Recursive Function Classes in Cartesian Categories

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MAP 2010, Logroño, November

Plan

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- What do we need for recursion in CT

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  - natural numbers object diagrams
  - representation
  - various Cartesian Categories

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  - free and syntactical structures
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- **Subrecursion**
  - translations to CT
  - Polarized Categories

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$$\begin{array}{ccccc} 1 & \xrightarrow{z} & N & \xrightarrow{s} & N \\ \downarrow & & \downarrow m & & \downarrow m \\ 1 & \xrightarrow{f} & A & \xrightarrow{g} & A \end{array}$$

where m is unique

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$$\begin{array}{ccccc} X & \xrightarrow{zX} & NX & \xrightarrow{sX} & NX \\ \downarrow & & \downarrow m & & \downarrow m \\ X & \xrightarrow{f} & A & \xrightarrow{g} & A \end{array}$$

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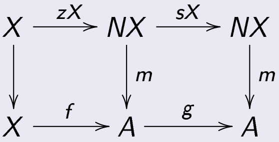

We can speak of a weak nno (wnno) if uniqueness is not required

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These definitions can be generalized

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A *left nno* (*lnno*) in a *Monoidal Category* $\mathcal{V} = (\mathcal{V}, \otimes, I, \alpha, \lambda, \varrho)$ **is** (N, z, s) **such that**

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\end{array}
$$

commutes

Representation

We define recursive function classes in $cc + nno$ by its *representation*

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Definition

We say that $f : \mathbb{N}^k \longrightarrow \mathbb{N}$ is *representable* in a $cc + nno$ $\mathcal{C}$ if there exists $\overline{f} : N^k \longrightarrow N$ in $\mathcal{C}$ such that

$$\overline{f}\,\langle \sharp n_1, \ldots, \sharp n_k \rangle = \sharp f(n_1, \ldots, n_k)$$

The results

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For total function classes in the form $f : \mathbb{N}^k \longrightarrow \mathbb{N}$ we have:

1. $$\mathcal{PR} = \{\textit{Representables in } cc{+}wpnno\}$$

2. $$\mathcal{PR} \subseteq \{\textit{Representables in Topos+nno}\} \subset \textit{TotalRec}$$

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Free and syntactical structures

Two constructions

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Peano-Lawvere Axiom in a category

A category satisfies PL Axiom (or it is PL) if every object has an nno

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PL categories are not in general cartesian nor are they endowed with a terminal object

However $P = F_{PL}(\cdot)$ has both

Free and syntactical structures

Let be $P = F_{PL}(\cdot)$ the *Category of $\mathcal{PR}$-formal functions*

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it will be called *precategory of $\mathcal{PR}$-programs*

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We call $\mathbb{P}$ *the image graph* $P \longrightarrow Set$

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- *its objects are* $\mathbb{N}^p$
- *its morphisms* $f : \mathbb{N}^p \longrightarrow \mathbb{N}^q$ *such that* $f = (f_0, f_1, ..., f_{q-1})$ *where* $f_i \in \mathcal{PR}$

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We call $\mathbb{P}$ *the image graph* $P \longrightarrow Set$

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$\mathbb{P}$ can (only?) be characterized by equivalence relations in P

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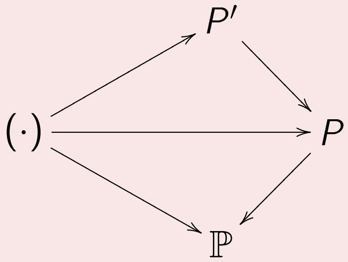

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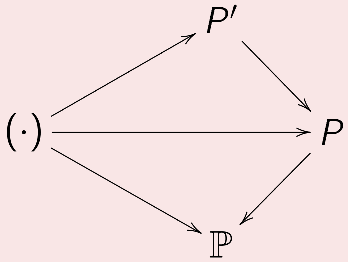

P' is a syntactical construction while $\mathbb{P}$ is a category with semantics

Categories of Algorithms

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Categories of Algorithms

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1. the implementations that we handle: programs
2. formalizations that are known: recursive functions

Categories of Algorithms

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We call this graph $\mathcal{PR}desc$
it has all descriptions about how to compute all $\mathcal{PR}$-functions

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$$\frac{\mathcal{PR}desc}{\sim} = \mathcal{PR}Alg$$

as the free initial category $F_{CatxN}(\emptyset)$ in $CatxN$

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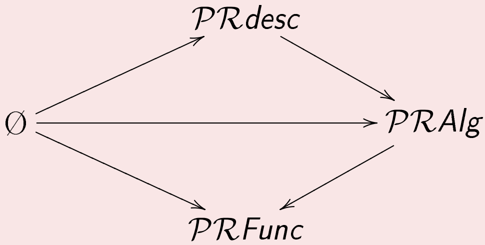

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- *free Monoidal Category* + *lnno* $\Phi(\emptyset)$ (Román-Paré)

Translations to CT

Classes in *Gzregorzcyk Hierarchy* can be defined by *bounding arithmetics*

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1. containing a derivation of every initial function 0, $Sx = x + 1$, $Px = max(0, x - 1)$ and *conditional* $C(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{else} \end{cases}$

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   $$f(\overline{x}) = h(g_1(\overline{x}^1), ..., g_m(\overline{x}^m))$$
   2. *full primitive recursiveness:* for g and h if $x \neq 0$ we derive
   $$f(x, \overline{y}) = h(x, \overline{y}, f(Px, \overline{y}))$$

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We use two kinds of arguments: *normal* and *safe*

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- $\rho(f) = max\{\rho(g), 1 + \rho(h)\}$ if f is defined by full primitive recursion of derivations g and h

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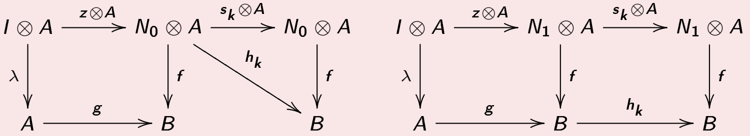

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$\mathcal{C}$ is a *cc* (the *opponent*) and $\mathcal{D}$ a category (the *player*)

- endowed with a *strong composition*

$$\frac{(C_1, D_1) \xrightarrow{f} D_2 \quad (C_2, D_2) \xrightarrow{g} D_3}{(C_1 \times C_2, D_1) \xrightarrow{f;g} D_3}$$

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$$\frac{(C,1) \stackrel{f}{\longrightarrow} D}{(F_o(C),1) \stackrel{F_{op}(f)}{\longrightarrow} F_p(D)}$$

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Example

$\otimes$ for $\mathcal{D}$ and $\times$ for $\mathcal{C}$ form a polarized functor

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Let $F^\star$ be a free algebra generated by

- contexts $C = 1$ and $D = Nat$
- constructors

$$Zero : 1 \longrightarrow Nat \text{ and } Succ : Nat \longrightarrow Nat$$

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Can we construct a *SPolCat* from this to get a general form of ramified recursion in CT?