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SPIN ROTOR STABILIZATION OF A DUAL-SPIN SPACECRAFT WITH TIME DEPENDENT MOMENTS OF INERTIA

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We consider a dual-spin deformable spacecraft, in the sense that one of the moments of inertia is a periodic function of time such that the center of mass is not altered. In the absence of external torques and spin rotors, by means of the Melnikov's method we prove that the body motion is chaotic. Stabilization is obtained by means of a spinning rotor about one of the principal axes of inertia.

1. Introduction

A dual spin spacecraft, also called a gyrostat, is a mechanical system $\mathcal{S}$ composed by many bodies; a rigid body $\mathcal{P}$, called platform or core body, and other axisymmetric bodies $\mathcal{R}$, called rotors or wheels, in such a way that the motion of the rotors does not modify the distribution of masses of the spacecraft. We will assume that the rotors are aligned with the principal axes of the platform and there are no external torques. This problem is an integrable case, and its solution, in the case of one rotor, is given in terms of elliptic functions (see e.g. [Ruminatsev, 1961; Cochran *et al.*, 1982]).

The dynamics of the gyrostat has been recently the object of interest in Astrodynamics, and it is used for controlling the attitude dynamics of spacecraft and for stabilizing their rotations [Hubert, 1980; Hall & Rand, 1994; Tsiotras & Longuski, 1994]. Other authors have used the Melnikov method to point out the chaotic behavior of a gyrostat under several kinds of perturbations. Holmes and Marsden [1983] consider certain asymmetry in the rotor, Koiller [1984] deals with an attachment with small imperfections and, recently, Tong *et al.* [1995] treat the gyrostat in the uniform gravity field. These studies are based on the premise that the rotating body can be considered to be perfectly rigid. Unfortunately, the rigid body is only a convenient approach to simplify the analysis. However, all real materials are elastic and deformable to some degree.

This fact has moved us to focus our attention on the dynamics of a deformable dual-spin spacecraft in the absence of external torques. Here, deformable means that one of the moments of inertia of the platform is a periodic function of time and that the center of mass of the spacecraft is not modified. This is a more realistic approximation to the attitude motion of a spacecraft but not exempt of considerable simplifications.

The problem is treated in noncanonical variables, the components of the angular momentum in the body frame. This treatment has the advantage that phase space reduces to a constant radius sphere and the phase flow is easily interpreted [Hubert, 1980]. In this set of variables the unperturbed system matches with several interesting

problems in nuclear and atomic physics [Elipe & Ferrer, 1994] and in optical problems [David *et al.*, 1990] and as it was demonstrated by Elipe [1996] the gyrostat model reduces to a generic quadratic Hamiltonian in a set of variables on the unit sphere. This kind of Hamiltonians has been widely studied in order to determine their equilibria, bifurcations and phase flow evolution [Lanchares & Elipe, 1995; Lanchares & Elipe, 1995; Lanchares *et al.*, 1995].

We demonstrate that in the absence of spinning rotors the system shows chaotic behavior in the sense that it exhibits Smale's horseshoes. We prove this statement by means of the Melnikov method (see e.g. [Ozorio de Almeida, 1990]). The presence of chaos may be viewed as a stochastic layer surrounding the unperturbed separatrix in a Poincaré surface of section.

The knowledge of the phase flow structure for the unperturbed system allows us to conjecture that the chaotic motion can be eliminated by means of a spinning rotor about one of the principal axes. The process of stabilization is well observed through a sequence of Poincaré surfaces of section for increasing values of the relative momentum of the rotor.

2. Hamiltonian and Equations of Motion

Let us consider a gyrostat, consisting of an asymmetric core body and three axisymmetric rotors aligned with the principal axis of the platform. Let us assume that the gyrostat has a fixed point O, identified with the center of mass of the gyrostat. Centered on it we will consider two orthonormal reference frames

— $\mathcal{S}$, the space frame $O\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3$ fixed in space.
— $\mathcal{B}$, the body frame $O\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3$ fixed in the platform.

The well known relations between the two reference frames result by means of three rotations involving the Euler angles (ψ, θ, ϕ) (see Fig. 1).

The total kinetic energy of the gyrostat (see e.g. [Tong *et al.*, 1995; Elipe, 1996; Elipe *et al.*, 1997]) is given by

$$\begin{aligned} T &= T_{\mathcal{P}} + T_{\mathcal{R}_1} + T_{\mathcal{R}_2} + T_{\mathcal{R}_3} \\ &= \frac{1}{2}\omega \cdot \mathbb{I}_{\mathcal{P}}\,\omega + \frac{1}{2}\omega_1 \cdot \mathbb{I}_{\mathcal{R}_1}\,\omega_1 \\ &\quad + \frac{1}{2}\omega_2 \cdot \mathbb{I}_{\mathcal{R}_2}\,\omega_2 + \frac{1}{2}\omega_3 \cdot \mathbb{I}_{\mathcal{R}_3}\,\omega_3 \end{aligned} \tag{1}$$

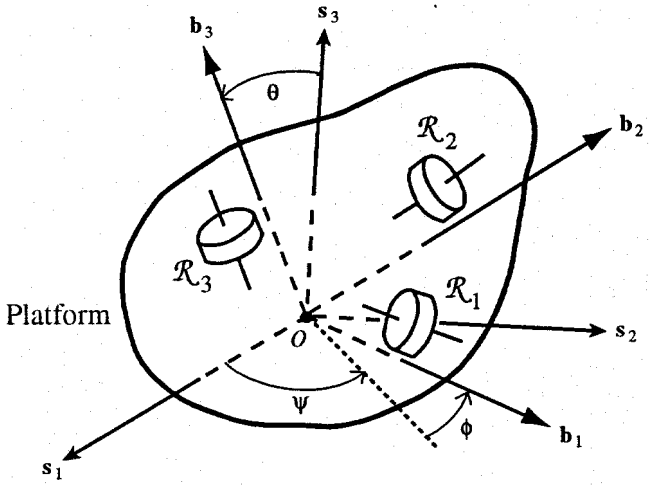

Fig. 1. Asymmetric gyrostat with three attached rotors.

where $\omega = (\omega_x, \omega_y, \omega_z)$, $\omega_1 = (\omega_x + \Omega_x, \omega_y, \omega_z)$, $\omega_2 = (\omega_x, \omega_y + \Omega_y, \omega_z)$, $\omega_3 = (\omega_x, \omega_y, \omega_z + \Omega_z)$; ω_x, ω_y, ω_z are the components of the angular velocity ω of the gyrostat in the body fixed reference frame; Ω_x, Ω_y, Ω_z are the relative spin speeds of the rotors with respect to the platform. $\mathbb{I}_{\mathcal{P}} = \mathrm{diag}\,(A_0, B_0, C_0)$, $\mathbb{I}_{\mathcal{R}_1} = \mathrm{diag}\,(A_1, B_1, B_1)$, $\mathbb{I}_{\mathcal{R}_2} = \mathrm{diag}\,(A_2, B_2, A_2)$, $\mathbb{I}_{\mathcal{R}_3} = \mathrm{diag}\,(A_3, A_3, C_3)$ are the tensor of inertia of the platform and the rotors, respectively. Denoting by $A = A_0 + A_1 + A_2 + A_3$, $B = B_0 + B_1 + B_2 + A_3$, $C = C_0 + B_1 + A_2 + C_3$ we obtain for the kinetic energy

$$\begin{aligned} T = \frac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2) + A_1\omega_x\Omega_x + B_2\omega_y\Omega_y \\ + C_3\omega_z\Omega_z + \frac{1}{2}(A_1\Omega_x^2 + B_2\Omega_y^2 + C_3\Omega_z^2)\,. \end{aligned} \tag{2}$$

We will assume that the relative angular momenta $h_x = A_1\Omega_x$, $h_y = B_2\Omega_y$, $h_z = C_3\Omega_z$ are constant.

The total angular momentum $\mathbf{G}$ of the gyrostat is given by

$$\mathbf{G} = \mathbf{G}_{\mathcal{P}} + \mathbf{G}_{\mathcal{R}_1} + \mathbf{G}_{\mathcal{R}_2} + \mathbf{G}_{\mathcal{R}_3}\,,$$

where, expressed in the body frame

$$\mathbf{G}_{\mathcal{P}} = \mathbb{I}_{\mathcal{P}}\boldsymbol{\omega}, \quad \mathbf{G}_{\mathcal{R}_1} = \mathbb{I}_{\mathcal{R}_1}\boldsymbol{\omega}_1\,,$$

$$\mathbf{G}_{\mathcal{R}_2} = \mathbb{I}_{\mathcal{R}_2}\boldsymbol{\omega}_2, \quad \mathbf{G}_{\mathcal{R}_3} = \mathbb{I}_{\mathcal{R}_3}\boldsymbol{\omega}_3\,.$$

Using the orthogonal basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ in the body frame we have

$$\mathbf{G} = \mathbb{I}_{\mathcal{G}}\boldsymbol{\omega} + h_x\mathbf{b}_1 + h_y\mathbf{b}_2 + h_z\mathbf{b}_3\,,$$

where $\mathbb{I}_{\mathcal{G}} = \mathrm{diag}(A, B, C)$ is the tensor of inertia of the whole gyrostat. Hence, the components of $\mathbf{G}$ in the body frame are

$$\begin{cases} g_1 = A\omega_x + h_x\,, \\ g_2 = B\omega_y + h_y\,, \\ g_3 = C\omega_z + h_z \end{cases}$$

and, by inversion,

$$\begin{cases} \omega_x = \dfrac{g_1}{A} - \dfrac{h_x}{A}\,, \\ \omega_y = \dfrac{g_2}{B} - \dfrac{h_y}{B}\,, \\ \omega_z = \dfrac{g_3}{C} - \dfrac{h_z}{C}\,. \end{cases}$$

The total kinetic energy, in terms of the components of the total angular momentum, takes the form

$$T = \frac{1}{2}\left(\frac{g_1^2}{A} + \frac{g_2^2}{B} + \frac{g_3^2}{C}\right) - \frac{1}{2}\left(\frac{h_x^2}{A} + \frac{h_y^2}{B} + \frac{h_z^2}{C}\right) + \frac{1}{2}\left(\frac{h_x^2}{A_1} + \frac{h_y^2}{B_2} + \frac{h_z^2}{C_3}\right).$$

The components of the total angular momentum can, also, be expressed in terms of the Euler angles (ψ, θ, ϕ) and the Euler angle velocities $(\dot\psi, \dot\theta, \dot\phi)$ as

$$\begin{cases} g_1 = A(\dot\psi \sin\theta \sin\phi + \dot\theta \cos\phi) + h_x\,, \\ g_2 = B(\dot\psi \sin\theta \cos\phi - \dot\theta \sin\phi) + h_y\,, \\ g_3 = C(\dot\psi \cos\theta + \dot\phi) + h_z\,. \end{cases} \tag{3}$$

As we consider a gyrostat in free rotation ($V = 0$), the Lagrangian $\mathcal{L}$ of the system is

$$\mathcal{L} = T - V = T\,,$$

that depends on $(\psi, \theta, \phi, \dot\psi, \dot\theta, \dot\phi)$ through g_1, g_2 and g_3 by (3).

The momenta conjugate of the Euler angles are defined by

$$\begin{cases} p_\psi = \dfrac{\partial \mathcal{L}}{\partial \dot\psi} = g_1 \sin\theta \sin\phi \\ \qquad\qquad + g_2 \sin\theta \cos\phi + g_3 \cos\theta\,, \\ p_\theta = \dfrac{\partial \mathcal{L}}{\partial \dot\theta} = g_1 \cos\phi - g_2 \sin\phi\,, \\ p_\phi = \dfrac{\partial \mathcal{L}}{\partial \dot\phi} = g_3 \end{cases}$$

and, by inversion we obtain

$$\begin{cases} g_1 = \left(\dfrac{p_\psi - p_\phi \cos\theta}{\sin\theta}\right) \sin\phi + p_\theta \cos\phi\,, \\ g_2 = \left(\dfrac{p_\psi - p_\phi \cos\theta}{\sin\theta}\right) \cos\phi - p_\theta \sin\phi\,, \\ g_3 = p_\phi\,. \end{cases}$$

The Hamiltonian $\mathcal{H}$ of the system is given by the Legendre transformation

$$\mathcal{H} = p_\psi \dot\psi + p_\theta \dot\theta + p_\phi \dot\phi - \mathcal{L}\,,$$

and we obtain

$$\mathcal{H} = \frac{1}{2}\left(\frac{g_1^2}{A} + \frac{g_2^2}{B} + \frac{g_3^2}{C}\right) - \left(\frac{g_1 h_x}{A} + \frac{g_2 h_y}{B} + \frac{g_3 h_z}{C}\right) + \frac{1}{2}\left(\frac{h_x^2}{A} + \frac{h_y^2}{B} + \frac{h_z^2}{C}\right) - \frac{1}{2}\left(\frac{h_x^2}{A_1} + \frac{h_y^2}{B_2} + \frac{h_z^2}{C_3}\right).$$

Since the relative angular momenta h_i are supposed to be constant, the Hamiltonian becomes, after dropping constant terms,

$$\mathcal{H} = \frac{1}{2}\left(\frac{g_1^2}{A} + \frac{g_2^2}{B} + \frac{g_3^2}{C}\right) - \left(\frac{g_1 h_x}{A} + \frac{g_2 h_y}{B} + \frac{g_3 h_z}{C}\right). \tag{4}$$

The Hamiltonian (4) is invariant under the group SO(2) of rotations $R(\phi, \mathbf{s}_3)$ about the space axis $\mathbf{s}_3$, since the angle ϕ is ignorable in $\mathcal{H}$ as it can be checked replacing Eqs. (3) into Eq. (4). Besides, the problem also is invariant under the group SO(3) of rotations about the origin O. Indeed, it is easy to derive from (3) that the Poisson brackets satisfy

$$\{g_1; g_2\} = -g_3, \quad \{g_2; g_3\} = -g_1\,, \quad \{g_3; g_1\} = -g_2\,. \tag{5}$$

These structural identities yield the Eulerian equations of the motion

$$\begin{aligned} \dot g_1 = \{g_1; \mathcal{H}\} &= (a_3 - a_2) g_2 g_3 \\ &\quad + a_2 h_y g_3 - a_3 h_z g_2\,, \\ \dot g_2 = \{g_2; \mathcal{H}\} &= (a_1 - a_3) g_1 g_3 \\ &\quad + a_3 h_z g_1 - a_1 h_x g_3\,, \\ \dot g_3 = \{g_3; \mathcal{H}\} &= (a_2 - a_1) g_1 g_2 \\ &\quad + a_1 h_x g_2 - a_2 h_y g_1, \end{aligned} \tag{6}$$

where $a_1 = 1/A$, $a_2 = 1/B$ and $a_3 = 1/C$; we will assume, from here on, that $a_1 < a_2 < a_3$.

The system (6) admits two integrals, the Hamiltonian $\mathcal{H}$ and the norm of the total angular momentum since

$$g_1\dot{g}_1 + g_2\dot{g}_2 + g_3\dot{g}_3 = 0$$

and the problem is, therefore, integrable. The phase space of (6) may be regarded as a foliation of invariant manifolds

$$S^2(G) = \{(g_1,\, g_2,\, g_3)\,|\,g_1^2 + g_2^2 + g_3^2 = G^2\}\,.$$

3. Chaotic Motion

We now suppose that the inverse of the maximum moment of inertia is the periodic function of time

$$a_1 = a_{10} + \epsilon\cos\nu t\,.$$

In the absence of spinning rotors the problem reduces to the Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \epsilon V(g_1,\, g_2,\, g_3;\, t)\,, \tag{7}$$

where the unperturbed term $\mathcal{H}_0$ is that of a rigid body in free rotation

$$\mathcal{H}_0 = \frac{1}{2}(a_{10}g_1^2 + a_2g_2^2 + a_3g_3^2)$$

and V is the periodic function of time

$$V = \frac{1}{2}g_1^2\cos\nu t$$

with period $T = 2\pi/\nu$.

For the unperturbed term there are two unstable equilibria located at the intersections of the $\mathbf{s}_2$ axis with the sphere $S^2(G)$. They are connected by four heteroclinic orbits as plotted in Fig. 2. We may expect that under the perturbation, the motion near the unperturbed separatrix behaves in an extremely complicated way in such a way that transverse heteroclinic (homoclinic) orbits may appear and chaotic motion occurs. Since the unperturbed system has one degree of freedom, we can use the Melnikov method (see e.g. [Ozorio de Almeida, 1990]) to prove the existence of heteroclinic (homoclinic) points in the Poincaré map of the perturbed problem.

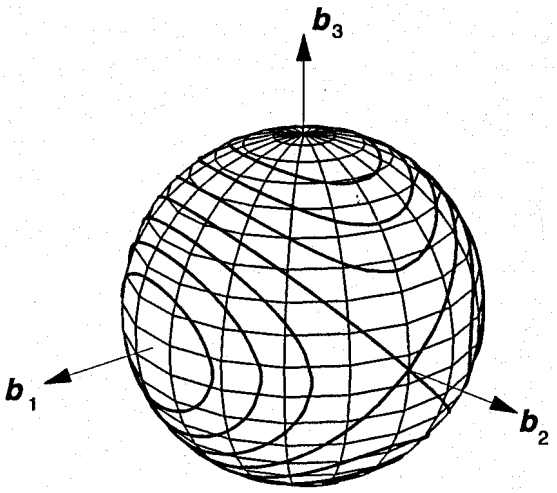

Fig. 2. The phase flow on an integral manifold for the rigid body in free rotation.

The Melnikov function for the Hamiltonian (7) is given by

$$M(t_0) = \int_{-\infty}^{\infty}\{\mathcal{H}_0(g_i(t-t_0));\, \epsilon V(g_i(t-t_0),\, t)\}\,dt\,.$$

The Poisson bracket $\{\mathcal{H}_0; \epsilon V\}$ can be evaluated taking into account the structural identities (5) and we obtain

$$\{\mathcal{H}_0; \epsilon V\} = (a_2 - a_3)\epsilon g_1g_2g_3\cos\nu t\,.$$

The Melnikov function results

$$M(t_0) = \int_{-\infty}^{\infty}(a_2 - a_3)\epsilon g_1g_2g_3\cos\nu t\,dt\,. \tag{8}$$

The solutions of the heteroclinic orbits for the unperturbed problem are (see e.g. [Deprit & Elipe, 1993])

$$\begin{cases} g_1 = (-1)^{[(k-1)/2]}\,G\sqrt{\dfrac{a_2 - a_3}{a_{10} - a_3}}\operatorname{sech}(n_2t)\,, \\ g_2 = (-1)^{k-1}\,G\tanh(n_2t)\,, \qquad\qquad k = 1,\, 2,\, 3,\, 4\,. \\ g_3 = (-1)^{[k/2]}\,G\sqrt{\dfrac{a_{10} - a_2}{a_{10} - a_3}}\operatorname{sech}(n_2t) \end{cases} \tag{9}$$

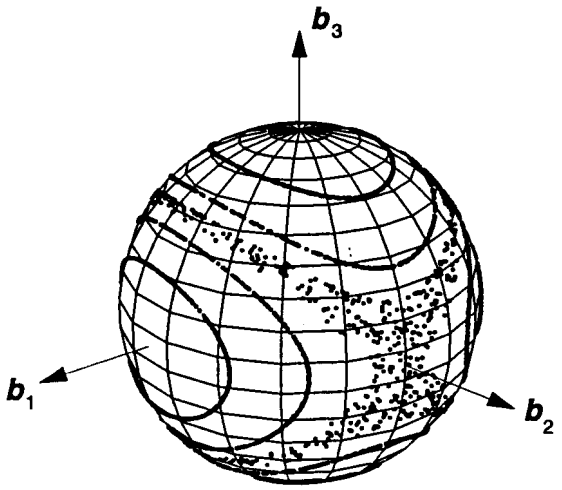

Fig. 3. Poincaré surface of section for $a_{10} = 0.1$, $a_2 = 0.2$, $a_3 = 0.3$, $G = 1$, $\epsilon = 0.005$ and $\nu = 0.1$.

where

$$n_2 = \sqrt{(a_{10} - a_2)(a_2 - a_3)} \tag{10}$$

and $[b]$ stands for the integer part of b.

By substitution of (9) into (8) we obtain

$$M(t_0) = \epsilon G^3 \frac{(a_2 - a_3)n_2}{(a_{10} - a_3)} \times \int_{-\infty}^{\infty} \frac{\sinh n_2(t - t_0)}{\cosh^3 n_2(t - t_0)} \cos \nu t \, dt \,.$$

Integrating by parts and using the table of integrals by Gradshteyn and Ryzhik [1980, p. 505] it results

$$M(t_0) = \frac{(a_2 - a_3)G^3 \pi \epsilon \nu^2}{2(a_{10} - a_3)n_2^2 \sinh \dfrac{\pi\nu}{2n_2}} \sin \nu t_0 \,. \tag{11}$$

We can conclude from (11) that the function $M(t_0)$ has simple zeroes and therefore the perturbation gives rise to chaotic motion in the sense that the system has Smale's horseshoes. Note that a formula similar to (11) would have been obtained if anyone of the moments of inertia (i.e. not necessarily the maximum moment of inertia) varies with time.

The chaotic behavior of the body is observed by means of a Poincaré surface of section. The surface consists of time sections of the fourth-dimensional (g_1, g_2, g_3, t) extended phase space. Figure 3 shows the presence of a stochastic layer around the unperturbed separatrix.

4. Spin Rotor Stabilization

Let us introduce, now, a spin rotor in the $\mathbf{s}_3$ axis with relative angular momentum h_z. The Hamiltonian of the problem becomes

$$\mathcal{H} = \frac{1}{2}(a_{10}g_1^2 + a_2 g_2^2 + a_3 g_3^2) - a_3 h_z g_3 + \frac{1}{2}\epsilon g_1^2 \cos \nu t \,. \tag{12}$$

The unperturbed part of (12) is

$$\mathcal{H}_0 = \frac{1}{2}(a_{10}g_1^2 + a_2 g_2^2 + a_3 g_3^2) - a_3 h_z g_3 \,. \tag{13}$$

After suitable transformations the unperturbed Hamiltonian may be reduced to

$$\mathcal{H}_0' = \frac{1}{2}(g_3^2 + P g_1^2) - Q g_3 \,, \tag{14}$$

where $P = (a_{10} - a_2)/(a_3 - a_2)$ and $Q = a_3 h_z/(a_3 - a_2)$. This type of Hamiltonian has been studied by Lanchares and Elipe [1995] and the equilibria solutions and the phase flow evolution are known in terms of the parameters P and Q. Since we are assuming $a_{10} < a_2 < a_3$, the parameter P is a negative constant quantity for fixed values of a_{10}, a_2 and a_3. Therefore, the unperturbed Hamiltonian depends on the parameter Q, that is, on h_z.

Let us recall the division of the parameter space for the biparametric quadratic Hamiltonian (14), as well as the equilibria solutions (see Table 1 and Fig. 4). It is observed that for negative P the effect of increasing h_z is a transition from region 6 to 4 and 2. At the same time this transition takes place, two pitchfork bifurcations occur when the boundaries between regions 6 and 4 first and 4 and

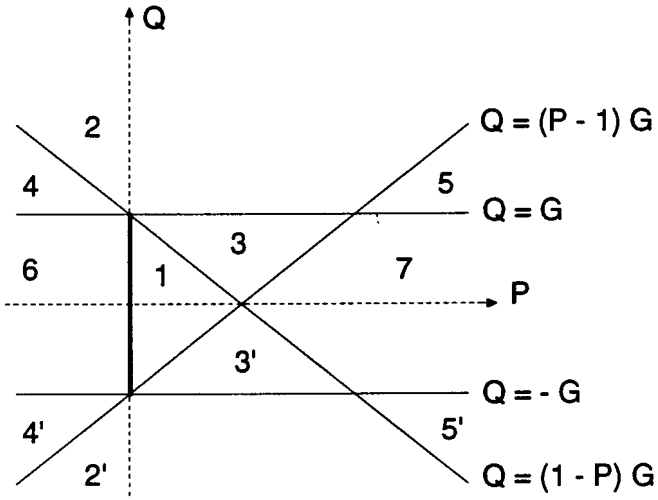

Fig. 4. Partition of the parametric plane PQ for the unperturbed Hamiltonian corresponding to one spinning rotor about the $\mathbf{b}_3$ axis.

Table 1. Equilibria points, their existence and stability for the biparametric quadratic Hamiltonian $\mathcal{H}_0' = 1/2 g_3^2 + 1/2 P g_1^2 - Q g_3$.

Equilibrium point in coordinates (g_1, g_2, g_3)	Existence	Stable
$(0, 0, G)$	Always	$(G-Q)[(1-P)G-Q] > 0$
$(0, 0, -G)$	Always	$(Q+G)[(1-P)G+Q] > 0$
$\left(\pm\sqrt{G^2 - \frac{Q^2}{(1-P)^2}}, 0, \frac{Q}{1-P}\right)$	$\lvert Q\rvert \le \lvert P-1\rvert G$	$P\frac{(P-1)^2G^2 - Q^2}{P-1} > 0$
$\left(0, \pm\sqrt{G^2-Q^2}, Q\right)$	$\lvert Q\rvert \le G$	$(G^2-Q^2)P > 0$

2 second are crossed. Once in region 2, only two equilibria exist and no separatrix exists in phase space.

Our task, now, is to compute the Melnikov function as h_z increases. It is clear that two situations have to be considered: one for the region 6 and the other for region 4. For the sake of simplicity (we do not want to perform excessive calculations) we will compute the Melnikov function in region 4 in order to investigate what happens when the region 2 is reached, that is, when the separatrix disappears.

To calculate the Melnikov function we need the solutions for g_1, g_2 and g_3 at the separatrix in region 4. Since the modulus of the angular momentum is constant the Hamiltonian (13) may be written as

$$\mathcal{H}_0 = \frac{1}{2}(a_{10}-a_2)g_1^2 + \frac{1}{2}(a_3-a_2)g_3^2 - a_3 h_z g_3 + \frac{1}{2}a_2 G^2 .$$

From the last equation we obtain

$$g_1^2 = \frac{1}{a_2 - a_{10}}(a_2 G^2 - 2\mathcal{H}_0 + (a_3-a_2)g_3^2 - 2a_3 h_z g_3) \tag{15}$$

and finally

$$g_2^2 = G^2 - g_1^2 - g_3^2 = \frac{1}{a_2 - a_{10}}(2\mathcal{H}_0 - a_{10}G^2 - (a_3 - a_{10})g_3^2 + 2a_3 h_z g_3) . \tag{16}$$

If we consider a value of h_z such that

$$\frac{a_3 h_z}{a_3 - a_{10}} < G < \frac{a_3 h_z}{a_3 - a_2} ,$$

that is, when we are in region 4, the homoclinic orbits stand for the energy value

$$\mathcal{H}_0 = \frac{1}{2}a_3 G^2 - a_3 G h_z .$$

Substitution of this value into Eqs. (15) and (16) yields

$$g_1^2 = \frac{1}{a_2 - a_{10}}(G - g_3)[2a_3 h_z - (a_3 - a_2)(g_3 + G)] ,$$

$$g_2^2 = \frac{1}{a_2 - a_{10}}(G - g_3)[(a_3 - a_{10})(g_3 + G) - 2a_3 h_z] .$$

Now, from the third differential equation of (6) it results

$$\dot{g}_3 = (G - g_3)\sqrt{[2a_3 h_z - (a_3 - a_2)(g_3 + G)][(a_3 - a_{10})(g_3 + G) - 2a_3 h_z]} .$$

This equation may be written as

$$\dot{g}_3 = \alpha(G - g_3)\sqrt{(g_3 - r_1)(r_2 - g_3)} \tag{17}$$

where

$$\alpha = \sqrt{(a_3 - a_{10})(a_3 - a_2)}, \quad r_1 = \frac{2a_3 h_z}{a_3 - a_{10}} - G ,$$

$$r_2 = \frac{2a_3 h_z}{a_3 - a_2} - G .$$

Let

$$z = \frac{1}{G - g_3}$$

then, we may write Eq. (17) as

$$\alpha\, dt = \frac{dz}{\sqrt{[z(G - r_1) - 1][z(r_2 - G) + 1]}} . \tag{18}$$

By introduction of the new quantities

$$M = (G - r_1)(r_2 - G)\,, \quad N = 2G - r_1 - r_2\,,$$
$$L^2 = \frac{4M + N^2}{4M^2}\,,$$

Eq. (18) is transformed into

$$\alpha\, dt = \frac{dz}{\sqrt{M}\sqrt{\left(z + \dfrac{N}{2M}\right)^2 - L^2}}\,.$$

and then

$$\alpha\sqrt{M}t = \operatorname{argch} \frac{z + \dfrac{N}{2M}}{L}\,.$$

Finally, we obtain for g_3 the following expression

$$g_3 = G - \frac{1}{L\cosh\left(\alpha\sqrt{M}t\right) - \dfrac{N}{2M}}\,. \qquad (19)$$

By means of Eqs. (15) and (16) it results

$$g_1^2 = \frac{a_3 h_z}{2[(a_3 - a_{10})G - a_3 h_z]} \frac{1 + \cosh(\alpha\sqrt{M}t)}{\left(L\cosh(\alpha\sqrt{M}t) - \dfrac{N}{2M}\right)^2}$$
$$g_2^2 = \frac{a_3 h_z}{2[a_3 h_z - (a_3 - a_2)G]} \frac{\cosh(\alpha\sqrt{M}t) - 1}{\left[L\cosh(\alpha\sqrt{M}t) - \dfrac{N}{2M}\right]^2}\,. \qquad (20)$$

The Poisson bracket $\{\mathcal{H}_0; \epsilon V\}$ can be evaluated taking into account the structural identities (5) and we obtain

$$\{\mathcal{H}_0; \epsilon V\} = [(a_2 - a_3)g_1 g_2 g_3 + a_3 h_z g_1 g_2]\epsilon\cos\nu t\,.$$

Now, we can construct the Melnikov function from the results above as

$$M(t_0) = \int_{-\infty}^{+\infty} [(a_2 - a_3)g_1 g_2 g_3 + a_3 h_z g_1 g_2]\epsilon\cos\nu(t - t_0)\, dt$$

and the substitution of g_1, g_2 and g_3 for their values at the separatrix yields

$$\begin{aligned} M(t_0) = {} & \epsilon(a_3 - a_2)K\cos\nu t_0 \int_{-\infty}^{+\infty} \frac{\sinh\left(\alpha\sqrt{M}t\right)\cos\nu t}{\left[L\cosh\left(\alpha\sqrt{M}t\right) - \dfrac{N}{2M}\right]^3}\, dt \\ & + \epsilon(a_3 - a_2)K\sin\nu t_0 \int_{-\infty}^{+\infty} \frac{\sinh\left(\alpha\sqrt{M}t\right)\sin\nu t}{\left[L\cosh\left(\alpha\sqrt{M}t\right) - \dfrac{N}{2M}\right]^3}\, dt \\ & + (a_3 h_z - (a_3 - a_2)G)\epsilon K\cos\nu t_0 \int_{-\infty}^{+\infty} \frac{\sinh\left(\alpha\sqrt{M}t\right)\cos\nu t}{\left[L\cosh\left(\alpha\sqrt{M}t\right) - \dfrac{N}{2M}\right]^2}\, dt \\ & + (a_3 h_z - (a_3 - a_2)G)\epsilon K\sin\nu t_0 \int_{-\infty}^{+\infty} \frac{\sinh\left(\alpha\sqrt{M}t\right)\sin\nu t}{\left[L\cosh\left(\alpha\sqrt{M}t\right) - \dfrac{N}{2M}\right]^2}\, dt \end{aligned} \qquad (21)$$

where

$$K = \frac{a_3 h_z}{2\sqrt{[(a_3 - a_{10})G - a_3 h_z][a_3 h_z - (a_3 - a_2)G]}}\,.$$

Since $\sin vt$, $\sinh\alpha\sqrt{M}t$ are odd functions as well as $\cos vt$, $\cosh\alpha\sqrt{M}t$ are even functions, the first and third terms in (21) vanish after integration. Thus

$$M(t_0) = \epsilon K\sin\nu t_0 \left\{ (a_3 - a_2)\int_{-\infty}^{+\infty} \frac{\sinh\left(\alpha\sqrt{M}t\right)\sin\nu t}{\left[L\cosh\left(\alpha\sqrt{M}t\right) - \dfrac{N}{2M}\right]^3}\, dt + (a_3 h_z - (a_3 - a_2)G)\int_{-\infty}^{+\infty} \frac{\sinh\left(\alpha\sqrt{M}t\right)\sin\nu t}{\left[L\cosh\left(\alpha\sqrt{M}t\right) - \dfrac{N}{2M}\right]^2}\, dt \right\}. \qquad (22)$$

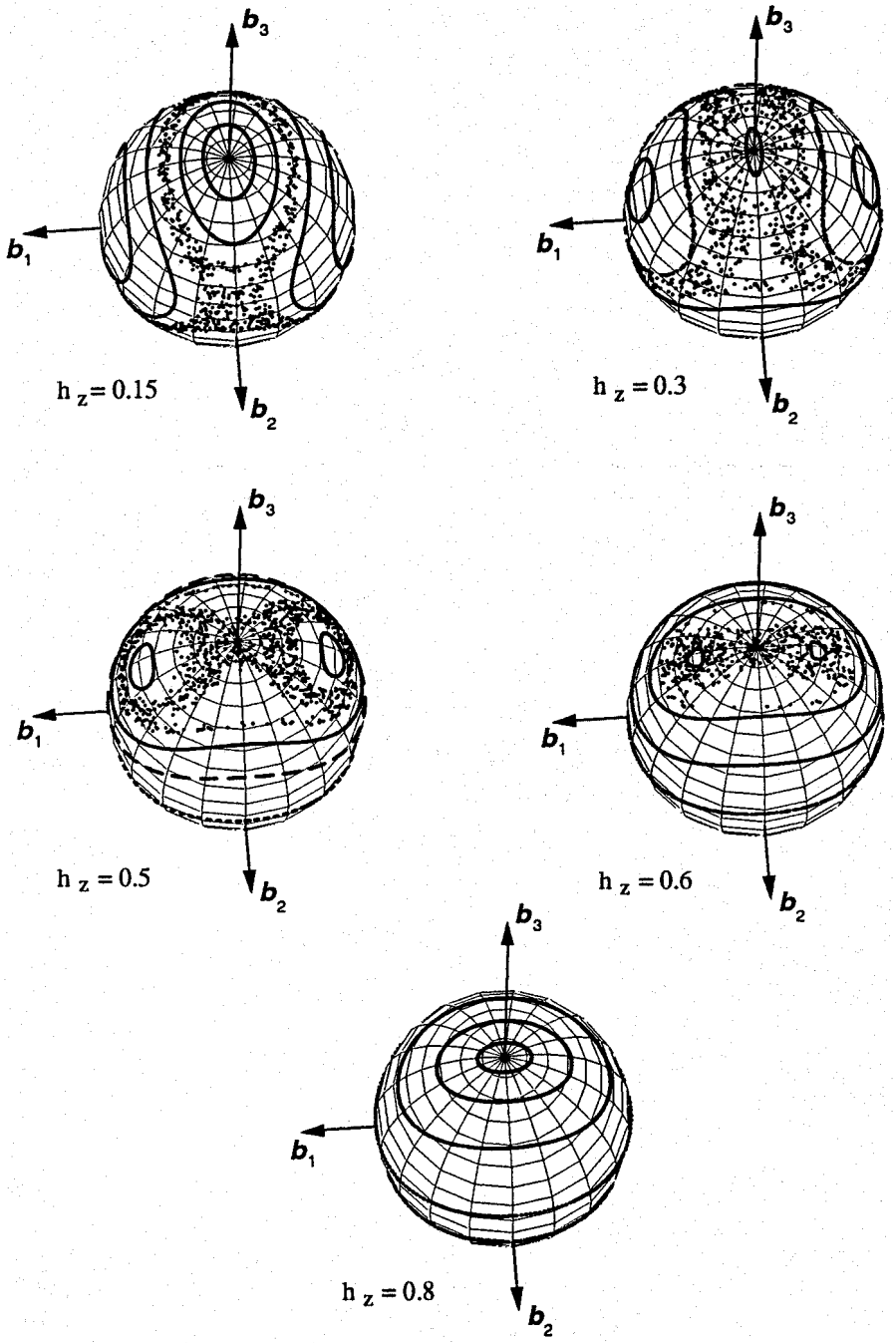

Fig. 5. Evolution of the Poincaré surfaces of section for $a_{10} = 0.1$, $a_2 = 0.2$, $a_3 = 0.3$, $G = 1$, $\epsilon = 0.005$ and $\nu = 0.1$ for increasing values of h_z.

The two expressions

$$\frac{\sinh\left(\alpha\sqrt{M}t\right)\sin\nu t}{\left[L\cosh\left(\alpha\sqrt{M}t\right)-\dfrac{N}{2M}\right]^3}$$

and

$$\frac{\sinh\left(\alpha\sqrt{M}t\right)\sin\nu t}{\left[L\cosh\left(\alpha\sqrt{M}t\right)-\dfrac{N}{2M}\right]^2}$$

are zero for $t = 0$ and decay exponentially as $t \longrightarrow +\infty$. Since $L > N/(2M)$ the two expressions above are larger than zero and, hence, for small values of the frequency ν, the two integrals in (22) are nonzero. Thus, we can conclude that the Melnikov function (22) has simple zeroes and chaotic motion can take place.

Nevertheless, as $h_z \longrightarrow G(a_3 - a_{10})/a_3$ the Melnikov function tend to zero because both g_1 and g_2 tend to zero as can be observed taking

limits in Eq. (20). This is a consequence of the disappearance of the unperturbed separatrix for $h_z = G(a_3 - a_{10})/a_3$. Hence, chaotic motion can be removed by increasing the relative angular momentum h_z. In this way, we obtain stabilization of the motion by means of increasing the spin of the rotor about the $\mathbf{s}_3$ axis. The transition from chaotic to stable motion may be observed through a sequence of Poincaré surfaces of section for increasing values of h_z. From a starting value $h_z = 0$ we get the plot depicted in Fig. 3. As h_z increases we observe, in Fig. 5, how the stochastic layer follows the evolution of the unperturbed separatrix. Once the unperturbed separatrix has disappeared no chaotic structures are observed in the Poincaré sections.

5. Conclusions

We have shown that for a deformable spacecraft in torque free rotation, chaotic motion can be removed by means of a spinning rotor about one of the principal axes of inertia. This is understood, in terms of the unperturbed Hamiltonian, as a consequence of the disappearance of the separatrix.

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