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ORIGINAL ARTICLE

Dynamics of a satellite orbiting a planet with an inhomogeneous gravitational field

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Abstract We study the dynamics of a satellite (artificial or natural) orbiting an Earth-like planet at low altitude from an analytical point of view. The perturbation considered takes into account the gravity attraction of the planet and in particular it is caused by its inhomogeneous potential. We begin by truncating the equations of motion at second order, that is, incorporating the zonal and the tesseral harmonics up to order two. The system is formulated as an autonomous Hamiltonian and has three degrees of freedom. After three successive Lie transformations, the system is normalised with respect to two angular co-ordinates up to order five in a suitable small parameter given by the quotient between the angular velocity of the planet and the mean motion of the satellite. Our treatment is free of power expansions of the eccentricity and of truncated Fourier series in the anomalies. Once these transformations are performed, the truncated Hamiltonian defines a system of one degree of freedom which is rewritten as a function of two variables which generate a phase space which takes into account all of the symmetries of the problem. Next an analysis of the system is achieved obtaining up to six relative equilibria and three types of bifurcations. The connection with the original system is established concluding the existence of various families of invariant 3-tori of it, as well as quasiperiodic and periodic trajectories. This is achieved by using KAM theory techniques.

Keywords Satellite dynamics · Zonal and tesseral harmonics · Delaunay normalisation · Reduction and invariant theories · Bifurcation lines · Non-linear stability · KAM theory · Invariant tori · Quasiperiodic and periodic orbits

1 Introduction

The gravity field of a planet is the most important perturbation affecting a satellite. In general, analytical theories are employed to provide fast and accurate calculation of ephemeris,

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although for a satellite orbiting at low altitude they are normally used to study the time variation of some tesseral coefficients of the gravity field. This paper deals with the influence of the tesserals in the motion of a satellite orbiting an Earth-like planet at low altitude. We have not taken into account the perturbations due to the a third orbit nor the atmospheric friction of a planet.

Analytical study of the effects of the potential upon satellite orbits around a planet has been an ongoing process, beginning with the original work of Brouwer (1959) and continuing to the present. Brouwer considered the main problem of the artificial satellite, that is, the problem defined as the two-body Hamiltonian with the perturbation due to the bulge of the planet caused by the oblateness coefficient of the gravity potential of the planet. Kozai (1962) generalised Brouwer's approach by considering a second-order theory for a Hamiltonian which included more terms in the perturbation (he took into consideration the perturbation of the oblateness coefficient plus the zonal harmonics up to J_8, i.e., $-C_{90}$). Improvements of Brouwer and Kozai's theories were tackled by Deprit and coworkers who pushed previous studies to higher orders (Coffey and Deprit 1982; Deprit 1981). They also analysed a broader problem considering more zonal harmonics in the perturbation (up to J_9) (Coffey et al. 1994). A common feature of these works is that all treatments are performed avoiding series expansions, thence they are valid for all elliptic motions. In Barrio and Palacián (2003) a generalisation of the previous research was done by taking into account the atmospheric friction.

One interesting area of this research has involved the contributions to the planet's potential of the longitude-dependent tesseral harmonics. The perturbations taking into account involve the rotation of the planet and are in general more complex than the theories which include only the zonal harmonics, the reason being that the tesseral problem is formulated by means of a three-degree-of-freedom autonomous Hamiltonian whereas the zonal problem is represented as a system of two degrees of freedom. While a numerical integration usually provides higher accuracy and even may gain in computation time, only an analytical theory can provide complete deeper insight into the nature of the perturbation. Indeed, this insight is often useful in the development of more efficient numerical integrators especially devised for the artificial satellite theory, see for instance (Segerman and Coffey 2000).

The tesseral problem has been studied by various authors since the pioneering work of Kaula (1966). In some papers, the entire potential, that is, the series containing the zonals and tesserals coefficients, is placed at first order. Then they expand the perturbation in power series of the eccentricity and Fourier series of a short-periodic angle, applying thereafter canonical transformations with the aim of eliminating the mean anomaly at first order of perturbation, see for instance Métris et al. (1993). These approaches are usually done in a phase space free of resonances.

More recently Segerman and Coffey (2000) have derived a theory for the tesseral problem for low altitude satellites based on the relegation algorithm of Deprit et al. (2001), with the purpose of producing ephemeris of the satellite's motion. However, they do not analyse the resulting Hamiltonian system. Periodic orbits for the tesseral problem have been investigated from a numerical point of view (Lara 2003; Lara and Elipe 2002). The authors have found families of periodic orbits emanating from the geostationary points as well as some bifurcation lines. Nevertheless, an analytical study of the possible periodic and quasiperiodic orbits as functions of the parameters of the problem remains an open but significant issue. Our treatment is based on a scaling of the Hamiltonian in a way that we take into account the relative values of the coefficients involved in the process. In particular we assume that the oblateness coefficient, i.e., the zonal term C_{20} is much bigger than the rest of terms as is the case of the Earth and other planets like Saturn or Mars, hence the reason of Earth-like planets.

Thus we are able to remove both short and long period terms from the equations of motion applying Lie transformations (Deprit 1969). We choose the small parameter as the quotient between the angular velocity of the planet and the mean motion of the satellite, arranging the initial Hamiltonian in a convenient way such that the transformations can be performed.

Once the Hamiltonian is simplified we use reduction techniques to express it in an adequate set of co-ordinates and its appropriate phase space, the so-called fully reduced phase space, where the flow may be discussed. In this sense we enlarge the work done for the main problem by Cushman (1983, 1988), Coffey et al. (1986) and Chang and Marsden (2003)—see also Coffey et al. (1994) for the zonal problem—finding the relative equilibria of the reduced Hamiltonian, analysing their non-linear stability and bifurcations and showing the existence of KAM 3-tori and quasiperiodic motions around some families of equilibria.

Our approach is new and by no means trivial as we are dealing with a three-degree-of-freedom conservative system. The amount of formulæ needed to achieve the simplifications and to obtain the critical points and bifurcation lines is enormous and cannot be made by hand as it may be done for the main and zonal problems (which are two-degree-of-freedom systems). Furthermore our analysis cannot be considered standard; for instance, the normalisation of the mean anomaly is highly technical and not very known, but is crucial to deal with perturbed Keplerian systems, especially when the perturbations depend upon negative powers of the radial distance of the object under study and the centre of mass of the system. Moreover, the proof of the existence of true 3-tori for the original (also called initial) Hamiltonian is not a feature which can be concluded straigthforwardly if one does not choose the right co-ordinates to check the non-degenerate hypotheses needed to achieve the existence and persistence of the invariant tori.

A previous work pointing out the some of the main features of our theory appears in the brief paper (Palacián, 2006). Here we extend this result, giving more details on the normalisation and reduction methods. Moreover, we focus on the global analysis of the fully reduced Hamiltonian, discussing the existence of its relative equilibria and the occurrence of the bifurcation lines and establishing the corresponding non-linear stability of the equilibria. Finally, we extract the consequences on the dynamics of the original system, finding families of invariant 3-tori and quasiperiodic orbits confined in them, some of which may be closed after using some geometric arguments.

The paper has seven sections. In Sect. 2 the equations of motion are given while Sect. 3 is devoted to the normalisations of the Hamiltonian. The reduction process is described in Sect. 4. The fully-reduced system is analysed in Sect. 5. The flow of the original system is reconstructed in Sect. 6 where the existence of the KAM tori of the original Hamiltonian as well as some periodic orbits is proved. In Sect. 7, we draw the main features of the paper.

2 Hamiltonian of the problem

We choose two sets of variables well suited to perform the normalisation of our original Hamiltonian, the so-called *polar-nodal* and *Delaunay* variables. For an explanation of both sets of canonical variables, see Deprit (1982). We start by fixing an inertial frame, say $0\,x\;y\;z$, centered at the centre of mass of the planet.

Polar-nodal variables, also known as Hill or Whittaker variables, is the set $(r, \vartheta, \nu, R, \Theta, N)$, where r is the radial distance from the centre of the planet to the satellite and its conjugate momentum R denotes the radial velocity. The action Θ is the magnitude of the angular momentum vector, $\boldsymbol{G}$, and its conjugate angle is the argument of the latitude $\vartheta \in [0, 2\pi)$. The argument of the node, ν, is the angle conjugate to the action N, which represents the projection of $\boldsymbol{G}$ onto the z-axis. The inclination of the instantaneous orbital plane with respect to the $x\,y$-plane (the so-called equatorial plane) is given by the angle $0 < I < \pi$ such that $N = \Theta \cos(I)$. We define $c = \cos(I)$ and $s = \sin(I)$.

On the other hand Delaunay variables, (ℓ, g, h, L, G, H), represent a set of action-angle variables for the Kepler problem, see Deprit (1981,1982) for details. The action L is related with the semimajor axis of the orbit, a, by the identity $L^2 = \mu\, a$ where μ is the gravitational parameter of the planet. Thence, if $\mathcal{H}_0$ stands for the Hamiltonian of the two-body problem, $\mathcal{H}_0 = -\mu^2/(2\,L^2)$. The action G is equal to Θ, whereas $H \equiv N$. The angle ℓ stands for the mean anomaly. The angle g is the argument of pericentre and $h \equiv \nu$. The eccentricity of the trajectory is designated by e and in terms Delaunay actions it is expressed as $e = \sqrt{1 - G^2/L^2}$.

If the planet is assumed to rotate with a uniform angular speed ω one can always choose a three-dimensional reference frame attached to the planet, $0\,x'\,y'\,z'$, such that its z'-component corresponds to the axis of rotation, thus $z \equiv z'$ (Palacián 2002a,b). The Hamiltonian of the problem is time-independent provided that ω is constant. Thus, it may be put as the sum $\mathcal{H} = \mathcal{T} + \mathcal{V} - \omega\, N$, where $\mathcal{T}$ and $\mathcal{V}$ represent, respectively, the kinetic and the potential energies while $-\omega\, N$ accounts for the Coriolis term caused by the fact that $\mathcal{H}$ is expressed in a rotating frame. Written in polar-nodal variables $\mathcal{T}$ is given through:

$$\mathcal{T} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right)$$

whereas

$$\mathcal{V} = -\frac{\mu}{r}\left[1 + \left(\frac{\alpha}{r}\right)^2 \mathcal{V}_2 + \left(\frac{\alpha}{r}\right)^3 \mathcal{V}_3 + \left(\frac{\alpha}{r}\right)^4 \mathcal{V}_4 + \ldots\right],$$

α being the mean equatorial radius of the planet. Besides, each $\mathcal{V}_i$ is a finite Fourier series in the angles ϑ and ν whose coefficients depend on C_{ij}, S_{ij} $(0 \le j \le i)$ and on c and s.

Now we drop the coefficients of order higher than two, retaining the most influent terms. Thence, the potential reduces to:

$$\begin{aligned}
\mathcal{V}_2 = {} & \frac{1}{4}(3s^2 - 2)C_{20} - \frac{3}{4}s^2 C_{20} \cos(2\,\vartheta) \\
& -\frac{3}{4}(c-1)s\,[S_{21}\cos(\nu - 2\,\vartheta) - C_{21}\sin(\nu - 2\,\vartheta)] \\
& +\frac{3}{2}cs\,[S_{21}\cos(\nu) - C_{21}\sin(\nu)] \\
& -\frac{3}{4}(c+1)s\,[S_{21}\cos(\nu + 2\,\vartheta) - C_{21}\sin(\nu + 2\,\vartheta)] \\
& +\frac{3}{4}(c-1)^2\,[C_{22}\cos(2\,\nu - 2\,\vartheta) + S_{22}\sin(2\,\nu - 2\,\vartheta)] \\
& +\frac{3}{2}s^2\,[C_{22}\cos(2\,\nu) + S_{22}\sin(2\,\nu)] \\
& +\frac{3}{4}(c+1)^2\,[C_{22}\cos(2\,\nu + 2\,\vartheta) + S_{22}\sin(2\,\nu + 2\,\vartheta)], \qquad (1)
\end{aligned}$$

see for example how it is derived in Kaula (1966) and Serrano (2003).

We emphasize that we have not considered the influence of higher-zonal harmonics as C_{30} or C_{40} because we have preferred to focus on the perturbation due to all the second-order coefficients and, in this sense, how the presence of the tesseral harmonics influences the dynamics of the main problem. A continuation of the present work could include these zonal coefficients in the gravity fields of the planet. The goal of the following paragraphs is to normalise and reduce the (autonomous) three-degree-of-freedom system defined by $\mathcal{H}$ into a new system of one degree of freedom.

3 Normalisations

We need to pass from $\mathcal{H}$ to a new Hamiltonian $\mathcal{K}$ using Lie transformations. We arrange the initial Hamiltonian as:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \frac{1}{2}\mathcal{H}_2 + \frac{1}{6}\mathcal{H}_3, \tag{2}$$

where $\mathcal{H}_0$ corresponds to the Hamiltonian of the two-body problem, that is,

$$\mathcal{H}_0 = \mathcal{T} - \frac{\mu}{r} = -\frac{\mu^2}{2\,L^2}.$$

Besides, $\mathcal{H}_1 = -\omega\,N$, $\mathcal{H}_2$ contains the terms factored by C_{20} while $\mathcal{H}_3$ have the terms related with the tesseral coefficients $C_{21}, C_{22}, S_{21}, S_{22}$. Higher-order terms are taken equal to zero. By doing so, one assumes that $|\mathcal{H}_{i+1}| \ll |\mathcal{H}_i|$ for $i \in \{0, \ldots, 4\}$. If n represents the mean motion of the satellite, i.e., $n = \mu^2/L^3$, the latter is satisfied whenever the quotient ω/n remains small (that is, it is of the size of a small parameter) and the tesseral harmonics are much smaller than C_{20}, see the details in Palacián (2002a). This is the typical situation of a satellite orbiting at low altitude and such that the corresponding planet's oblateness coefficient prevails over the rest of the zonal and tesseral coefficients in the gravity field.

Next we identify $\mathcal{K}_0 \equiv \mathcal{H}_0$ and apply three Lie transformations with the task of obtaining:

$$\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1 + \frac{1}{2}\mathcal{K}_2 + \frac{1}{6}\mathcal{K}_3 + \frac{1}{24}\mathcal{K}_4 + \frac{1}{120}\mathcal{K}_5.$$

The mean anomaly is removed through two successive Lie transformations, whereas the argument of the node is removed from the equations by means of another Lie transformation.

(i) Following Deprit (1981), we apply the technique called *the elimination of the parallax*. This transformation is not a normalisation procedure in the sense that no angle is averaged in the process. Nevertheless, it is useful to apply it in order to alleviate the number of terms in the resulting Hamiltonian and is strongly recommended when a normalisation process needs to be carried out to high order, as has been repeatedly by Deprit and his coworkers (Deprit 1981; Coffey et al. 1986, 1994; Deprit and Miller 1988). The elimination of the parallax is performed in closed form.
The homological equation that needs to be resolved at each order i $(1 \leq i \leq 5)$ is written in terms of polar-nodal co-ordinates as follows:

$$\frac{\Theta}{r^2}\,\frac{\partial\,\mathcal{P}_i}{\partial\,\vartheta} + \mathcal{A}_i = \bar{\mathcal{H}}_i, \tag{3}$$

for the unknowns $\mathcal{A}_i$ (the new Hamiltonian) and $\mathcal{P}_i$ (the generating function). In the equation, each $\bar{\mathcal{H}}_i$ is known in terms of $\bar{\mathcal{H}}_j$ and of $\mathcal{P}_j$ (for $1 \leq j \leq i-1$). Indeed, the simple form acquired by the homological equation is due to the fact that $\bar{\mathcal{H}}_i$ is arranged adequately to depend explicitly on sines and cosines of ϑ and possibly on r only through powers of $(\alpha/r)^2$ but not on R, see the details in Deprit (1981). Then one takes $\mathcal{A}_i$ as the average of $\bar{\mathcal{H}}_i$ over the argument of the latitude and solve Eq. (3) is solved in $\mathcal{P}_i$ by calculating a quadrature with respect to ϑ. Thus, $\mathcal{P}$ is a periodic function in ϑ and ν, equivalently a periodic function in the three Delaunay angles.

A different approach to the elimination of the parallax is the so-called *elimination of the latitude* (Deprit and Ferrer 1987; Coppola and Palacián 1994). The aim of this transformation is the elimination of the terms depending on ϑ with the aim of preparing the transformed Hamiltonian for the subsequent transformation. After applying this transformation, one obtains a different intermediate Hamiltonian, but after removing the terms depending on ℓ (following the procedure explained in (ii)) the resulting final Hamiltonian is the same.

(ii) Next the mean anomaly is eliminated through a *Delaunay normalisation* (Deprit 1982). This time, the homological equation solved at each order i $(1 \leq i \leq 5)$ can be written in terms of Delaunay co ordinates as:

$$\frac{\mu^2}{L^3}\,\frac{\partial\,\mathcal{Q}_i}{\partial\,\ell} + \mathcal{B}_i \;=\; \bar{\mathcal{A}}_i, \tag{4}$$

for the unknowns $\mathcal{B}_i$, i.e., the transformed Hamiltonian, and $\mathcal{Q}_i$ (the generating function). The terms $\bar{\mathcal{A}}_i$ are known at each order as functions of $\bar{\mathcal{A}}_j$ and of $\mathcal{Q}_j$ $(1 \leq j \leq i-1)$. After taking $\mathcal{B}_i$ as the average of $\bar{\mathcal{A}}_i$ with respect to ℓ, Eq. (4) is solved in $\mathcal{Q}_i$ by calculating the corresponding primitive in ℓ.

The removal of ℓ avoiding Taylor and Fourier expansions has been possible thanks to the introduction of the polylogarithmic function of orders two and three in the generating function of the procedure for $i = 4$ and $i = 5$. Specifically, the polylogarithm of $z \in \vec{C}$ of order n is defined as:

$$\mathrm{Li}_n(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

Besides, some other combinations involving logarithmic terms (Osácar and Palacián 1994; Palacián 2002b) are used to preserve the closure of the expressions, making the transformation valid for any type of elliptic motions. This step is essential to construct our theory and should not be underestimate. Moreover, $\mathcal{Q}$ is a periodic function in the three angles ℓ, g and h.

We stress that the polylogarithms come out as the solutions of some of the quadratures needed for obtaining $\mathcal{Q}_i$ in closed form. More precisely, the intermediate Hamiltonian $\bar{\mathcal{A}}_4$ depends on the angle called the equation of the centre, e.g., the subtraction of the true and the mean anomalies, and the quadrature corresponding to the equation of the centre is obtained in closed form by means of the dilogarithmic function (i.e., the polylogarithm of order two).

Alternatively we could have used the methodology of Cushman and van de Meer (van der Meer and Cushman 1986; Cushman 1992) to carry out the normalisation over ℓ through regularisation techniques and identification to the geodesic flow on the sphere S^4. However, it is doubtful to obtain closed-form expressions of the generating functions associated with the average over ℓ by means of the Moser or of the Kustaanheimo–Stiefel transformation.

(iii) The third Lie transformation corresponds to the *elimination of the argument of the node*. it is removed from the equations via a standard average, also without introducing series expansions. First, as $\mathcal{B}_0 = -\mu^2/(2\,L^2)$ is a formal integral, Hamiltonian $\mathcal{B}_i$ can be arranged so that $-\omega\,H$ appears at zero order while the different perturbations appear at orders ranging from one to four. So, for $1 \leq i \leq 4$, the homological equation becomes now:

$$-\,\omega\,\frac{\partial\,\mathcal{R}_i}{\partial\,h}+\mathcal{K}_i\;=\;\bar{\mathcal{B}}_i, \tag{5}$$

for the unknowns $\mathcal{K}_i$ (i.e., the transformed Hamiltonian) and the generating function $\mathcal{R}_i$. Now, each $\bar{\mathcal{B}}_i$ is known in terms of $\bar{\mathcal{B}}_j$ and of $\mathcal{R}_j$ $(1\le j\le i-1)$. Then, one chooses $\mathcal{K}_i$ as the average of $\bar{\mathcal{B}}_i$ with respect to the argument of the node while $\mathcal{R}_i$ is determined solving the resulting quadrature of (5). The function $\mathcal{R}$ is periodic in g and h.

We do not write down the explicit expressions of the generating functions nor of the intermediate Hamiltonians as they are rather big formulæ. However they have been generated using MATHEMATICA$^{5.2}$ and are available upon request. We stress that all the generating functions are $2\,\pi$-periodic functions in the angular co-ordinates.

After truncating higher-order terms, the resulting Hamiltonian is independent of ℓ and h, hence L and H are integrals of motion for it. Thus, $\mathcal{K}$ defines a system of one degree of freedom. The reason for pushing the calculations to fifth order is that we need $\mathcal{K}_5$ to capture the influence of C_{21}, C_{22}, S_{21} and S_{22}. In particular the Hamiltonians $\mathcal{K}_i$ are given by:

$$\begin{aligned}
\mathcal{K}_0 &= -\frac{\mu^2}{2\,L^2},\\
\mathcal{K}_1 &= -\omega\,H,\\
\mathcal{K}_2 &= -\frac{\alpha^2\,\mu^4\,C_{20}\,(G^2-3\,H^2)}{2\,L^3\,G^5},\\
\mathcal{K}_3 &= 0,\\
\mathcal{K}_4 &= -\frac{9\,\alpha^4\,\mu^6\,C_{20}^2}{16\,L^5\,G^{11}}\Big[35\,L^2\,H^4+36\,L\,G\,H^4+5\,(2\,L^2+H^2)\,G^2\,H^2\\
&\qquad -24\,L\,G^3\,H^2-(5\,L^2+18\,H^2)\,G^4+4\,L\,G^5+5\,G^6\\
&\qquad +2\,(G^4-16\,G^2\,H^2+15\,H^4)\,(L^2-G^2)\,\cos{(2\,g)}\Big],\\
\mathcal{K}_5 &= \frac{135\,\alpha^4\,\mu^8\,H}{\omega\,L^6\,G^{10}}\Big[\left(C_{21}^2+S_{21}^2\right)\left(G^2-2\,H^2\right)+\,2\,\left(C_{22}^2+S_{22}^2\right)\left(-G^2+H^2\right)\Big].
\end{aligned} \tag{6}$$

We use the same names for the three-times transformed variables as the original ones in order to avoid cumbersome notation; however, the passage from $\mathcal{H}$ to $\mathcal{K}$ needs three Lie transformations and therefore three changes of variables which are constructed using the three generating functions. We remark that the reason why the tesseral harmonics appear for the first time is due to the initial arrangement made to $\mathcal{H}$, and in particular to the choice of the small parameter and the form of $\mathcal{H}_1$.

The reader should notice the absence of the third and fourth powers of C_{20} what would be compatible with C_{22}^2 and S_{22}^2. The reason is that the terms factored by C_{20}^3 would appear for the first time at order six once elimination of the parallax is performed. Hence, some of them would remain after the successive application of the Delaunay normalisation and the elimination of the node. However, we have stopped the computations at order five because it is in this order when the occurrence of the tesseral coefficients take place.

4 Reductions

Our aim is to use the continuous and discrete symmetries of $\mathcal{K}$ in order to obtain the simplest possible expression for this normal form Hamiltonian. If we put $\mathcal{K}$ in Cartesian co-ordinates, say (x, y, z, P_x, P_y, P_z) (writing L, G and H explicitly in terms of Cartesian elements and transforming $\cos(g)$ and $\sin(g)$ conveniently through the introduction of the argument of the latitude and the true anomaly; see Palacián (2002a)), it is easy to prove that two independent symmetries of $\mathcal{K}$ are given by:

$$\begin{aligned} \mathcal{R}_1 &: (x, y, z, P_x, P_y, P_z) \rightarrow (x, -y, -z, -P_x, P_y, P_z), \\ \mathcal{R}_2 &: (x, y, z, P_x, P_y, P_z) \rightarrow (x, -y, z, -P_x, P_y, -P_z). \end{aligned} \tag{7}$$

We stress that $\mathcal{R}_1$ and $\mathcal{R}_2$ are not inherited from $\mathcal{H}$, so these are only discrete symmetries of the (truncated) normal form Hamiltonian.

Inspired by Cushman and his collaborators (Cushman and Sadovskií 2000; Efstathiou et al. 2004), we define a couple of variables, σ_1 and σ_2, to reflect the occurrence of these symmetries. The relationship between the Delaunay and the new variables was derived in Iñarrea et al. (2004) (see also Iñarrea et al. 2006). It is as follows:

$$\begin{aligned} \sigma_1 &= (L - |H|)^2 + (1 - L^2/G^2)\,(G^2 - H^2)\,\sin^2(g), \\ \sigma_2 &= G, \end{aligned} \tag{8}$$

so σ_2 represents the modulus of the angular momentum vector whereas σ_1 depends upon G and g. On the other hand we get:

$$\cos(g) = \pm\sqrt{\frac{4\,\sigma_2^4 - 4\,\sigma_1\,\sigma_2^2 - 2\,L\,|H|\,(\sigma_1 + 2\,\sigma_2^2) + L^2\,H^2}{4\,\sigma_2^4 - 4\,(L^2 + H^2)\,\sigma_2^2 - 2\,L\,|H|\,(L^2 + H^2 - 2\,\sigma_2^2) + 5\,L^2\,H^2}}$$

whereas for the sine we get:

$$\sin(g) = \pm\sqrt{\frac{-4\,(L^2 + H^2 - \sigma_1)\,\sigma_2^2 - 2\,L\,|H|\,(L^2 + H^2 - \sigma_1 - 4\,\sigma_2^2) + 4\,L^2\,H^2}{4\,\sigma_2^4 - 4\,(L^2 + H^2)\,\sigma_2^2 - 2\,L\,|H|\,(L^2 + H^2 - 2\,\sigma_2^2) + 5\,L^2\,H^2}}.$$

These new co-ordinates generate the fully-reduced phase space, $\mathcal{U}_{L,H}$, which depend on the integrals L and H. For $|H| > 0, \mathcal{U}_{L,H}$ is given by:

$$\mathcal{U}_{L,H} = \left\{(\sigma_1, \sigma_2) \in \boldsymbol{R}^2 \;\middle|\; \frac{(\sigma_2^2 - L\,|H|)^2}{\sigma_2^2} \le \sigma_1 \le (L - |H|)^2,\ |H| \le \sigma_2 \le L\right\},$$

whereas for $H = 0, \mathcal{U}_{L,0}$ is defined through:

$$\mathcal{U}_{L,0} = \left\{(\sigma_1, \sigma_2) \in \boldsymbol{R}^2 \mid \sigma_2^2 \le \sigma_1 \le L^2,\quad 0 \le \sigma_2 \le L\right\}.$$

The space $\mathcal{U}_{L,H}$ has two singular points: $((L - |H|)^2, H|)$ and $((L - |H|)^2, L)$ while $\mathcal{U}_{L,0}$ has three singularities: $(L^2, 0)$, (L^2, L) and $(0, 0)$. The singularities of $\mathcal{U}_{L,H}$ and $(L^2, 0)$, (L^2, L) are spurious as they have been introduced by reducing out the discrete symmetries $\mathcal{R}_1$ and $\mathcal{R}_2$. In Fig. 1, we have depicted the fully reduced phase spaces.

Applying the changes above to $\mathcal{K}$, dropping the terms $\mathcal{K}_0$ and $\mathcal{K}_1$ which are integrals and simplifying a bit, the resulting Hamiltonian is defined on $\mathcal{U}_{L,H}$ (for $0 \le |H| \le L$) and it

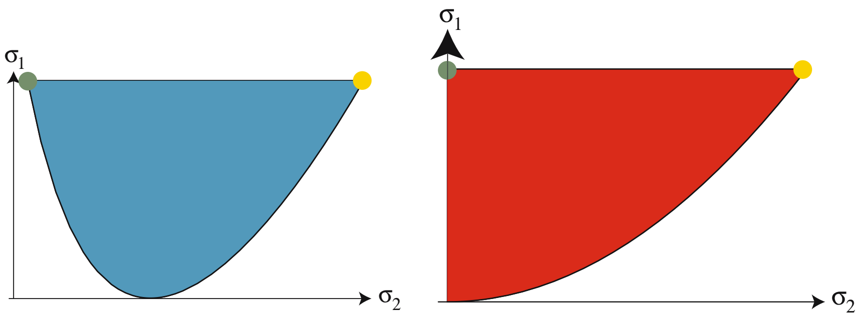

Fig. 1 On the left, phase space for $|H|>0$. The co-ordinates of the extreme points of $\mathcal{U}_{L,H}$ are $((L-|H|)^2, |H|)$ (equatorial motions, the green point) and $((L-|H|)^2, L)$ (circular motions, the yellow point). The space reaches its lowest point at $(0, \sqrt{L\,|H|})$. On the right, phase space for $H = 0$. The co-ordinates of the extreme points of $\mathcal{U}_{L,0}$ are $(L^2, 0)$ (polar rectilinear motions, the green point), (L^2, L) (polar circular motions, the yellow point) and $(0, 0)$ (the non-spurious singular point of $\mathcal{U}_{L,0}$). The segment with extremes $(0, 0)$ and $(L^2, 0)$ corresponds to rectilinear motions

yields:

$$\begin{aligned} \mathcal{S} = \frac{1}{128\,\omega\,L^6\,\sigma_2^{11}} \Big\{ &-32\,\omega\,C_{20}\,L^3\,\sigma_2^6\,(\sigma_2^2-3\,H^2) - 3\,\omega\,C_{20}^2\,L\Big[3\,\sigma_2^6+4\,L\,\sigma_2^5 \\ &-(7\,L^2-10\,H^2-8\,L\,|H|-4\,\sigma_1)\,\sigma_2^4-24\,L\,H^2\,\sigma_2^3 \\ &+H^2\,(38\,L^2+35\,H^2-120\,L\,|H|-60\,\sigma_1)\,\sigma_2^2+36\,L\,H^4\,\sigma_2 \\ &+65\,L^2\,H^4\Big]-144\,(C_{21}^2+S_{21}^2)\,H\,\sigma_2\,(\sigma_2^2-2\,H^2) \\ &+288\,(C_{22}^2+S_{22}^2)\,H\,\sigma_2\,(\sigma_2^2-H^2)\Big\}. \end{aligned} \tag{9}$$

Note that $\mathcal{S}$ is not defined at the segment $\sigma_2 = 0$ (then $0 \le \sigma_2 \le L^2$). Thus, to avoid collisions we suppose from now on that $\sigma_2 \ge \sigma_{20} > 0$ for a fixed value of the magnitude of the angular momentum vector. Note that this value must be chosen so that the satellite does not collapse with the planet, and this happens for $a\,(1-e) < 1$. Hence, as $a = L^2/\mu$ and $e = \sqrt{1-\sigma_2^2/L^2}$, we choose $\sigma_{20} = \sqrt{2\mu-\mu^2/L^2}$.

In the above Hamiltonian we have scaled time and distance conveniently, by setting $\mu = \alpha = 1$ in order to simplify the forthcoming expressions. Thence, $L > 1$ but it never exceeds the value of, say 2, in order to preserve the feature of low altitude orbits. Besides, the coefficients C_{ij} and S_{ij} are usually quantities smaller than 10^{-2} in absolute value and ω is also small (about 0.0588321 ... for the Earth in the units where $\mu = \alpha = 1$).

5 Analysis of the reduced system

5.1 Relative equilibria

The flow defined by $\mathcal{S}$ is analysed in $\mathcal{U}_{L,H}$ and in $\mathcal{U}_{L,0}$. Equilibria are determined by the extrema of (9) on $\mathcal{U}_{L,H}$ and on $\mathcal{U}_{L,0}$ when $H = 0$.

First of all we examine the possibility of locating extrema in the interior of the phase space. These points should be obtained as the roots of the system formed by equating to zero the partial derivatives of $\mathcal{S}$ with respect to σ_1 and σ_2. In particular, from (9) we readily obtain:

$$\frac{\partial \mathcal{S}}{\partial \sigma_1} = -\frac{3\, C_{20}^2\, (\sigma_2^2 - 15\, H^2)}{32\, L^5\, \sigma_2^9},$$

which only vanishes at $\sigma_2 = \sqrt{15}\,|H|$. Notice that we have to discard $H = 0$ because then $\sigma_2 = 0$ and σ_2 cannot be zero. Now, the partial derivative replaced at $\sigma_2 = \sqrt{15}\,|H|$ yields $\frac{\partial^2 \mathcal{S}}{\partial \sigma_2\, \partial \sigma_1}(\sigma_1, \sqrt{15}\,|H|) = -C_{20}^2/(270000\, L^5\, H^8)$, which is strictly negative in the interior of $\mathcal{U}_{L,H}$. Besides, $\frac{\partial \mathcal{S}}{\partial \sigma_2}$ is negative for $\sigma_2 = \sqrt{15}\,|H|$ and for σ_1 evaluated at $(\sigma_2^2 - L\,|H|)^2/\sigma_2^2 = (L - 15\,|H|)^2/15$ provided that $|C_{20}|$ is much bigger than the tesseral coefficients; thus $\frac{\partial \mathcal{S}}{\partial \sigma_2}(\sigma_1, \sqrt{15}\,|H|)$ is a strictly negative function. Thence we conclude that there are no stationary points in the interior of $\mathcal{U}_{L,H}$ and, consequently, all possible extrema are located on the boundary.

We commence with the points where the two curves delimiting the boundary of $\mathcal{U}_{L,H}$ meet. Their co-ordinates are:

$$E_1 \equiv \Big((L - |H|)^2\,,\ |H|\Big), \qquad E_2 \equiv \Big((L - |H|)^2\,,\ L\Big),$$

which correspond to the class of equatorial and circular "orbits", respectively. The point E_2 exists for all $0 \le |H| \le L$ and for the rest of parameters while E_1 occurs excepting for $H = 0$ because in this case the equatorial motions would be also rectilinear, a situation which has been excluded previously as neither $\mathcal{H}$ nor $\mathcal{K}$ (and $\mathcal{S}$) are defined for rectilinear orbits.

To determine the rest of the equilibria, two cases must be considered:

(a) The equilibria located on the rectilinear part of the boundary given by the curve $\sigma_1 = (L - |H|)^2$, under the restriction $|H| \le \sigma_2 \le L$.
(b) Those equilibria located on the curved part of the boundary defined by $\sigma_1\, \sigma_2^2 = (\sigma_2^2 - L\,|H|)^2$ and $|H| \le \sigma_2 \le L$.

For both cases we use the Lagrange multipliers technique in order to discuss the possible relative extrema of $\mathcal{S}$ on the boundary of $\mathcal{U}_{L,H}$ and $\mathcal{U}_{L,0}$. We get the following results.

5.1.1 Case (a).

If $\sigma_1 = (L - |H|)^2$ then (9) turns into a single-valued real function. Hence, its extrema are reached either at $\sigma_2 = |H|$, $\sigma_2 = L$ or at those points satisfying:

$$\frac{d\, \mathcal{S}((L - |H|)^2, \sigma_2)}{d\, \sigma_2} = \frac{3\, P_1(\sigma_2)}{128\, \omega\, L^6\, \sigma_2^{12}} = 0, \tag{10}$$

where

$$
\begin{aligned}
P_1(\sigma_2) = {} & 32\,\omega\, C_{20}\, L^3\, \sigma_2^8 - 5\,\omega\, C_{20}\, L\,(32\, L^2\, H^2 - 3\, C_{20})\,\sigma_2^6 \\
& + 24\,\omega\, C_{20}^2\, L^2\, \sigma_2^5 - 7\,\omega\, C_{20}^2\, L\,(3\, L^2 - 14\, H^2)\,\sigma_2^4 \\
& - 192\, H \left[\omega\, C_{20}^2\, L^2\, H + 2\,(C_{21}^2 - 2\, C_{22}^2 + S_{21}^2 - 2\, S_{22}^2)\right]\sigma_2^3 \\
& - 9\,\omega\, C_{20}^2\, L\, H^2\,(22\, L^2 + 25\, H^2)\,\sigma_2^2 \\
& + 120\, H^3 \left[3\,\omega\, C_{20}^2\, L^2\, H + 8\,(C_{21}^2 - C_{22}^2 + S_{21}^2 - S_{22}^2)\right]\sigma_2 \\
& + 715\,\omega\, C_{20}^2\, L^3\, H^4.
\end{aligned}
$$

As $((L-|H|)^2, |H|)$ and $((L-|H|)^2, L)$ are equilibria of $\mathcal{S}$, it suffices to analyse the possible roots of P_1 when σ_2 belongs to $[|H|, L]$. Then, changes in the number of roots may occur either when P_1 has multiple roots in $(|H|, L)$ or when σ reaches the extremes $|H|$ and L.

As P_1 is a polynomial of degree eight in σ_2 depending on eight parameters, the discussion of the possible roots is a very complicated matter. However, we can restrict ourselves to the cases of interest, that is, when the zonal and tesseral coefficients are small quantities. This is achieved by introducing a small parameter $\epsilon < 0$ and redefining new coefficients through:

$$
\begin{aligned}
& C_{20} = \epsilon, \\
& C_{21} = \epsilon^3\, c_{21}, \quad C_{22} = \epsilon^2\, c_{22}, \quad S_{21} = \epsilon^3\, s_{21}, \quad S_{22} = \epsilon^2\, s_{22}.
\end{aligned} \tag{11}
$$

This rescaling of parameters is compatible with the relative values of all parameters involved in P_1 and with the scaling of Hamiltonian $\mathcal{H}$, see Serrano (2003). We also assume that $C_{20} < 0$ ($\epsilon < 0$) which means that the polar radius of the planet is smaller than the equatorial one, a feature shared by the gravity fields of most of the planets and natural satellites. In the case of the Earth $C_{20} = -0.1082630 \times 10^{-2}$ whereas $C_{21} = 0.1342634 \times 10^{-8}$, $S_{21} = -0.3137117 \times 10^{-8}$, $C_{22} = 0.1574742 \times 10^{-5}$ and $S_{22} = -0.9023759 \times 10^{-6}$, see Segerman and Coffey (2000), Serrano (2003). These values are in accordance with the scaling (11). The quotient ω/n may reach the value $0.1664024\ldots$ for $L \equiv 2$, so in this case $|\epsilon|$ is about 434 times smaller than ω/n while if $L \equiv 1$, $|\epsilon|$ is about 54 smaller than ω/n.

After inserting the "new harmonics" in P_1 we discuss the possible multiple roots of P_1, calculating the resultant between P_1 and $d\,P_1/d\,\sigma_2$ with respect to σ_2. Thus, we obtained a (huge) polynomial relation relating the parameters c_{ij}, s_{ij}, ϵ, L, H and ω which does not vanish whenever $0 < -\epsilon \ll 1$ excepting for $H = 0$, a situation which will be discussed later. Besides, we calculate $P_1(|H|)$, arriving at a polynomial which cannot be zero for any combination of all the parameters provided that $0 < -\epsilon \ll 1$. Then, we discard possible bifurcations arising from equatorial motions in the upper boundary of $\mathcal{U}_{L,H}$.

Next, we explore the possibility of bifurcations coming from circular orbits. Thus we evaluate P_1 in L, replacing at the same time the zonal and tesseral coefficients using (11). We introduce Γ_1 as $P_1(L)$ where:

$$
\begin{aligned}
\Gamma_1 = 2\, L\, \epsilon \Big\{ & -96\, H \Big[-(c_{21}^2 + s_{21}^2)\,(2\, L^2 - 5\, H^2)\,\epsilon^5 + (c_{22}^2 + s_{22}^2)\,(4\, L^2 - 5\, H^2)\,\epsilon^3 \Big] \\
& + \omega\, L^2\,(9\, L^4 - 146\, L^2\, H^2 + 425\, H^4)\,\epsilon + 16\,\omega\, L^8\,(L^2 - 5\, H^2) \Big\}.
\end{aligned} \tag{12}
$$

We want to solve $\Gamma_1 = 0$ for H. We can achieve it by means of the Newton–Raphson algorithm, putting H in terms of the rest of parameters. Starting with $H_0 = L/\sqrt{5}$ and using three iterations of the Newton–Raphson procedure we expand the result in terms of ϵ, getting:

$$H^* = \frac{L}{\sqrt{5}} - \frac{\epsilon}{10\sqrt{5}\,L^3} - \frac{7\,\epsilon^2}{200\sqrt{5}\,L^7} + \left(\frac{7}{800\sqrt{5}\,L^5} + \frac{9\,c_{22}^2}{5\,\omega} + \frac{9\,s_{22}^2}{5\,\omega}\right)\frac{\epsilon^3}{L^6} + \mathcal{O}(\epsilon^4), \tag{13}$$

which corresponds to one of the two branches of $\Gamma_1 = 0$. Using $H_0 = -L/\sqrt{5}$, we get the other branch as:

$$H_* = -\frac{L}{\sqrt{5}} + \frac{\epsilon}{10\sqrt{5}\,L^3} + \frac{7\,\epsilon^2}{200\sqrt{5}\,L^7} + \left(-\frac{7}{800\sqrt{5}\,L^5} + \frac{9\,c_{22}^2}{5\,\omega} + \frac{9\,s_{22}^2}{5\,\omega}\right)\frac{\epsilon^3}{L^6} + \mathcal{O}(\epsilon^4), \tag{14}$$

When $\epsilon = 0$ we recover the values of H corresponding to the critical inclination value of H, while dropping powers of ϵ bigger than two we arrive at the expressions obtained by Coffey et al. (1986) and by Cushman (1988). Thus, the presence of the tesseral coefficients refines the curves obtained for the main problem (i.e., the Hamiltonian model in which the tesseral coefficients are taken zero). We stress that the influence of the terms c_{21} and s_{21} in H^* and H_* appears at ϵ^5. Given other initial conditions $H_0 \in [-L, L]$ no acceptable values of H are obtained in the sense that the series in ϵ do not converge.

Now the sequence of valid roots of P_1 for $\sigma_2 \in [|H|, L]$ can be obtained after solving $P_1 = 0$ for σ_2, using the Newton–Raphson procedure and giving initial values for σ_2. In particular, we arrive at the following description.

- If $H \in [-L, H_*]$, P_1 has no acceptable root, therefore there is no equilibrium point coming from P_1.
- If $H \in (H_*, 0)$, there is one equilibrium, say

$$E_3 = \left((L - |H|)^2\,,\ \sigma_{2*}\right),$$

such that σ_{2*} is computed recursively by means of the Newton–Raphson procedure, starting with $\sigma_{20} = -\sqrt{5}\,H$ and using three iterations. After expanding the result in powers of ϵ up to ϵ^3, we obtain:

$$\begin{aligned}\sigma_{2*} = {}& -\sqrt{5}\,H - \frac{(L^2 - 4\,H^2)\,\epsilon}{10\sqrt{5}\,L^2\,H^3} + \frac{(L^2 - 4\,H^2)\,(14\sqrt{5}\,L^2 - 30\,L\,H - 25\sqrt{5}\,H^2)\,\epsilon^2}{25000\,L^4\,H^7}\\ &+ \frac{1}{25000000\,\omega\,L^6\,H^{11}}\Big[\omega\,(L^2 - 4\,H^2)\,(353\sqrt{5}\,L^4 + 1680\,L^3\,H\\ &\qquad - 2665\sqrt{5}\,L^2\,H^2 - 3000\,L\,H^3 + 1650\sqrt{5}\,H^4)\\ &\qquad - 9000000\,(c_{22}^2 + s_{22}^2)\,L^3\,H^8\Big]\epsilon^3 + \mathcal{O}(\epsilon^4).\end{aligned} \tag{15}$$

The presence of c_{21} and s_{21} occurs at ϵ^5. As we are considering $H \in (H_*, 0)$ it might be close to zero. Then the value of σ_{2*} would have no sense for small H. To get a lower bound for H we balance in (15) the powers of H in the denominators with the powers of ϵ in the numerators. We get the bound $\bar{H} = -|\epsilon|^{3/11}$. If $H \in (H_*, \bar{H}]$ then (15) is still a power series in ϵ with positive powers of ϵ. If H is closer to zero then the above expression of σ_{2*} gets invalid and then another Newton–Raphson procedure should be implemented taking into account the smallness of H.

- If $H = 0$, there is no equilibrium point from P_1.
- If $H \in (0, H^*)$, there is one equilibrium,

$$E_3 = \Big((L - |H|)^2\,,\ \sigma_2{}^*\Big),$$

where $\sigma_2{}^*$ is obtained recursively starting with $\sigma_{20} = \sqrt{5}\, H$. The first terms of $\sigma_2{}^*$ are:

$$\sigma_2^* = \sqrt{5}\, H + \frac{(L^2 - 4\, H^2)\, \epsilon}{10\, \sqrt{5}\, L^2\, H^3} - \frac{(L^2 - 4\, H^2)\,(14\, \sqrt{5}\, L^2 + 30\, L\, H - 25\, \sqrt{5}\, H^2)\, \epsilon^2}{25000\, L^4\, H^7}$$
$$- \frac{1}{25000000\, \omega\, L^6\, H^{11}} \Big[\omega\,(L^2 - 4\, H^2)\,(353\, \sqrt{5}\, L^4 - 1680\, L^3\, H - 2665\, \sqrt{5}\, L^2\, H^2 + 3000\, L\, H^3 + 1650\, \sqrt{5}\, H^4) + 9000000\,(c_{22}^2 + s_{22}^2)\, L^3\, H^8\Big]\, \epsilon^3 + \mathcal{O}(\epsilon^4). \tag{16}$$

As before, the occurrence of c_{21} and s_{21} is at degree five in ϵ. Similarly as in the case $H \in (H_*, 0)$, the value of $\sigma_2{}^*$ would have no sense for small H. Balacing in (16) the powers of H in the denominators with the powers of ϵ in the numerators we obtain the bound $\bar{H} = |\epsilon|^{3/11}$. If H is closer to zero then the above expression of $\sigma_2{}^*$ gets invalid and the Newton–Raphson procedure should be applied taking into account that H is smaller than $|\epsilon|^{3/11}$.
- If $H \in [H^*, L]$, there is no stationary point coming from P_1.

5.1.2 Case (b)

When $\sigma_1\, \sigma_2^2 = (\sigma_2^2 - L\,|H|)^2$ then (9) is again a single-valued real function. Thus, its extrema are reached either at $\sigma_2 = |H|$, $\sigma_2 = L$ or at those points such that:

$$\frac{d\, \mathcal{S}((L - |H|)^2, \sigma_2)}{d\, \sigma_2} = \frac{3\, P_2(\sigma_2)}{128\, \omega\, L^6\, \sigma_2^{12}} = 0, \tag{17}$$

where

$$\begin{aligned} P_2(\sigma_2) = {} & 32\, \omega\, C_{20}\, L^3\, \sigma_2^8 - 5\, \omega\, C_{20}\, L\,(32\, L^2\, H^2 - 7\, C_{20})\, \sigma_2^6 \\ & + 24\, \omega\, C_{20}^2\, L^2\, \sigma_2^5 - 7\, \omega\, C_{20}^2\, L\,(7\, L^2 + 50\, H^2)\, \sigma_2^4 \\ & - 192\, H \Big[\omega\, C_{20}^2\, L^2\, H + 2\,(C_{21}^2 - 2\, C_{22}^2 + S_{21}^2 - 2\, S_{22}^2)\Big]\, \sigma_2^3 \\ & + 63\, \omega\, C_{20}^2\, L\, H^2\,(6\, L^2 + 5\, H^2)\, \sigma_2^2 \\ & + 120\, H^3 \Big[3\, \omega\, C_{20}^2\, L^2\, H + 8\,(C_{21}^2 - C_{22}^2 + S_{21}^2 - S_{22}^2)\Big]\, \sigma_2 \\ & + 55\, \omega\, C_{20}^2\, L^3\, H^4. \end{aligned}$$

As circular and equatorial trajectories are already equilibria, we focus on the analysis the possible roots of P_2 for $\sigma_2 \in [|H|, L]$. Changes in the number of roots can occur if P_2 has multiple roots in $(|H|, L)$ or when σ reaches the extremes $|H|$ or L.

Polynomial P_2 has degree eight in σ_2 and as P_1, it depends on eight parameters, so we introduce the new “harmonics" c_{ij}, s_{ij} and the discussion of the roots of P_2 becomes simpler.

We start by computing the resultant of P_2 and $d\,P_2/d\,\sigma_2$ with respect to σ_2, obtaining a polynomial relating the parameters c_{ij}, s_{ij}, ϵ, L, H, ω, which cannot vanish whenever $0 < -\epsilon \ll 1$ excepting for $H = 0$, a particular case that will be treated later. Similarly, we calculate $P_2(|H|)$, arriving at a polynomial which cannot be zero for any combination of the parameters if ϵ is restricted to be a small parameter. So we discard possible bifurcations arising from equatorial motions in the lower boundary of $\mathcal{U}_{L,H}$.

Next, we evaluate P_2 in L, replacing at the same time the zonal and tesseral coefficients using (11). We introduce Γ_2 as $P_2(L)$. It yields:

$$\Gamma_2 = 2\,L\,\epsilon\,\Big\{96\,H\Big[-(c_{21}^2+s_{21}^2)\,(2\,L^2-5\,H^2)\,\epsilon^5+(c_{22}^2+s_{22}^2)\,(4\,L^2-5\,H^2)\,\epsilon^3\Big] + \omega\,L^2\,(5\,L^4-82\,L^2\,H^2+365\,H^4)\,\epsilon+16\,\omega\,L^8\,(L^2-5\,H^2)\Big\}. \quad (18)$$

Our aim is to get an explicitly approximation of H in terms of the other parameters from the resolution of $\Gamma_2 = 0$. We can achieve it by means of the Newton–Raphson algorithm. Starting with $H_0 = L/\sqrt{5}$ and using three iterations of the Newton–Raphson procedure, we expand the resulting expression in terms of ϵ, getting one branch of $\Gamma_2 = 0$:

$$H^{\&} = \frac{L}{\sqrt{5}} + \frac{\epsilon}{10\,\sqrt{5}\,L^3} + \frac{3\,\epsilon^2}{40\,\sqrt{5}\,L^7} + \left(\frac{299}{4000\,\sqrt{5}\,L^5} + \frac{9\,c_{22}^2}{5\,\omega} + \frac{9\,s_{22}^2}{5\,\omega}\right)\frac{\epsilon^3}{L^6} + \mathcal{O}(\epsilon^4), \quad (19)$$

and using $H_0 = -L/\sqrt{5}$ we obtain the other valid branch of $\Gamma_2 = 0$:

$$H_{\&} = -\frac{L}{\sqrt{5}} - \frac{\epsilon}{10\,\sqrt{5}\,L^3} - \frac{3\,\epsilon^2}{40\,\sqrt{5}\,L^7} + \left(-\frac{299}{4000\,\sqrt{5}\,L^5} + \frac{9\,c_{22}^2}{5\,\omega} + \frac{9\,s_{22}^2}{5\,\omega}\right)\frac{\epsilon^3}{L^6} + \mathcal{O}(\epsilon^4). \quad (20)$$

The influence of the terms c_{21} and s_{21} in H^* and H_* appears for the first time through terms factored by ϵ^5. When $\epsilon = 0$ we recover again the values of H corresponding to the critical inclination. Furthermore, if we drop the powers of ϵ bigger than two we get the expressions of Coffey et al. (1986) and Cushman (1988). For other initial conditions of $H_0 \in [-L, L]$ the corresponding series in ϵ do not converge.

The sequence of valid roots of P_2 for $\sigma_2 \in [|H|, L]$ is obtained after solving the equation $P_2 = 0$ for σ_2 using the Newton–Raphson procedure with different initial values for σ_2. We arrive at:

- If $H \in [-L, H_{\&}]$, there is no equilibrium point related with P_2.
- If $H \in (H_{\&}, 0)$ there is one equilibrium,

$$E_4 = \Big(\frac{(\sigma_{2\&}^2 - L\,|H|)^2}{\sigma_{2\&}^2},\ \sigma_{2\&}\Big),$$

where $\sigma_{2\&}$ is calculated using the Newton–Raphson algorithm with three steps, starting with $\sigma_{20} = -\sqrt{5}\,H$. The result, expanded in powers of ϵ, is given by:

$$\begin{aligned}\sigma_{2\&} = -\sqrt{5}\,H + \frac{(9\,L^2{-}35\,H^2)\epsilon}{100\,\sqrt{5}\,L^2\,H^3} + \frac{(9\,L^2{-}35\,H^2)(29\,\sqrt{5}\,L^2{+}12\,L\,H{-}63\,\sqrt{5}\,H^2)\,\epsilon^2}{100000\,L^4\,H^7}\\ + \frac{1}{500000000\,\omega\,L^6\,H^{11}} \times \Big[\omega\,(9\,L^2 - 35\,H^2)\,(29701\,\sqrt{5}\,L^4 + 17400\,L^3\,H\\ - 131930\sqrt{5}L^2\,H^2{-}37800\,L\,H^3{+}128625\,\sqrt{5}\,H^4)\\ - 180000000\,(c_{22}^2{+}s_{22}^2)L^3\,H^8\Big]\epsilon^3{+}\mathcal{O}(\epsilon^4). \qquad (21)\end{aligned}$$

The presence of c_{21} and s_{21} occurs at ϵ^5. Since $H \in (H_\&, 0)$, it can be close to zero, therefore the value of $\sigma_{2\&}$ would have no sense for small H and we need a lower bound for H. Thus we balance in (21) the powers of H in the denominators with the powers of ϵ in the numerators, getting the bound $\bar{H} = -|\epsilon|^{3/11}$. If $H \in (H_\&, \bar{H}]$ then (15) is still a power series in ϵ with positive powers of ϵ. If $H \in (\bar{H}, 0)$ then the above expression of $\sigma_{2\&}$ gets invalid and then another Newton–Raphson procedure should be implemented assuming that H is also a small parameter.

- If $H = 0$ there is no equilibrium point coming from P_2.
- If $H \in (0, H^\&)$, there is one equilibrium,

$$E_4 = \Big(\frac{(\sigma_2{}^{\&2} - L\,|H|)^2}{\sigma_2{}^{\&2}}\,,\ \sigma_2{}^\&\Big).$$

The value of $\sigma_2{}^\&$ is obtained recursively from $\sigma_{20} = \sqrt{5}\,H$ by means of the Newton–Raphson algorithm through three steps. The result, after expansion in powers of ϵ, is given through terms factored by ϵ^3:

$$\begin{aligned}\sigma_2^\& = \sqrt{5}\,H - \frac{(9\,L^2 - 35\,H^2)\,\epsilon}{100\,\sqrt{5}\,L^2\,H^3} - \frac{(9\,L^2{-}35\,H^2)\,(29\,\sqrt{5}\,L^2 - 12\,L\,H{-}63\,\sqrt{5}\,H^2)\,\epsilon^2}{100000\,L^4\,H^7}\\ + \frac{1}{500000000\,\omega\,L^6\,H^{11}}\Big[\omega\,(9\,L^2 - 35\,H^2)\,(-29701\,\sqrt{5}\,L^4 + 17400\,L^3\,H\\ + 131930\,\sqrt{5}\,L^2\,H^2 - 37800\,L\,H^3 - 128625\,\sqrt{5}\,H^4)\\ - 180000000\,(c_{22}^2 + s_{22}^2)\,L^3\,H^8\Big]\,\epsilon^3 + \mathcal{O}(\epsilon^4). \qquad (22)\end{aligned}$$

The appearance of c_{21} and s_{21} starts at degree five in ϵ. As in the case $H \in (H_\&, 0)$, the value of $\sigma_2{}^\&$ would have no sense for small H. Balacing in (22) the powers of H in the denominators with the powers of ϵ in the numerators we obtain the bound $\bar{H} = |\epsilon|^{3/11}$. If H is closer to zero then the above expression of $\sigma_2{}^\&$ gets invalid and the Newton–Raphson procedure should be applied taking into account that H is smaller than $|\epsilon|^{3/11}$.

- If $H \in [H^\&, L]$ there is no stationary point related to P_2.

We stress that according to (13) and (19), the branch of $\Gamma_2 = 0$ corresponding to positive values of H is located over the line $L = \sqrt{5}\,H$ whereas the branch of $\Gamma_1 = 0$ for $H > 0$ is placed below $L = \sqrt{5}\,H$. On the other hand, by inspection of (14) and (20), we conclude that the branch of $\Gamma_2 = 0$ for $H < 0$ is below the line $L = -\sqrt{5}\,H$ while the corresponding branch of $\Gamma_1 = 0$ goes over the line $L = -\sqrt{5}\,H$. However $\Gamma_1 = 0$ never touches $\Gamma_2 = 0$

provided that $0 < -\epsilon \ll 1$ and the other values of c_{ij}, s_{ij}, L, H and ω remain in interval of physical significance. Thus, we conclude that the chain of inequalities

$$-L < H_* < -L/\sqrt{5} < H_\& < 0 < H^\& < L/\sqrt{5} < H^* < L$$

holds along our study.

We observe that for the four roots of σ_2, that is, for σ_{2*}, $\sigma_2{}^*$, $\sigma_{2\&}$ and $\sigma_2{}^\&$ we calculate the eccentricities $\sqrt{1-\sigma_2^2/L^2}$ and the corresponding inclinations given by $\cos(I) = H/\sigma_2$, obtaining in all cases expressions of the form

$$e = \frac{\sqrt{|L^2 - 5\,H^2|}}{L} + \mathcal{O}(\epsilon), \quad |\cos(I)| = \frac{1}{\sqrt{5}} + \mathcal{O}(\epsilon)$$

which means that the orbits related to the relative equilibria E_3 and E_4 have inclinations close to the critical inclination, i.e., $\arccos(1/\sqrt{5})$, and eccentricities which tend to zero as soon as H approaches either H_*, H^*, $H_\&$ or $H^\&$.

In summary, we can conclude the following:

- the relative equilibrum E_1 corresponding to equatorial orbits appears for $H \in [-L, L] \setminus \{0\}$;
- the relative equilibrium E_2 corresponding to circular orbits is present for all $H \in [-L, L]$;
- if $H \in [-L, H_*]$ or $H \in [H^*, L]$, there are no more equilibria;
- if $H \in (H_*, H^*) \setminus \{0\}$, the point E_3 is an equilibrium;
- if $H \in (H_\&, H^\&) \setminus \{0\}$, the point E_4 is an equilibrium.

5.2 Non-linear stability

In order to analyse the stability of the relative equilibria it is convenient to make a change of co-ordinates so that the resulting Hamiltonian may be written in a phase space well suited to apply the techniques of non-linear analysis based on Morse functions and index theory. Thus, we use a set of three co-ordinates $\{\pi_1, \pi_2, \pi_3\}$ which define the two-dimensional space:

$$\mathcal{X}_{L,H} = \left\{(\pi_1, \pi_2, \pi_3) \in \boldsymbol{R}^3 \mid \pi_2^2 + \pi_3^2 = [(L+\pi_1)^2 - H^2]\,[(L-\pi_1)^2 - H^2]\right\}, \quad (23)$$

where $0 \le |H| \le L$ and $L > 0$. Note that π_2 and π_3 lie in the interval $[H^2 - L^2, L^2 - H^2]$ whereas π_1 is in $[|H| - L, L - |H|]$. The explicit relationship between the π_i's and the Delaunay elements g and G is:

$$G^2 = \frac{1}{2}\,(L^2 + H^2 - \pi_1^2 + \pi_3),$$

$$\cos(g) = \frac{-\pi_2}{\sqrt{(L^2 - H^2)^2 - (\pi_1^2 - \pi_3)^2}},$$

$$\sin(g) = \pi_1 \sqrt{\frac{2\,(L^2 + H^2 - \pi_1^2 + \pi_3)}{(L^2 - H^2)^2 - (\pi_1^2 - \pi_3)^2}}.$$

Cushman (1983) proved that $\{\pi_1, \pi_2, \pi_3\}$ are the appropriate variables that should be used for the reduction process of a Keplerian Hamiltonian for which L and H are integrals of motion and that $\mathcal{X}_{L,H}$ is its corresponding phase space. Moreover, in Cushman (1983,1992) it is proven that whether $0 < |H| < L$,$\mathcal{X}_{L,H}$ is diffeomorphic to a two-sphere S^2 and therefore the reduction is regular in that region of the phase space. However, when $H = 0$ then $\mathcal{X}_{L,0}$ is a topological two sphere with two singular points at the vertices $(\pm L, 0, 0)$. Finally, when$|H| = L$ the phase space $\mathcal{X}_{\pm L,L}$ gets reduced to a point.

The relationship between the π_i's and the σ_i's is given through:

$$\sigma_1 = (L - |H|)^2 - \pi_1^2, \qquad \sigma_2 = \frac{\sqrt{L^2 + H^2 - \pi_1^2 + \pi_3}}{\sqrt{2}} \tag{24}$$

and

$$\pi_1 = \pm\sqrt{(L - |H|)^2 - \sigma_1}, \qquad \pi_3 = -\sigma_1 + 2\,\sigma_2^2 - 2\,L\,|H|. \tag{25}$$

From (25) and the constraint of (23), it is readily deduced that a single point in the interior of $\mathcal{U}_{L,0}$ or of $\mathcal{U}_{L,H}$ is in correspondence with four points in the space $\mathcal{X}_{L,0}$ or in $\mathcal{X}_{L,H}$, respectively. Besides, a single point in the regular part of the boundaries of either $\mathcal{U}_{L,0}$ or $\mathcal{U}_{L,H}$ is related to two points of $\mathcal{X}_{L,0}$ or of $\mathcal{X}_{L,H}$. In addition, to each of the two singular points of the boundary of $\mathcal{U}_{L,H}$, it corresponds one point of $\mathcal{X}_{L,H}$. Finally, the points of $\mathcal{U}_{L,0}$ with co-ordinates $(L^2, 0)$ and (L^2, L) are related, respectively, with the points $(0, 0, -L^2)$ and $(0, 0, L^2)$ on $\mathcal{X}_{L,0}$ whereas the point whose co-ordinate is $(0, 0)$ in $\mathcal{U}_{L,0}$ corresponds to the singular points $(\pm L, 0, 0)$ of $\mathcal{X}_{L,0}$. Thus the number of equilibria of $\mathcal{S}$ remains the same in $\mathcal{U}_{L,H}$ if only equatorial and circular orbits are present while each equilibrium point in the regular part of the boundary of $\mathcal{U}_{L,H}$ is doubled when considering the reduced Hamiltonian in $\mathcal{X}_{L,H}$, as we shall see later on.

Next, Hamiltonian $\mathcal{S}$ written in terms of the new co-ordinates yields:

$$\begin{aligned}\Pi = \frac{1}{1024\,\omega\,L^6\,\pi_4^{11}} \times \Big\{ &- 16\,\omega\,C_{20}\,L^3\,(\pi_1^2 - \pi_3 - L^2 - H^2)^3\,(\pi_1^2 - \pi_3 - L^2 + 5\,H^2) \\ &+ 24\,\omega\,C_{20}^2\,L\Big[-65\,L^2\,H^4 - 36\,L\,H^4\,\pi_4 - H^2\,(60\,\pi_1^2 - 22\,L^2 - 25\,H^2)\,\pi_4^2 \\ &\qquad + 24\,L\,H^2\,\pi_4^3 + (4\,\pi_1^2 + 3\,L^2 - 14\,H^2)\,\pi_4^4 - 4\,L\,\pi_4^5 - 3\,\pi_4^6\Big] \\ &- 576\,(C_{21}^2 + S_{21}^2)\,H\,(\pi_1^2 - \pi_3 - L^2 + 3\,H^2)\,\pi_4 \\ &+ 1152\,(C_{22}^2 + S_{22}^2)\,H\,(\pi_1^2 - \pi_3 - L^2 + H^2)\,\pi_4\Big\},\end{aligned} \tag{26}$$

where we have introduced $\pi_4 = \sqrt{(L^2 + H^2 - \pi_1^2 + \pi_3)/2}$.

We remark that $\mathcal{X}_{L,H}$ is the reduced phase space corresponding to Hamiltonian (6) if one considers only the continuous integrals introduced through the Lie transformations but not the discrete symmetries of the problem. However, we have preferred to apply all possible symmetries of the normalised problem at once, simplifying it as much as possible in order to study its dynamics and make use of the expression of the Hamiltonian in the π_i's only in the discussion concerning the stability of the system.

For the non-linear stability analysis we use the Lagrange multipliers technique. First, we define $\mathcal{F}$ as follows:

$$\mathcal{F}(\pi_1, \pi_2, \pi_3) = \Pi(\pi_1, \pi_2, \pi_3) + \lambda\left\{\pi_2^2 + \pi_3^2 - [(L + \pi_1)^2 - H^2]\,[(L - \pi_1)^2 - H^2]\right\}.$$

Then we calculate the Hessian matrix associated to $\mathcal{F}$ and evaluate it at each critical point. Note that each critical point of $\mathcal{F}$ is equivalent to an equilibrium of Π. We consider an arbitrary vector (h_1, h_2, h_3) in the tangent space to $\mathcal{X}_{L,H}$ at the critical point, i.e., at a point $(\pi_1^0, \pi_2^0, \pi_3^0)$ satisfying $h_1\,\pi_1^0 + h_2\,\pi_2^0 + h_3\,\pi_3^0 = 0$. Next, we calculate the bilinear form associated to the Hessian matrix applied on a tangent vector at the critical point. This bilinear form yields $\alpha_1\,h_1^2 + \alpha_2\,h_2^2 + \alpha_3\,h_3^2$, where one of the coefficients is zero and the other two are non-null. The signs of the non-null coefficients determines the linear stability of the critical points. Moreover, the non-linear stability follows immediately from Morse Lemma (Abraham and Marsden 1978, Verhulst 1996) in case of non-degeneracy, that is, if two of the three α_i's do not vanish at the same time.

We begin by analysing the circular equilibrium, which in $\mathcal{U}_{L,H}$ is given through E_2 whereas in $\mathcal{X}_{L,H}$ it is located at $(0, 0, L^2 - H^2)$. We also calculate the appropriate value of λ so that $(0, 0, L^2 - H^2)$ becomes a critical point of $\mathcal{F}$. After some algebra, the computation of the bilinear form yields:

$$\beta(h_1, h_2) = -\frac{3\,\Gamma_2}{128\,\omega\,L^{17}\,(L^2 - H^2)}\,h_1^2 - \frac{3\,\Gamma_1}{512\,\omega\,L^{19}\,(L^2 - H^2)}\,h_2^2, \tag{27}$$

which is well defined if $|H| < L$. The cases $L = H$ and $L = -H$ are excluded since the corresponding phase space reduces to a unique point, consequently circular and equatorial motions coincide at this point, so $L = |H|$ are bifurcation lines.

Thus, supposing that the coefficients of h_1 and h_2 do not vanish at the same time, we might conclude that the equilibrium related to circular orbits is stable (centre) in the sense of Lyapunov if these coefficients have the same sign. (The Lyapunov stability of the relative equilibria in the reduced system is translated into the orbital stability of the corresponding periodic and quasiperiodic orbits in the original truncated Hamiltonian.) If the coefficients have opposite signs the conclusion is that the point accounting for circular motions is a saddle and hence, linearly and non-linearly unstable.

If $H \in (-L, H_*)$ then $\Gamma_1, \Gamma_2 > 0$ whenever $0 < -\epsilon \ll 1$, and so the bilinear form (27) is given by $\alpha_1\,h_1^2 + \alpha_2\,h_2^2$ with $\alpha_1, \alpha_2 < 0$, thus $(0, 0, L^2 - H^2)$ is a non-linearly stable point (elliptic). When $H \in (H_*, H_\&)$, Γ_1 changes sign while Γ_2 remains positive, so the bilinear form $\alpha_1\,h_1^2 + \alpha_2\,h_2^2$ is such that $\alpha_1 < 0 < \alpha_2$ and $(0, 0, L^2 - H^2)$ is an unstable point (saddle). Next, when H belongs to $(H_\&, H^\&)$, Γ_2 becomes negative whereas Γ_1 remains negative, therefore the circular equilibrium is of elliptic character (non-linear centre) since $\alpha_1, \alpha_2 > 0$. If $H \in (H^\&, H^*)$, Γ_1 is still negative but Γ_2 becomes positive, hence $(0, 0, L^2 - H^2)$ is unstable (a saddle). If H is in (H^*, L), Γ_2 is positive and Γ_1 becomes also positive, thus the circular point is again a non-linear centre.

The critical cases, $H \in \{H_*, H_\&, H^\&, H^*\}$, deserve a special treatment. We follow the approach of Cushman (1988).

As $\Gamma_1 = 0$ and $\Gamma_2 = 0$ do not intersect each other, the two coefficients of (27) cannot be zero at the same time. After writing π_3 in terms of π_1 and π_2 (we use the positive value of π_3 but we might use its negative counterpart, arriving at the same expressions), we calculate a Taylor series expansion of Π up to powers of degree four in π_1 and π_2, around $(0, 0)$ (i.e., the projection corresponding to the stationary point $(0, 0, L^2 - H^2)$). This 4-jet can be written as:

$$\Pi^{4-\mathrm{jet}}(\pi_1, \pi_2) = a_0\,\pi_1^2 + a_1\,\pi_2^2 + a_2\,\pi_1^4 + a_3\,\pi_1^2\,\pi_2^2 + a_4\,\pi_2^4, \tag{28}$$

where the coefficients a_i depend upon the parameters of the problem. In particular:

$$a_0 = -\frac{3\,\Gamma_2}{256\,\omega\,L^{17}\,(L^2 - H^2)}, \qquad a_1 = -\frac{3\,\Gamma_1}{1024\,\omega\,L^{19}\,(L^2 - H^2)}.$$

Whenever $H = H_\&$ or $H = H^\&$, a_0 vanishes but $a_1 \neq 0$, while if $H = H_*$ or $H = H^*$, $a_1 = 0$ and $a_0 \neq 0$.

When $H \in \{H_{\&}, H^{\&}\}$, we introduce ρ_1 and ρ_2 such that:

$$\rho_1 = \pi_1, \qquad \rho_2 = \pi_2 \sqrt{1 + \frac{a_3}{a_1}\,\pi_1^2 + \frac{a_4}{a_1}\,\pi_2^2}$$

and choose its inverse change as:

$$\pi_1 = \rho_1, \qquad \pi_2 = \sqrt{-\frac{a_1}{2\,a_4} - \frac{a_3}{2\,a_4}\,\rho_1^2 + \frac{\sqrt{(a_1 + a_3\,\rho_1^2)^2 + 4\,a_1\,a_4\,\rho_2^2}}{2\,a_4}}$$

then (28) is rewritten in terms of the ρ_i's obtaining $\Pi^{4-\mathrm{jet}}(\rho_1, \rho_2) = a_2\,\rho_1^4 + a_1\,\rho_2^2$.

Replacing H by $H_{\&}$ or by $H^{\&}$ and expanding $\Pi^{4-\mathrm{jet}}$ in series of ϵ up to powers of degree three, we arrive at:

$$\Pi^{4-\mathrm{jet}}(\rho_1, \rho_2) = \left[-\frac{3\,\epsilon\,(431\,\epsilon^2 + 640\,\epsilon\,L^4 - 400\,L^8)}{4096\,L^{18}} + \mathcal{O}(\epsilon^4)\right]\rho_1^4 + \left[\frac{3\,\epsilon^2\,(3\,\epsilon + 10\,L^4)}{640\,L^{18}} + \mathcal{O}(\epsilon^4)\right]\rho_2^2.$$

The coefficient of ρ_1^4 is negative and that of ρ_2^2 is positive, provided that $0 < -\epsilon \ll 1$ and $L > 0$. Hence, because of the different signs of ρ_1^4 and ρ_2^2, the equilibrium point is a non-linear saddle and we may conclude that $(0, 0, L^2 - H^2)$ (and subsequently E_2) corresponds to an unstable point in the degenerate cases where H is either $H_{\&}$ or $H^{\&}$. It agrees with Cushman's result for the degenerate cases of circular orbits (Cushman 1988) when H is a root of $\Gamma_2 = 0$; thus the effect of adding the tesseral coefficients does not alter the stability character.

For the cases $H \in \{H_*, H^*\}$ we make a local change of co-ordinates introducing ρ_1 and ρ_2 as follows:

$$\rho_1 = \pi_1 \sqrt{1 + \frac{a_4}{a_1}\,\pi_1^2 + \frac{a_3}{a_1}\,\pi_2^2}, \qquad \rho_2 = \pi_2,$$

and its inverse change defined through:

$$\pi_1 = \sqrt{-\frac{a_1}{2\,a_4} - \frac{a_3}{2\,a_4}\,\rho_2^2 + \frac{\sqrt{4\,a_1\,a_4\,\rho_1^2 + (a_1 + a_3\,\rho_2^2)^2}}{2\,a_4}}, \qquad \pi_2 = \rho_2,$$

then (28) is rewritten in terms of the ρ_i's as $\Pi^{4-\mathrm{jet}}(\rho_1, \rho_2) = a_1\,\rho_1^2 + a_2\,\rho_2^4$.

Substituting H by H_* or by H^* and expanding $\Pi^{4-\mathrm{jet}}$ in series of ϵ up to powers of degree three, we get:

$$\Pi^{4-\mathrm{jet}}(\rho_1, \rho_2) = \left[\frac{3\,\epsilon^2\,(3\,\epsilon - 10\,L^4)}{160\,L^{16}} + \mathcal{O}(\epsilon^4)\right]\rho_1^2 + \left[\frac{3\,\epsilon\,(189\,\epsilon^2 + 240\,\epsilon\,L^4 + 400\,L^8)}{65536\,L^{22}} + \mathcal{O}(\epsilon^4)\right]\rho_2^4.$$

It is readily deduced that the coefficients of ρ_1^2 and ρ_2^4 are both negative, provided that $0 < -\epsilon \ll 1$. Thus, because of the equal signs of ρ_1^2 and ρ_2^4 the stationary point behaves like a non-linear centre, hence $(0, 0, L^2 - H^2)$ (and E_2) corresponds to a non-linearly stable point in the degenerate cases where $H \in \{H_*, H^*\}$, a result which is in agreement with Cushman's treatment of the degenerate points for the main problem Cushman (1988) when H is one of the two valid roots of $\Gamma_1 = 0$. As before, the inclusion of the tesseral harmonics does not change the stability of these degenerate situations.

We study now the non-linear stability of the equatorial equilibrium, which in $\mathcal{U}_{L,H}$ is given through E_1 whereas in $\mathcal{X}_{L,H}$ it is located at $(0, 0, H^2 - L^2)$. We exclude the situations $L = |H|$ as they correspond to bifurcation lines and then the phase space gets reduced to a point where circular and equatorial motions coincide. We also discard the line $H = 0$ because for equatorial orbits, $G = |H|$ and $H = 0$ means that $G = 0$, so the trajectories are rectilinear and the satellite collapses with the planet. Then, we restrict ourselves to $0 < |H| < L$.

Proceeding as in the circular case, we compute the bilinear form, getting an expression of the type $A_1 h_1^2 + A_2 h_2^2$, where A_1 and A_2 are functions which depend on all parameters of the problem. Expanding them in power series of ϵ up to ϵ^3, one gets:

$$
\begin{aligned}
A_1 &= \frac{3\,\epsilon\,[3\,\epsilon\,(2\,L\,|H| + H^2) - 2\,L\,H^4\,|H|]}{2\,L^4\,H^8\,(L^2 - H^2)} + \mathcal{O}(\epsilon^4),\\
A_2 &= \frac{3\,\epsilon\,[\epsilon\,(31\,L^2\,|H| + 12\,L\,H^2 - 7\,H^2\,|H|) - 8\,L^2\,H^4\,|H|]}{32\,L^5\,H^{10}\,(L^2 - H^2)} + \mathcal{O}(\epsilon^4).
\end{aligned}
\tag{29}
$$

Thence, if $0 < -\epsilon \ll 1$ and $0 < |H| < L$, A_1 and A_2 are positive quantities which implies that $(0, 0, H^2 - L^2)$ (and therefore E_1) is non-linearly stable (centre), after application of Morse Lemma to the Morse function $A_1 h_1^2 + A_2 h_2^2$ plus higher order terms in h_1 and h_2.

It is time of analysing the non-linear stability of E_3 and E_4. Starting with E_3, we take first the case $H \in (H_*, 0)$, therefore σ_2 is given through (15). The two corresponding points of E_3 in $\mathcal{X}_{L,H}$ are given by the co-ordinates $(0, \pm\bar{\pi}_2, \bar{\pi}_3)$ where $\bar{\pi}_3$ is obtained from (25) after replacing σ_2 by (15) and $\bar{\pi}_2$ is calculated using the constraint of (23). Note that $\bar{\pi}_2$ and $\bar{\pi}_3$ belong to $[H^2 - L^2, L^2 - H^2]$. The computation of the bilinear form is similar to that of the circular and equatorial motions, but we need to replace the asymptotic expression of (15). We get a rather big expression since it depends on $\bar{\pi}_2$, $\bar{\pi}_3$ and all the parameters involved the problem. After expanding it in powers of ϵ up to degree two we arrive at:

$$
\beta(h_1, h_2) = \left[-\frac{3\,\epsilon^2\,|H|}{1000\,\sqrt{5}\,L^5\,H^8} + \mathcal{O}(\epsilon^3)\right] h_1^2 + \left[\frac{3\,\epsilon\,|H|}{4000\,\sqrt{5}\,L^3\,H^8} + \mathcal{O}(\epsilon^2)\right] h_3^2. \tag{30}
$$

This formula is valid for $(0, \bar{\pi}_2, \bar{\pi}_3)$ and for $(0, -\bar{\pi}_2, \bar{\pi}_3)$. The sign of the term factored by h_1^2 is negative and the one corresponding to h_3^2 is also negative. Both conditions imply that $(0, \pm\bar{\pi}_2, \bar{\pi}_3)$ and E_3 are non-linear stable points (non-linear centres).

If $H \in (0, H^*)$ we follow similar steps to the previous paragraph, but replacing σ_2 by its formula (16). We arrive at the same expression (30) for the bilinear form. Thus, $(0, \pm\bar{\pi}_2, \bar{\pi}_3)$ and E_3 are non-linear stable points.

Now we study E_4 with H in $(H_\&, 0)$ or in $(0, H^\&)$. The value of σ_2 is given by (21) or by (22) depending if H is negative or positive. The two points in $\mathcal{X}_{L,H}$ related with E_4 have co-ordinates $(\pm\bar{\pi}_1, 0, \bar{\pi}_3)$ where $\bar{\pi}_3$ is obtained from (25) after replacing σ_2 by (21) for $H \in (H_\&, 0)$ and by (22) if $H \in (0, H^\&)$. The co-ordinate $\bar{\pi}_1$ is calculated using the constraint of (23). We compute the bilinear form using either formula (21) or formula (22). We obtain a quite cumbersome expression because it is a function of $\bar{\pi}_1$, $\bar{\pi}_3$ and all the parameters of the problem. After expanding it in powers of ϵ up to degree two, it results in:

$$\beta(h_1, h_2) = \left[\frac{3\,\epsilon^2}{20000\,\sqrt{5}\,L^5\,H^8\,|H|} + \mathcal{O}(\epsilon^3)\right] h_2^2 + \left[\frac{6\,\epsilon\,(L^2 - 5\,H^2)}{\sqrt{5}\,L^3\,(L^2 - 25\,H^2)^2\,H^2\,|H|} + \mathcal{O}(\epsilon^2)\right] h_3^2. \tag{31}$$

This expansion is valid for positive and negative values of H. Note that the factor $L^2 - 25\,H^2$ cannot vanish for H in $(H_{\&}, 0) \cup (0, H^{\&})$. Moreover, $L^2 - 5\,H^2$ is very close to zero but it is a positive quantity as $H^{\&} < L/\sqrt{5}$ and $H_{\&} > -L/\sqrt{5}$. Thus, the coefficient of h_2^2 is positive whereas that of h_3^2 is negative for all admissible values of H. Both conditions imply that $(\pm\bar{\pi}_1, 0, \bar{\pi}_3)$ and E_4 are unstable points (non-linear saddles).

5.3 Bifurcation lines

Once the non-linear stability of the stationary points has been established with detail, the sequence of bifurcation lines becomes clear.

Starting with $H = -L$, the phase spaces $\mathcal{U}_{L,H}$ and $\mathcal{X}_{L,H}$ become a point. So, a unique equilibrium, corresponding to circular and equatorial motions, is present in the equations, that is, E_1 and E_2 coincide. When one moves H increasing its value, $-L < H \leq H_* < 0$, then both E_1 and E_2 exist and are Lyapunov stable equilibria. Thus, $\Gamma_3 \equiv H = -L$ is the first bifurcation line.

When $H = H_*$, it corresponds with the branch of $\Gamma_1 = 0$ near $H = -L/\sqrt{5}$, then the point $(0, 0, L^2 - H^2)$ (and E_2) becomes degenerate but is still non-linearly stable. However, once H passes the value H_*, then $(0, 0, L^2 - H^2)$ bifurcates into three points: $(0, 0, L^2 - H^2)$ and the two equilibria of Π corresponding to E_3, the points $(0, \pm\bar{\pi}_2, \bar{\pi}_3)$. Besides E_2 changes its stability and becomes unstable whereas $(0, \pm\bar{\pi}_2, \bar{\pi}_3)$ (and E_3) are stable. On the other hand, E_1 (and $(0, 0, H^2 - L^2)$) remains stable. So, if $H \in (H_*, H_{\&})$, Π has four relative equilibria. We may conclude that the line $H = H_{\&}$ is a pitchfork bifurcation of circular trajectories.

If one keeps on increasing H, reaching its next critical value corresponding with the branch of $\Gamma_2 = 0$ near $H = -L/\sqrt{5}$, that is $H = H_{\&}$, then $(0, 0, L^2 - H^2)$ (and E_2) becomes degenerate but still non-linearly unstable. However, once H passes the value $H_{\&}$, $(0, 0, L^2 - H^2)$ bifurcates into three points, namely $(0, 0, L^2 - H^2)$ and the two points of Π corresponding to E_4: $(\pm\bar{\pi}_1, 0, \bar{\pi}_3)$. Moreover E_2 changes its stability again and becomes stable while $(\pm\bar{\pi}_1, 0, \bar{\pi}_3)$ (and E_4) are unstable. The points $(0, \pm\bar{\pi}_2, \bar{\pi}_3)$ (and E_3) and E_1 remain all stable. Thus, when $H \in (H_{\&}, 0)$, Π has six relative equilibria. We deduce that the line $H = H_{\&}$ is also a pitchfork bifurcation of the circular equilibrium.

In the straight line $H = 0$, then $(0, 0, L^2 - H^2)$ is the only equilibrium and it keeps its stability, that is, it remains as a non-linear centre. On this occasion, $(0, 0, H^2 - L^2)$ is discarded because Π becomes singular at $G = |H| = 0$. Hence, $\Gamma_4 \equiv H = 0$ is a new bifurcation line.

Once H becomes positive the sequence of bifurcations reverts although the system is not symmetric with respect to $H = 0$. If $H > 0$ (but still $H < H^{\&}$) the other five equilibria appear keeping the same stability character as in $(H_{\&}, 0)$.

In the line $H = H^{\&}$, i.e., the branch of $\Gamma_2 = 0$ near $H = L/\sqrt{5}$, then $(0, 0, L^2 - H^2)$ is degenerate and together with $(\pm\bar{\pi}_1, 0, \bar{\pi}_3)$ collide into $(0, 0, L^2 - H^2)$. Then $(0, 0, L^2 - H^2)$ (and E_2) becomes unstable whereas the rest of equilibria, i.e., the points $(0, 0, H^2 - L^2)$,

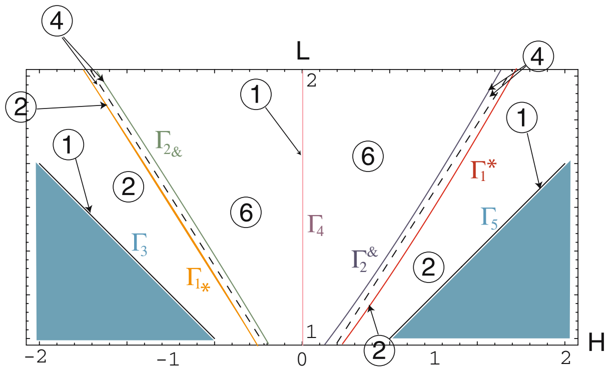

Fig. 2 Bifurcation diagram corresponding to $\mathcal{X}_{L,H}(0 \le |H| \le L)$, with the number of relative equilibria in each region encircled. The action L varies from 1 to 2 so that collisions with the planet are avoided. Dashed lines correspond to the critical inclination values for the main problem of the artificial satellite, i.e., they are the lines $L = \sqrt{5}\,|H|$. We stress that the curves Γ_{1*}, $\Gamma_1{}^*$, $\Gamma_{2\&}$, and $\Gamma_2{}^{\&}$ are not straight lines

$(0, \pm\bar{\pi}_2, \bar{\pi}_3)$ (and E_3) exist with the same stability behaviour. Hence, $H = H^{\&}$ is a curve representing a pitchfork bifurcation of the circular equilibrium. The other three equilibria remain untouched with the same stability for all $H \in (H^{\&}, H^*)$.

Increasing H, it reaches the value H^* (corresponding with the branch of $\Gamma_1 = 0$ near $H = L/\sqrt{5}$) then $(0, 0, L^2 - H^2)$ together with $(0, \pm\bar{\pi}_2, \bar{\pi}_3)$ collide into $(0, 0, L^2 - H^2)$. Besides $(0, 0, L^2 - H^2)$ (and E_2) becomes stable whereas the other point E_1 keeps on being stable. Hence, $H = H^*$ is a pitchfork bifurcation of the circular equilibrium. Now, if H increases a bit and leaves H^*, the two equilibria of Π are $(0, 0, L^2 - H^2)$ and $(0, 0, H^2 - L^2)$.

Finally, when $H = L$, the phase spaces $\mathcal{U}_{L,H}$ and $\mathcal{X}_{L,H}$ become a point, that is, $(0, 0, L^2 - H^2)$ and $(0, 0, H^2 - L^2)$ are the same (also E_1 and E_2 coincide). So, $\Gamma_5 \equiv H = L$ is the last bifurcation line.

We remark that the equatorial and circular orbits (i.e., the ones corresponding with the lines $|H| = L$) could be studied in the reduced phase space $S_L \times S_L$ and there the two relative equilibria are always stable points, see for instance (Iñarrea et al. 2004, 2006).

We have depicted in Fig. 2 the bifurcation lines as well as the number of relative equilibria in each region. We have identified the lines Γ_{1*} with the curve $H = H_*$, $\Gamma_1{}^*$ with the curve $H = H^*$, $\Gamma_{2\&}$ with $H = H_{\&}$ and $\Gamma_2{}^{\&}$ with $H = H^{\&}$. The Euler characteristic of $\mathcal{X}_{L,H}$ (sum of the indexes of the relative equilibria) is 2 whether $|H| < L$. This feature is verified excepting the case $H = 0$ because when $H = 0$ we have excluded the part of the phase space corresponding to rectilinear motions.

No saddle-connection bifurcation takes place in the Hamiltonian equations as the only possibility to have two saddles is when $(0, \pm\bar{\pi}_2, \bar{\pi}_3)$ exist in $(H_{\&}, 0) \cup (0, H^{\&})$, but then both points coincide as a unique equilibrium of the system associated with $\mathcal{S}$, thus no heteroclinic connection can occur.

Summing up, the stability of the relative equilibria goes as follows:

- the relative equilibrum E_1 is always stable in $H \in [-L, L] \setminus \{0\}$, the same holds for $(0, 0, H^2 - L^2)$ in $\mathcal{X}_{L,H}$;
- the relative equilibrium E_2 is stable when $H \in [-L, H_*]$, unstable for $H \in (H_*, H_{\&}]$, stable when $H \in (H_{\&}, H^{\&})$, unstable for $H \in [H^{\&}, H^*)$ and finally stable for $H \in [H^*, L]$, the same holds for $(0, 0, L^2 - H^2)$ in $\mathcal{X}_{L,H}$;
- the equilibrium E_3 is stable for all values of H where it exists, that is, for $H \in (H_*, H^*) \setminus \{0\}$, in $\mathcal{X}_{L,H}$ the points $(0, \pm\bar{\pi}_2, \bar{\pi}_3)$ are both stable;
- the equilibrium E_4 is unstable for all values of H where it exists, that is, for $H \in (H_{\&}, H^{\&}) \setminus \{0\}$, in $\mathcal{X}_{L,H}$ the points $(\pm\bar{\pi}_1, 0, \bar{\pi}_3)$ are both unstable.

6 Implications for the original system

6.1 Flow of the initial Hamiltonian

Our next purpose is to approximate the invariant sets of the initial system $\mathcal{H}$ from the critical points of the reduced one (either $\mathcal{S}$ or Π). We know (see Palacián 2003) that for the case of toroidal symmetries—as the ones related to the appearance of the integrals L and H—we can generically continue a certain invariant p-torus of Π to a family of invariant $(p+m-s)$-tori corresponding to a truncation of the initial Hamiltonian (2); here m designates the number of degrees of freedom of the original system and s the number of degrees of freedom of the reduced one (i.e., the reduced system expressed in the orbit space). This truncation of the initial Hamiltonian (2) is obtained by inverting back $\mathcal{K}$ through the direct changes of co-ordinates associated with the three Lie transformations. We name the resulting Hamiltonian $\bar{\mathcal{H}}$. It is verified that $\mathcal{H} - \bar{\mathcal{H}} = \mathcal{O}(\varepsilon^{n+1})$, so $\bar{\mathcal{H}}$ means an $\mathcal{O}(\varepsilon^n)$-approximation of the initial Hamiltonian.

For our particular case $p = 0$ (the only invariants determined in the fully-reduced Hamiltonian are stationary points), $m = 3$ and $s = 1$. Thus (regular) critical points of Π (i.e, of $\mathcal{S}$) correspond to invariant 2-tori of $\mathcal{H}$, which are densely filled up with quasiperiodic orbits. Once we have an equilibrium of Π, we can determine a family of approximate invariant 2-tori of the truncatation of the original Hamiltonian, parameterised by L and H.

The procedure is as follows. Since $\mathcal{S}$ has been obtained after three Lie transformations and a reduction process, in order to pass to the Hamiltonian $\mathcal{K}$, we should attach either a family of invariant 2-tori (with parameters L and H) to any point of $\mathcal{X}_{L,H}$ if $|H| > 0$ or either a family of periodic orbits (parameterised by L) to the singular points $(\pm L, 0, 0)$ of $\mathcal{X}_{L,0}$. However we notice that the points $(\pm L, 0, 0)$ must be discarded as all our normal form Hamiltonians (perturbations of the two-body system) are singular for rectilinear orbits. We also need to exclude those critical points whose linearisation has null eigenvalues. If $|H| < G < L$, the invariant 2-tori are defined by the angles ℓ and h. In case of equatorial ($G = |H|$) or circular motions ($G = L$) it is still possible to define other action and angle variables and perform the reconstruction of the invariant tori similarly, as we will see later on.

Notice that we obtain true invariant 2-tori of $\mathcal{K}$, since $\mathcal{K}$ does not contain the tail of the Lie transformations. These families of tori are related with approximate families of invariant tori of $\mathcal{H}$, depending on the parameters L and H but also on the external parameters. This is equivalent to saying that the Hamiltonian $\bar{\mathcal{H}}$ has true invariant 2-tori and quasiperiodic orbits with the same type of stability that that corresponding to the relative equilibria obtained in Sect. 5. These 2-tori bifurcate according to the bifurcation lines analysed in Sect. 5. Moreover, the 2-tori of $\bar{\mathcal{H}}$ coming from the relative equilibria of elliptic type are surrounded by 3-tori corresponding with the periodic orbits around the centres of $\mathcal{X}_{L,H}$. Thence, these 3-tori are true tori of $\bar{\mathcal{H}}$ and approximated tori of $\mathcal{H}$.

An equilibrium on the fully reduced phase space, whose linearisation has no null eigenvalue, must be in correspondence with one, two or four families of invariant 2-tori in $\boldsymbol{R}^6$, depending on where these points are placed in the fully-reduced phase space. For those equilibria of $\mathcal{U}_{L,H}$ or of $\mathcal{U}_{L,0}$ where the linearisation yields null eigenvalues, a specific analysis should be performed. This situation occurs here only for the pitchfork bifurcations as the bifurcations happening at $H \in \{-L, 0, L\}$ does not carry out stationary points whose linearisations have null eigenvalues (in that sense, the lines $H = 0$ and $L = |H|$ are not real bifurcation lines). For a detailed analysis and reconstruction of the flow we address to Ferrer (2002). Note that in our case, the (circular) degenerate equilibria are of parabolic type, and particular, they are stable if $H \in \{H_*, H^*\}$ and unstable if $H \in \{H_\&, H^\&\}$.

We might even compute the explicit formulæ of the approximations of the invariant 2-tori using the direct changes of the Lie transformations constructed through the three generating functions, $\mathcal{R}$, $\mathcal{Q}$ and $\mathcal{P}$, inserting thereafter the co ordinates of E_i in these changes. One must use first the change related with the elimination of h, then the one related with the Delaunay normalisation and finally the change accounting for the elimination of the parallax. On the one hand, we have detected families of quasiperiodic orbits of equatorial and circular type. However, on the other hand, Delaunay co-ordinates are not defined for these orbits. Thus, we need to resort to a different collection of action and angle co-ordinates well defined for all kind of inclinations or for all eccentricities $e \in [0, 1)$. Although it is only needed for circular and equatorial motions, we can use these new variables for all type of trajectories.

Thus, we introduce Henrard's co-ordinates, (Henrard 1974):

$$\begin{aligned} q_1 &= \tfrac{1}{2}(\ell + g - h), & q_2 &= \tfrac{1}{2}(\ell + g + h), & q_3 &= \ell, \\ p_1 &= G - H, & p_2 &= G + H, & p_3 &= L - G, \end{aligned} \tag{32}$$

which are valid for small eccentricities and inclinations close to zero ($G \approx H$) or to π ($G \approx -H$), provided that all the formulæ satisfy the d'Alembert characteristic, which is true for all Hamiltonians involved in the paper.

Now we construct the changes corresponding to the three Lie transformations using (32). Hence, an invariant torus related with a specific stationary point E_i (or any of the possible six equilibria in $\mathcal{X}_{L,H}$ with $0 \leq |H| \leq L$) is defined through the angles q_1 and q_3. Note that G and g are functions of the parameters L and H and consequently we have obtained a (two-parameter) family of invariant 2-tori for a given critical point of $\mathcal{X}_{L,H}$. In this case, a fifth-order process has been enough to study the qualitative dynamics of the original problem. Nevertheless, provided that the global error after truncation be maintained small enough, the higher the order we reach with the Lie transformations, the more accurate the invariant tori of $\bar{\mathcal{H}}$ are.

6.2 Persistence of invariant 3-tori in $\mathcal{H}$

Now we analyse when the invariant 3-tori of $\bar{\mathcal{H}}$ persists under perturbation and are therefore present for the Hamiltonian $\mathcal{H}$.

Our aim now is to demonstrate the existence (and persistence) of true invariant 3-tori of Hamiltonian (2) associated with the non-degenerate elliptic relative equilibria of $\mathcal{X}_{L,H}$. For that we use a slight modification of the theory established for the isoenergetic version of the KAM theorem, the reason being that our normal form Hamiltonian $\mathcal{K}$ has all its main frequencies given by $\partial\mathcal{K}_0/\partial L$, $\partial\mathcal{K}_1/\partial H$ and $\partial\mathcal{K}_2/\partial G$ at different orders of perturbations.

The theorem is due to Britta Sommer (2003) who stated a generalisation of Arnold's theorem (Arnold 1988) for the case of a Hamiltonian having three different time-scales. She applied the theory for a special case of the spatial circular restricted three-body problem but we can adapt it to our Hamiltonians. Her result can be stated as follows.

Consider an analytic family of real analytic Hamiltonian systems

$$\mathcal{D} = d(J) + \varepsilon\, f_\varepsilon(J, \phi, \varepsilon)$$

where d is defined by

$$d(J_1, J_2, J_3) = J_1 + \beta\, d_1(J_1, J_2) + \beta^2\, d_2(J_1, J_2, J_3; \beta),$$

the pair $(J; \phi) \in D \times \mathbf{T}$ are action-angle variables of d such that $J = (J_1, J_2, J_3)$, $\phi = (\phi_1, \phi_2, \phi_3)$. The given set D is supposed to be an open, bounded and connected subset of

$\mathbf{R}^3$. For small and positive β define $\varepsilon = k\,\beta^l$ for some constants $k > 0$ and $l > 3$. Assume that d is real analytic in auniform, complex neighbourhood of $D \times \mathbf{T}$ and that the following holds

$$\left|\frac{\partial^2 d_1}{\partial(J_1, J_2)}\right| > 0, \quad \left|\frac{\partial^2 d_2}{\partial J_3^2}\right| > 0.$$

Then there exist positive constants δ, κ and χ with $\chi = \beta^2\,\kappa$ and $\beta^{l-3} \leq \delta\,\kappa^2$ such that for all frequency vectors $\Omega = (\partial d/\partial J_1, \partial d/\partial J_2, \partial d/\partial J_3)$ the 3-tori $(\mathbf{T}; \Omega)$ of the unperturbed system $d(J)$ survive for any fixed $\tau > 2$ as slightly deformed Lagrangian tori for the Hamiltonian defined through $\mathcal{D}$. They depend on Ω in a smooth manner and fill $D \times \mathbf{T}^3$ up to a set of measure $\mathcal{O}(\kappa)$.

In order to apply Sommer's theorem to our Hamiltonian (6) we have to seek the actions J_i's and the parameter β. It is clear that we have to set $J_1 \equiv -\mu^2/(2L^2)$. Moreover, as the small parameter is of the size of ω/n, we can rewrite $\mathcal{K}_1$ as $\beta\,\mu^2\,H/L^3$ where $\beta = \omega/n$ (recall that $n = \mu^2/L^3$). Now, putting L in terms of J_1 we get $d_1 = -2\sqrt{2}\,\beta\,J_1\sqrt{-J_1}\,J_2/\mu$ where we have identified H with J_2 and we have taken $d_1 = \mathcal{K}_1/\beta$. Thus, we compute the determinant of the Hessian of d_1 with respect to J_1 and J_2 arriving at:

$$\det\begin{pmatrix} \dfrac{\partial^2 d_1}{\partial J_1^2} & \dfrac{\partial^2 d_1}{\partial J_1\,\partial J_2} \\ \dfrac{\partial^2 d_1}{\partial J_2\,\partial J_1} & \dfrac{\partial^2 d_1}{\partial J_2^2} \end{pmatrix} = \frac{18\,J_1}{\mu^2},$$

which never vanishes as J_1 remains bounded below zero. Next, we identify J_3 with G and express $\mathcal{K}_2$ in terms of the J_i's and β. (Notice that with the choices of the J_i's we can take $\phi_2 = h$, $\phi_3 = g$ and ϕ_1 an appropriate function of ℓ so that ϕ_1 becomes the angle conjugated to J_1, for instance, $\phi_1 = (L^3/\mu^2)\,\ell$; hence, $(J; \phi)$ is a set of action-angle coordinates.) We define $d_2 = \mathcal{K}_2/(2\,\beta^2)$ where we first put $\mathcal{K}_2$ in terms of the actions J_i's:

$$\mathcal{K}_2 = \frac{8\sqrt{2}\,\alpha^2\,C_{20}\sqrt{-J_1}\,J_1^4\,(J_3^2 - 3\,J_2^2)}{\mu\,\omega^2\,J_3^5}.$$

Thus, we compute the second derivative of d_2 with respect to J_3 obtaining:

$$\frac{\partial^2 d_2}{\partial J_3^2} = -\frac{24\sqrt{2}\,\alpha^2\,C_{20}\sqrt{-J_1}\,J_1^4\,(-15\,J_2^2 + 2\,J_3^2)}{\mu\,\omega^2\,J_3^7}. \tag{33}$$

The above expression vanishes at $J_3 = \sqrt{15/2}\,|J_2|$, that is when $G = \sqrt{15/2}\,|H|$. However, excluding the equatorial and the circular motions the relative equilibria satisfy that G is close to $\sqrt{5}\,|H|$ so, for these elliptic points we can assure that the partial derivative $\partial^2 d_2/\partial J_3^2$ does not vanish. Besides, for equatorial trajectories $G = |H|$ hence $\partial^2 d_2/\partial J_3^2$ cannot be zero. Thus, it remains to check if G can get the value $\sqrt{15/2}\,|H|$ for circular trajectories. For this kind of orbits, $G = L$, and then the partial derivative (33) is zero when $L = \sqrt{15/2}\,|H|$. Thence, this KAM theorem cannot be applied for the circular type of relative equilibria when $L = \sqrt{15/2}\,|H|$ and in these lines of the parametric plane the circular orbits are always centres. We notice that the lines $L = \sqrt{15/2}\,|H|$ are inside the region $H \in (H_{\&}, H^{\&})$.

Now we set $k = 1$, $l = 4$ and then $\varepsilon = \beta^4$. Thus conditions of Sommer's theorem hold (excepting for circular orbits when $L = \sqrt{15/2}\,|H|$) and we may conclude that the majority of the unperturbed 3-tori of Hamiltonian $\mathcal{K}$ will persist to the whole system $\mathcal{H}$ defined by (2). More precisely, this majority is in the sense that the persistent 3-tori form a set whose complement has a measure $\mathcal{O}(\kappa)$ and since $\beta \leq \delta\kappa^2$ we can take κ of the order of $\sqrt{\beta}$. Generically, the true tori can be refined using either analytical or numerical continuation techniques. Finally, the bifurcations of relative equilibria studied in Sect. 5 are translated into bifurcations of families of invariant 2-tori or quasiperiodic orbits.

We remark that our Hamiltonian model is free of resonances. The reason is that as the quotient ω/n is a small quantity and the resonances between the mean anomaly and the argument of the node would appear when one takes into account the harmonics C_{ij}, S_{ij} with $i\,n - j\,\omega \approx 0$ (leading to small denominators), but with the range of validity of the parameters this situation cannot happen, thus the KAM tori cannot be destroyed due to possible resonances.

6.3 Periodic orbits

We show how some of the quasiperiodic orbits of the original system confined in the invariant 3-tori of Sect. 6.2 can be turned into periodic orbits under certain assumptions. The reader might have a look to the method we use to establish the existence of periodic orbits in the framework of the restricted three-body problems, see Palacián et al. (2006). The true invariant 3-tori shrink down to two-parameter families of elliptic invariant 2-tori as soon as the periodic orbits surrounding the non-degenerate elliptic relative equilibria of $\mathcal{X}_{L,H}$ get closer to these equilibria (see a similar example in Ferrer et al. 2002). The idea is to compute the main frequencies of the 2-tori surrounded by the KAM 3-tori with the aim of obtaining a rational quotient between the two frequencies.

The quasiperiodic orbits fill up the invariant 2-tori which are determined by the angles q_1 and q_3. If we are able to find a rational relation between the two main frequencies defining a family of tori (where p_1 and p_3 vary while the rest of parameters are fixed), we will get a family of periodic orbits confined into these tori. Thus, we look for a relation between the angles q_1 and q_3 so that a quasiperiodic orbit of Hamiltonian $\mathcal{H}$ be periodic. The easiest relation we can look for is a linear one.

A linear relation between the angles q_1 and q_3 is: $q_3 = q_3^0 + s\,(q_1 - q_1^0)$ for some initial conditions q_1^0 and q_3^0 on the invariant torus. The determination of the slope s is made through the quotient:

$$s = \frac{d\,q_3}{d\,q_1} = \frac{d\,q_3/d\,t}{d\,q_1/d\,t},$$

where we need to go back to the averaged Hamiltonian (6). Next, as we know that

$$\frac{d\,\ell}{d\,t} = \frac{\partial\,\mathcal{K}}{\partial\,L}, \qquad \frac{d\,g}{d\,t} = \frac{\partial\,\mathcal{K}}{\partial\,G}, \qquad \frac{d\,h}{d\,t} = \frac{\partial\,\mathcal{K}}{\partial\,H},$$

we arrive at:

$$\frac{d\,q_3}{d\,t} = \frac{\partial\,\mathcal{K}}{\partial\,L} \quad \text{and} \quad \frac{d\,q_1}{d\,t} = \frac{1}{2}\left(\frac{\partial\,\mathcal{K}}{\partial\,L} + \frac{\partial\,\mathcal{K}}{\partial\,G} - \frac{\partial\,\mathcal{K}}{\partial\,H}\right).$$

Now, $\cos(g)$, $\sin(g)$ and G are written in terms of the σ_i's and finally, the co-ordinates of the equilibrium are substituted into the resulting expressions. In this way we compute the possible slopes s for the critical points of elliptic character corresponding to $\mathcal{X}_{L,H}(0 \leq |H| \leq L)$, s being a function of the parameters L, H, and the rest of external parameters.

For example, for circular and equatorial orbits we obtain, respectively, the slopes s_c and s_e as:

$$s_c = \frac{n_{sc}}{d_{sc}}, \qquad s_e = \frac{n_{se}}{d_{se}}$$

where:

$$
\begin{aligned}
n_{sc} &= \omega\, C_{20}\, L^2 \Big[16\, L^6\, (L^2 - 3\, H^2) + C_{20}\, (13\, L^4 - 78\, L^2\, H^2 + 137\, H^4)\Big] \\
&\quad - 144\, (C_{21}^2 + S_{21}^2)\, H\, (L^2 - 2\, H^2) + 288\, (C_{22}^2 + S_{22}^2)\, H\, (L^2 - H^2), \\
d_{sc} &= 2\, \omega\, C_{20}\, L^2 \Big[8\, L^6\, (L^2 - L\, H - 4\, H^2) \\
&\qquad + C_{20}\, (5\, L^4 - 8\, L^3\, H - 48\, L^2\, H^2 + 38\, L\, H^3 + 133\, H^4)\Big] \\
&\quad - 12\, (C_{21}^2 + S_{21}^2)\, (L^3 + 14\, L^2\, H - 6\, L\, H^2 - 32\, H^3) \\
&\quad + 24\, (C_{22}^2 + S_{22}^2)\, (L + H)\, (L^2 + 13\, L\, H - 16\, H^2),
\end{aligned}
$$

and

$$
\begin{aligned}
n_{se} &= 2\, \omega\, C_{20}\, L\, H \Big[8\, L^2\, H^4\, |H| - 8\, C_{20}\, (15\, L^2\, |H| + 8\, L\, H^2 - 5\, H^2\, |H|)\Big] \\
&\quad - 72\, (C_{21}^2 + S_{21}^2)\, H^2, \\
d_{se} &= \omega\, C_{20}\, L \Big\{8\, L^2\, H^4\, (2\, L\, H + L\, |H| + H\, |H|) \\
&\qquad - C_{20} \Big[L\, H\, (55\, L^2 + 12\, L\, H + H^2) \\
&\qquad\qquad + (20\, L^3 + 39\, L^2\, H - 2\, L\, H^2 - 5\, H^3)\, |H|\Big]\Big\} \\
&\quad - 6\, (C_{21}^2 + S_{21}^2)\, (5\, L\, H + 12\, L\, |H| + 6\, H^2) \\
&\quad + 24\, (C_{22}^2 + S_{22}^2)\, L\, (H + |H|).
\end{aligned}
$$

For the remaining equilibria one proceeds in a similar manner, using the asymptotic values of σ_2 (i.e., σ_2^*, σ_{2*}, $\sigma_2{}^{\&}$ or $\sigma_{2\&}$) in the corresponding expressions of the partial derivatives of $\mathcal{K}$ with respect to L, G and H.

Then, $q_3 - q_3^0 = s\, (q_1 - q_1^0)$ for some initial conditions q_1^0 and q_3^0 on the corresponding invariant torus.

Given a concrete system, i.e., after fixing the values of the external parameters are fixed, we impose that $s \in \vec{Q}$, which means that a constraint between L and H ought to be satisfied. Hence, with this choice of q_3 as a linear function of q_1, the trajectory is determined by q_1, it is closed and its period depends on s. Thus, related to the invariant 2-torus associated to a relative equilibrium of $\mathcal{X}_{L,H}$, one can find an infinite (but discrete) number of periodic orbits of the Hamiltonian $\mathcal{H}$.

Finally, we note that we have computed families of quasiperiodic orbits having any eccentricity $e \in [0, 1)$ and any inclination $I \in [0, \pi]$ by means of the co-ordinates (32). However the associated periodic orbits will exist only in the case of non-degenerate centres in $\mathcal{X}_{L,H}$ $(0 \leq |H| \leq L)$.

6.4 Summary of the invariant objects

The main consequences drawn along this section are:

- We have established the existence of invariant 2-tori and quasiperiodic orbits of an $\mathcal{O}(\beta^5)$-approximation of Hamiltonian (2). These invariant objects inherite the non-linear stability character of the (non-degenerate) relative critical points of $\mathcal{X}_{L,H}(0 \leq |H| \leq L)$, so bifurcations of these equilibria turn into bifurcations of 2-tori.

- There exist true 3-tori of the Hamiltonian (2) around the approximate 2-tori mentioned above. These persistent 3-tori form a set whose complement has a measure $\mathcal{O}(\kappa)$ where κ is of the order of $\sqrt{\beta}$. This is equivalent to saying that the majority of the invariant 3-tori of $\mathcal{K}$ (or of the truncation of $\mathcal{H}$) persist in (2). The 3-tori shrink down into families of 2-tori, exactly the tori which get reduced to the relative equilibria when applying the reduction process.
- Some of the quasiperiodic orbits that fill the true 2-tori (obtained from the 3-tori) of the Hamiltonian (2) can be closed to get families of true periodic orbits of (2). This can be achieved selecting the main frequencies of the 2-tori in a way that their quotient is forced to be a rational number.

7 Concluding remarks

The dynamics of a satellite orbiting a planet at low altitude under the influence of the gravitation field is studied. We have focused on the case where the main force acting over the particle is the purely Keplerian term. Moreover, the satellite is supposed to be orbiting the planet at low altitudes and the oblateness coefficient of the planet must be significantly bigger than the rest of the zonal and tesseral coefficients. For instance, this is typical situation for the geodetic satellites around the Earth.

The features of our study can be summarized as follows:

(i) We have made a rigorous analysis of the problem, establishing the existence of true invariant 3-tori and quasiperiodic orbits. As well, a complete analysis about the relative equilibria and their non-linear stability has been performed. The occurrence and type of stability of the equilibria depend on two internal parameters (the formal integrals L and H), and six external parameters. Besides we have determined analytically the bifurcation lines, i.e., the relations satisfied by the parameters so that a change in the number of equilibria and stability happens. In this respect our analysis is new.

(ii) The analysis has been possible through a severe simplification of the original Hamiltonian. However, many of the dynamical features of the initial Hamiltonian have been preserved through the simplifications. First of all, we have been able to pass from the original Hamiltonian of three degrees of freedom to a Hamiltonian of one degree of freedom, by means of three Lie transformations. These transformations are rather technical but we have drawn their main features. It is indeed the first time these operations are executed in closed form, making the subsequent analysis valid for all eccentricities in the elliptic domain. Then, we have applied reduction theory to give the averaged Hamiltonian its simplest possible form and in its appropriate phase space. It has been achieved because we have taken into account all the continuous and discrete symmetries of Hamiltonian $\mathcal{K}$.

(iii) We have enlarged the studies done in (Coffey et al. 1986, 1994; Cushman, 1983, 1988; Chang and Marsden 2003), finding a similar picture of the bifurcation diagram. However that we have included rigorously the influence of the tesserals. We stress that the analogy between the flows of the reduced systems corresponding to the main and to the tesseral problems of the artificial satellite is not a feature that should be expected a priori and cannot be achieved by a simple inspection of the equations of motion of the main problem. Moreover, the different relative equilibria and bifurcation lines discussed along this paper are a refinement of the equilibria and bifurcations of the main problem. The expressions have been obtained after a careful analysis of the fifth-order normal form Hamiltonian (we recall that the small parameter is of the order of the quotient between the frequency of the planet—i.e., its angular speed—and the mean motion of the satellite).

(iv) We have proved the existence of invariant KAM 3-tori of the original Hamiltonian related with the elliptic relative equilibria of the fully reduced Hamiltonian. This has been possible thanks to a special KAM theorem of Sommer (2003), which uses a generalisation of the isoenergetic conditions for proving the persistence of the invariant 3-tori. We have excluded the tori related to the circular orbits when $L = \sqrt{15/2}\,|H|$ as there the hypotheses of Sommer's theorem do not hold. These true tori can be approximated in the sense that we may compute their analytic expressions accurately up to order five in the small parameter β using the changes of co-ordinates provided by the different Lie transformations.

(v) Some true periodic orbits of the original Hamiltonian have been approximated to fifth order using the main frequencies of the 2-tori surrounded by the true 3-tori around the non-linear centres. These periodic orbits cross the equatorial plane of the planet's orbit. More concretely there are three types of families of periodic orbits of the original problem: (i) families with inclination close to 0 or to π having any eccentricity $e \in [0, 1)$ (assuming that the satellite does not collide with the planet); (ii) circular or almost circular periodic orbit having any inclination; and (iii) families of periodic orbits whose inclination is very near the critical inclination of the main problem of the artificial satellite and whose eccentricity has any value in the elliptic domain but which goes to zero as soon as H approaches H_*, H^*, $H_\&$ or $H^\&$. The orbital stability of these families corresponds to the non-linear stability of the relative equilibria associated with them which has been established in Sect. 5.2.

Besides, the periodic orbits can be either continued numerically using standard methods (Lara 2003; Lara and Elipe 2002) or analytically approximated pushing the normalisation to higher orders combined with an approach based on Poincaré–Lindstedt perturbation method (Viswanath 2001).

The techniques and tools used here may be of interest for many other systems modelled by the two-body problem to which one attaches a small perturbation, a typical situation in many problems of celestial and classical mechanics. Our approach is not standard because normalisation of the initial system requires very sophisticated routines for perturbed Keplerian systems we have developed. Besides, the non-linear analysis of the relative equilibria has been possible after applying Lagrange multiplier techniques and computing Morse functions using a symbolic manipulator.

The analysis of the tesseral problem may be utilized by the mission analysts of the space agencies, in order to use some of the periodic and quasiperiodic orbits when designing a specific mission of a satellite around a planet, comet or asteroid.

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