Hip–Hop solutions of the $2N$–Body problem with eccentricity close to 1

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Abstract

Hip–Hop solutions of the $2N$–body problem with equal masses are shown to exist using a topological argument. These solutions are close to a planar regular $2N$–gon homographic configuration with values of the eccentricity close to 1, plus a small vertical oscillations in which each mass.

1 Introduction

We are interested in hip-hop orbits, which are solutions of the $2N$-body problem with equal masses such that all bodies stay for all time on the vertices of an anti-prism (see [6], [1]). The $2N$ bodies can be arranged in two groups of N, each group moving on a rotating regular N–gon configuration in a plane perpendicular to a given axis of the anti-prism. The motion of each group oscillate along this axis, and coincide with opposite velocities at regular intervals in the same plane. The orthogonal projection of both N–gons on the plane perpendicular to the axis of symmetry is always a regular rotating $2N$–gon.

Some results related to hip-hop solutions were obtained mostly by means of variational methods, which make it possible to find solutions that do not depend on a small parameter ([4], [3]). In [1], the authors show that Poincare's argument of analytic continuation can be used to add vertical oscillations to the circular motion of $2N$ bodies of equal mass occupying the vertices of a regular $2N$-gon, and prove the existence of families of hip-hop solutions with eccentricity close to zero.

The aim of this work is to prove the existence of hip-hop solutions with high values of the eccentricity and close to 1. In this case, the analytical continuation method cannot be used, and has been substituted by a topological reasoning. The details can be found in [2].

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2 Main result

Let us consider a suitable reference system of coordinates (x, y, z) such that the two N-gon are in planes perpendicular to the vertical z axis. Let $\mathbf{r}$ be the position of one body. The position (and velocity) of the other $2N-1$ bodies are given by $R^{i-1}\mathbf{r}$, for $i = 2, \cdots, 2N$, where R is a rotation plus a reflection. The problem can be reduced to a system with three degrees of freedom (see [1]) given by

$$\ddot{\mathbf{r}} = \sum_{k=1}^{2N-1} \frac{(R^k - I)\mathbf{r}}{|(R^k - I)\mathbf{r}|^3}. \tag{1}$$

We introduce cylindrical coordinates (r, ϕ, d), and their corresponding momenta (p_r, p_ϕ, p_d). The Hamiltonian of the problem does not depend on ϕ, which means that $\dot{p_\phi} = 0$. We fix the angular momentum $p_\phi = \Phi$, and we consider the *reduced problem* in a rotational frame given by the equations

$$\begin{aligned} \ddot{r} &= \frac{\Phi^2}{r^3} - 2r\sum_{k=1}^{2N-1} \frac{\sin^2\left(\frac{k\pi}{2N}\right)}{\left(4r^2\sin^2\left(\frac{k\pi}{2N}\right) + \left((-1)^k - 1\right)^2 d^2\right)^{3/2}}, \\ \ddot{d} &= -\frac{d}{2}\sum_{k=1}^{2N-1} \frac{\left((-1)^k - 1\right)^2}{\left(4r^2\sin^2\left(\frac{k\pi}{2N}\right) + \left((-1)^k - 1\right)^2 d^2\right)^{3/2}}, \end{aligned} \tag{2}$$

Notice that the initial conditions $d(0) = \dot{d}(0) = 0$ will result in a planar (in the $z = 0$ plane) homographic motion of the $2N$ bodies. We want to add a small vertical motion (in the z direction) and show the existence of periodic 3-dimensional hip-hop solutions. If the vertical motion is small enough, the motion can be decoupled into a planar plus a vertical motion system. The uncoupling can be accomplished through a rescaling of the variable d. Substituting d by εd into the Eq. (2) we obtain

$$\ddot{r} = \frac{\Phi^2}{r^3} - \frac{K_N^2}{r^2} + O(\epsilon^2), \qquad \ddot{d} = -\frac{S_N^2}{r^3}\, d + O(\epsilon^2), \tag{3}$$

where $K_N^2 = \frac{1}{4}\sum_{k=1}^{2N-1} \sin^{-1}\left(\frac{k\pi}{2N}\right), \qquad S_N^2 = \frac{1}{64}\sum_{k=1}^{2N-1}((-1)^k - 1)^4 \sin^{-3}\left(\frac{k\pi}{2N}\right).$

For $\epsilon = 0$, the system (3) uncouples into a planar Kepler motion plus a vertical oscillation. Using similar techniques as in [5], the existence of symmetric periodic solutions of the problem given by Eq. (3) can be shown for a sequence of values of the eccentricity tending to 1. The main result is

Theorem 1 *Let N, m be positive integers, $N \geq 2$. Consider a homographic motion of a system of $2N$ bodies of equal mass of semimajor axis a^*. There exists an increasing infinite sequence of eccentricities $e_{n,m}$, $n \geq 1$, converging to 1, and an infinite sequence of positive numbers $\epsilon_{n,m}$ such that, for any $\epsilon \in (0, \epsilon_{n,m}]$ there exist a_ϵ and e_ϵ in such a way that $a_\epsilon \to a^*$, $e_\epsilon \to e_{n,m}$ and the solution of system (3) with initial conditions $r(0) = a_\epsilon(1 - e_\epsilon)$, $\dot{r}(0) = 0$, $d(0) = 0$, $\dot{d}(0) = \epsilon$ is a Hip-Hop solution of period $T^* = 2m\frac{\pi a^{*3/2}}{K_N}$ of the reduced problem (2).*

References

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