory's result: $\quad L_{\neg} = \bigcup_n \int_n (L)$

Grätzer, *General lattice theory*, 2nd, 2003.

Link dimension and B-dimension

$$\text{The image of } \int_n : L_{(n)} \to L_\neg \text{ is } \int_n (L) \subseteq L_\neg$$

$\bigcup_n \int_n (L) = L_\neg$

$$\int_n (L),\ \neg \int_n (L) \hookrightarrow \int_{n+1} (L)$$

$$\int_n (L) = \neg \int_n (L) \ \Rightarrow\ \int_n (L) = L_\neg$$

$$LdimL \leq n \ \Leftrightarrow\ \int_{n+1} (L) = L_\neg \ \Leftrightarrow\ BdimL \leq n$$

Universal property

Universal property of $\qquad \ell_{\mathbf{a}} : L \to L_{\mathbf{a}},\ \ \ell_{\mathbf{a}}(x) = 0_{\mathbf{a}} \vee (1_{\mathbf{a}} \wedge x^{n+1})$

$$\ell_{\mathbf{a}} \circ \mathbf{a} \in \mathcal{C}(L_{\mathbf{a}}) : \ \begin{cases} \ell_{\mathbf{a}}(a_j) = \mathbf{a}_j \\ \neg \ell_{\mathbf{a}}(a_j) = (0, \mathbf{a}_{\hat{j}}, 1) \end{cases}$$

Push-out in $\mathcal{L}$:

$$\begin{array}{ccc} Ch_n & \xrightarrow{\ \mathbf{a}\ } & L \\ \downarrow & & \downarrow{\scriptstyle \ell_{\mathbf{a}}} \\ (Ch_n)_{\neg} & \longrightarrow & L_{\mathbf{a}} \end{array}$$

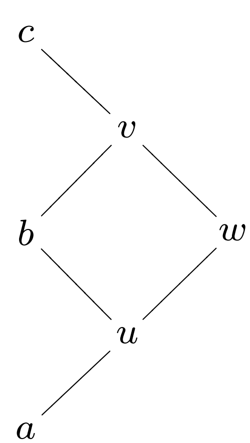
By category theory, enough to prove :

$$\begin{array}{ccc} L & & \\ \downarrow{\scriptstyle \ell_{\mathbf{a}}} & \searrow^{incl} & \\ L_{\mathbf{a}} & \xrightarrow{\ s_{\mathbf{a}}\ } & L_{\neg} \end{array}$$

First: $\ell_{\mathbf{a}}$ is mono and epi

$$\mathbf{x} = \int_{2n+1} \ell_{\mathbf{a}} \circ (\mathbf{a} \triangleleft \mathbf{x})$$

inserting x :



Then: . . .

Calculating with a valuation

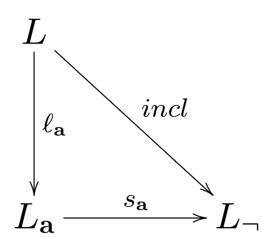
$$\int_n : L_{(n)} \to L_\neg \quad \text{is a valuation}$$

$$\int_n \mathbf{0} = 0, \qquad \int_n (\mathbf{a} \wedge \mathbf{b}) + \int_n (\mathbf{a} \vee \mathbf{b}) = \int_n \mathbf{a} + \int_n \mathbf{b}$$

$$\ell_{\mathbf{a}}(x) = 0_{\mathbf{a}} \vee (1_{\mathbf{a}} \wedge x^{n+1}); \quad \int_{n+1} \ell_{\mathbf{a}}(x) = x + \int_n \mathbf{a}$$

$s_{\mathbf{a}}(\mathbf{x}) = \int_{2n+1} \mathbf{a} \triangleleft \mathbf{x}$

defines a morphism such that



. . . the universal property follows.

Moreover:

$$\text{Linked } n\text{-chains}: \quad \mathbf{x} \diamond \mathbf{y} \quad \Leftrightarrow \quad \begin{cases} (0, \mathbf{x}) \le (\mathbf{y}, 1) \\ (0, \mathbf{y}) \le (\mathbf{x}, 1) \\ \int_n \mathbf{x} + \int_n \mathbf{y} = 1 \end{cases}$$

Dimension of a chain

$$LdimCh_n = n, \qquad LdimLink_n = n - 1$$

Definition: $dim\mathbf{a} = dimL_{\mathbf{a}}$ (for all dimensions)

Push-out :
$$\begin{array}{ccc} Ch_n & \xrightarrow{\mathbf{a}} & L \\ \downarrow & & \downarrow \ell_{\mathbf{a}} \\ (Ch_n)_{\neg} & \longrightarrow & L_{\mathbf{a}} \end{array}$$

Theorem:

$$1 + LdimL \leq (1 + n)(1 + LdimL_{\mathbf{a}})$$

Corollary:

$$\exists \mathbf{a} \in L_n, \ \ L_{\mathbf{a}} \text{ boolean } \ \Rightarrow \ LdimL \leq n$$

THAT'S ALL

THANKS