HOMOTOPY CATEGORIES FOR SIMPLY CONNECTED TORSION SPACES

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ABSTRACT. For each $n > 1$ and each multiplicative closed set of integers S, we study closed model category structures on the pointed category of topological spaces, where the class of weak equivalences are classes of maps inducing isomorphism on homotopy groups with coefficients in determined torsion abelian groups, in degrees higher than or equal to n. We take coefficients either on all the cyclic groups $\mathbb{Z}/s$ with $s \in S$, or in the abelian group $\mathbb{C}[S^{-1}] = \mathbb{Z}[S^{-1}]/\mathbb{Z}$ where $\mathbb{Z}[S^{-1}]$ is the group of fractions of the form $\frac{z}{s}$ with $s \in S$. In the first case, for $n > 1$ the localized category $\mathbf{Ho}(\mathcal{T}_n S{-}\mathbf{Top}^*)$ is equivalent to the ordinary homotopy category of $(n-1)$-connected CW-complexes whose homotopy groups are S-torsion. In the second case, for $n > 1$ we obtain that the localized category $\mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n} S{-}\mathbf{Top}^*)$ is equivalent to the ordinary homotopy category of $(n-1)$-connected CW-complexes whose homotopy groups are S-torsion and the n^{th} homotopy group is divisible.

These equivalences of categories are given by colocalizations $X^{\mathcal{T}_n S} \to X$, $X^{\mathcal{T}_{\mathcal{D}_n} S} \to X$ obtained by cofibrant approximations on the model structures. These colocalization maps have nice universal properties. For instance, the map $X^{\mathcal{T}_{\mathcal{D}_n} S} \to X$ is final (in the homotopy category) among all the maps of the form $Y \to X$ with Y an $(n-1)$-connected CW-complex whose homotopy groups are S-torsion and its n^{th} homotopy group is divisible. The spaces $X^{\mathcal{T}_n S}$, $X^{\mathcal{T}_{\mathcal{D}_n} S}$ are constructed using the cones of Moore spaces of the form $M(T, k)$, where T is a coefficient group of the corresponding structure of models, and homotopy colimits indexed by a suitable ordinal.

If S is generated by a set P of primes and S^p is generated by a prime $p \in P$ one has that for $n > 1$ the category $\mathbf{Ho}(\mathcal{T}_n S{-}\mathbf{Top}^*)$ is equivalent to the product category $\prod_{p\in P} \mathbf{Ho}(\mathcal{T}_n S^p{-}\mathbf{Top}^*)$. If the multiplicative system S is generated by a finite set of primes, then localized category $\mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n} S{-}\mathbf{Top}^*)$ is equivalent to the homotopy category of n-connected Ext-S-complete CW-complexes and a similar result is obtained for $\mathbf{Ho}(\mathcal{T}_n S{-}\mathbf{Top}^*)$.

1. INTRODUCTION

In this paper we use model structures [17] on the category $\mathbf{Top}^*$ of pointed topological spaces, to study the ordinary homotopy category of simply connected torsion spaces. For

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the model structures analysed in this paper, the localized category is equivalent to the standard homotopy category of cofibrant spaces. In our study we give some algebraic characterization of these classes of cofibrant spaces, we also analyse some equivalences of homotopy categories and develop some methods to compute homology and homotopy groups.

In this work, for each $n > 1$ and for a closed multiplicative set of integers S, we introduce a new closed model structure that is denoted by $\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*$. The class of weak equivalences of the new structure $\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*$ is the class of maps $X \to Y$ inducing isomorphism of homotopy groups at degrees $\geq n$ with coefficients in the abelian group $\mathbb{C}[S^{-1}] = \mathbb{Z}[S^{-1}]/\mathbb{Z}$, where we use the $\mathbb{C}$ to recall that $\mathbb{C}[S^{-1}]$ is a subgroup of the circle group of complex numbers of module 1.

We show that for $n > 1$ the category $\mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*)$ is equivalent to the standard homotopy category of $(n-1)$-connected CW-complexes whose homotopy groups are S-torsion and the n^{th} homotopy group is S-divisible. The equivalence of categories is given by the CW-approximation $X^{\mathcal{T}_{\mathcal{D}_n}S}$, which is the cofibrant approximation of X in the closed model structure.

The space $X^{(S,n)}$ (in this paper denoted by $X^{\mathcal{T}_nS}$) was constructed in [3] using as building blocks the cones of Moore spaces, which are finite complexes of the form $M(\mathbb{Z}/s; m)$ with $s \in S$, and inductive colimits indexed by the ordinal of positive integers. Nevertheless, to construct the space $X^{\mathcal{T}_{\mathcal{D}_n}S}$, we use cones of Moore spaces of the form $M(\mathbb{C}[S^{-1}], m)$ and colimits indexed by a higher limit ordinal. The reason of this fact is that an infinite number of cells is needed to construct the Moore space $M(\mathbb{C}[S^{-1}], m)$. The relation between these constructions is given in section 7 by a fibration

$$K(T_S(\pi_nX)/D_ST_S(\pi_nX), n-1) \to X^{\mathcal{T}_{\mathcal{D}_n}S} \to X^{\mathcal{T}_nS}$$

where D_SA denotes the maximal S-divisible subgroup of an abelian group A and T_SA denotes the S-torsion subgroup of A .

The following paragraphs of this introduction contain a selection of the main results of this paper.

In section 4, we give up to weak equivalence the following algebraic characterization of $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant spaces.

Theorem 4.1 Let X be a pointed space, then the following statements are equivalent

(i) X is weakly equivalent to an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space,

(ii) X is a $(n-1)$-connected space, for every S-uniquely divisible abelian group B the reduced singular cohomology groups $\tilde{H}^q(X; B)$ are trivial and for any abelian group C with no S-divisible (nontrivial) subgroups the singular cohomology group $H^n(X; C)$ is trivial,

(iii) X is an $(n-1)$-connected space, for every $s \in S$ the singular homology groups $H_n(X; \mathbb{Z}/s)$ are trivial and for $q \geq n$, $H_q(X; \mathbb{Z}[S^{-1}]) \cong 0$.

(iv) X is a $(n-1)$-connected space, H_nX is an S-torsion divisible group and for $q>n$, H_qX is an S-torsion group.

(v) X is a $(n-1)$-connected space, π_nX is an S-torsion divisible group and for $q>n$, π_qX is an S-torsion group.

In section 5, we describe how the colocalized spaces can also be constructed by using homotopy fibres of Quillen-Sullivan localization maps $FX\to X\to X[S^{-1}]$, n-connective coverings Y^n of a space Y and homotopy fibres of maps which represent some distinguished cohomological classes. We see that $X^{\mathcal{T}_nS}$ is weakly equivalent to $(FX^n)^n$ and the following result is proved:

Theorem 5.3 Consider the homomorphism of abelian groups

$$a\colon \pi_n(X^{\mathcal{T}_nS})\to \mathrm{Tor}(\pi_nX,\mathbb{C}[S^{-1}])\cong T_S(\pi_nX)\to T_S(\pi_nX)/D_S(T_S(\pi_nX)),$$

the corresponding cohomological element

$$A\colon (FX^n)^n\to K(T_S(\pi_nX)/D_ST_S(\pi_nX),n)$$

and denote by $(\bar{F}X^n)^n$ the homotopy fibre of A . Then $X^{\mathcal{T}_{\mathcal{D}_n}S}$ is weakly equivalent to $(\bar{F}X^n)^n$. Moreover, for $k\geq n+1$ the following sequence is exact

$$0\to \pi_{k+1}X\otimes\mathbb{C}[S^{-1}]\to \pi_k(X^{\mathcal{T}_{\mathcal{D}_n}S})\to \mathrm{Tor}(\pi_kX,\mathbb{C}[S^{-1}])\to 0,$$

and for $k=n$, we have the exact sequence

$$0\to \pi_{n+1}X\otimes\mathbb{C}[S^{-1}]\to \pi_n(X^{\mathcal{T}_{\mathcal{D}_n}S})\to D_ST_S(\pi_nX)\to 0.$$

In 1972, Bousfield and Kan [1] introduced techniques of homology localization. For instance, for the ring $\mathbb{Z}/p$ with p a prime and for a 1-connected space X they constructed a localization map $X\to(\mathbb{Z}/p)_\infty X$ that induces isomorphism on the homology functors $H_q(-;\mathbb{Z}/p)$. The space $(\mathbb{Z}/p)_\infty X$ is also 1-connected and its homotopy groups are Ext-p-complete abelian groups.

In this preprint, an n-connected space X is said to be Ext-S-complete, if its homotopy groups are Ext-S-complete and an abelian group π is Ext-S-complete if $\mathrm{Ext}(\mathbb{C}[S^{-1}],\pi)\cong\pi$. The case of Ext-p-complete is obtained when S is generated by a prime p .

For a multiplicative system S generated by a finite number of primes $p_1,\cdots,p_r$ and for the ring $R=\mathbb{Z}/p_1\times\cdots\times R=\mathbb{Z}/p_r$, in section 9, for $n>1$, the following equivalence of categories is given:

Theorem 9.1 The left derived functor
$R^L_\infty\colon \mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*)\longrightarrow \mathbf{Ho}(\text{Ext-}S\text{-complete } n\text{-connected spaces})$
is left adjoint to
$(.)^{\mathcal{T}_{\mathcal{D}_n}S}\colon \mathbf{Ho}(\text{Ext-}S\text{-complete } n\text{-connected spaces})\longrightarrow \mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*)$.
Moreover, the pair of functors above give an equivalence categories.

2. Preliminaries

2.1. **Closed model categories.** A closed model category $\mathcal{C}$ is a category endowed with three distinguished classes of maps called cofibrations, fibrations and weak equivalences satisfying certain axioms. We refer the reader to [17] , [18], [11], [12], [4], [5] for any properties, notation and results concerning closed model categories.

In this paper, the following closed model category (CMC) structure will be considered:

The closed model category $Q-\mathbf{Top}$ of topological spaces with the following classes: Given a map $f\colon X \longrightarrow Y$ in **Top**, f is said to be a fibration if it is a fibre map in the sense of Serre; f is a weak equivalence if f induces isomorphism $\pi_q(f)$ for $q \geq 0$ and for any choice of base point and f is a cofibration if it has the LLP with respect to all trivial fibrations. For the study of this structure and its properties we refer the reader to Quillen [17]. We also recall that $Q-\mathbf{Top}^*$ has also an induced closed model category structure: A pointed map $f\colon (X,*) \to (Y,*)$ is said to be a fibration (resp., weak equivalence, cofibration) if in the non pointed setting the map $f\colon X \to Y$ is a fibration (resp., weak equivalence, cofibration.) We recall that both categories of spaces and pointed spaces have compatible simplicial structures, see [17], [8] . For instance for pointed spaces, if K is a finite simplicial object and X is a pointed space then $X \otimes K$ is defined to be

$$X \otimes K = X \times |K|^+/(X \times * \cup * \times |K|^+)$$

where $|K|^+$ is the disjoint union of $|K|$ and the one point space $*$.

In particular we have the standard pointed cylinder

$$X \otimes I = X \otimes \Delta[1] \ .$$

Let $\mathbf{Ho}(Q-\mathbf{Top}^*)$ denote the corresponding localized category obtained by formal inversion of of weak equivalences defined above.

In this subsection we recall a CMC structure on the category of pointed spaces that will be used to prove the main theorems of this paper. A particular case of this CMC structure was given in [6], and the general construction can be seen in [9] , where the reader is referred for proofs, notations and results. Nevertheless, we include some significant facts and properties of this CMC structure that are used in the following sections.

In the category of pointed topological spaces and continuous maps, $\mathbf{Top}^*$, let $\mathcal{F} = \{M_\lambda | \lambda \in \Lambda\}$ be a family of spaces which are suspensions of CW-complexes ($M_\lambda = \Sigma N_\lambda$ where N_λ is a CW-complex).

We consider the following classes of maps:

Definition 2.1. *Let $f : X \longrightarrow Y$ be a map in* $\mathbf{Top}^*$ *,*

(i) *f is an $\mathcal{F}$-weak equivalence if the induced map*

$$[\Sigma^k M_\lambda, f]\colon [\Sigma^k M_\lambda, X] \longrightarrow [\Sigma^k M_\lambda, Y]$$

is an isomorphism for each $k \geq 0$ and $\lambda \in \Lambda$, where $[-,-]$ denotes the standard set of pointed homotopy classes.

(ii) *f is an $\mathcal{F}$-fibration if it has the RLP in the category of pointed spaces with respect to the family $\mathcal{T}(\mathcal{F})$ of inclusions*

$$(C\Sigma^k N_\lambda \times 0) \cup (\Sigma^k N_\lambda \otimes I) \longrightarrow C\Sigma^k N_\lambda \otimes I$$

for every $k \geq 0$ and $\lambda \in \Lambda$.

A map which is both an $\mathcal{F}$-fibration and an $\mathcal{F}$-weak equivalence is said to be a $\mathcal{F}$-trivial fibration.

(iii) *f is an $\mathcal{F}$-cofibration if it has the LLP with respect to any trivial $\mathcal{F}$-fibration.*

A map which is both an $\mathcal{F}$-cofibration and an $\mathcal{F}$-weak equivalence is said to be a $\mathcal{F}$-trivial cofibration.

A pointed space X is said to be $\mathcal{F}$-fibrant if the map $X \longrightarrow *$ is an $\mathcal{F}$-fibration, and X is said to be $\mathcal{F}$-cofibrant if the map $* \longrightarrow X$ is an $\mathcal{F}$-cofibration.

Remark 2.1. *Let C be the path-component of the given base point of X . Note that the inclusion $C \longrightarrow X$ is always an $\mathcal{F}$-weak equivalence. It as also clear that all objects in* **Top*** *are $\mathcal{F}$-fibrant.*

In order to see the difference with the CMC structures given in [11] we have included the following characterization of the family of $\mathcal{F}$-fibrations . Notice that the family of $\mathcal{F}$-fibrations of our CMC structure is larger than the class of Serre fibrations.

We refer the reader to [9] to see a proof of the following characterizations:

Theorem 2.1. *Suppose that $\mathcal{F}$ has at least a non trivial CW-complex, and for a map $f\colon X \longrightarrow Y$ in* **Top****, denote by $f_0\colon X_0 \longrightarrow Y_0$ the induced map on the path-components of the given base points. Then f is an $\mathcal{F}$-fibration if and only if f_0 is a Serre fibration.*

Proposition 2.1. *For a map $f\colon X \longrightarrow Y$ in* **Top****, the following statements are equivalent:*

(i) *f is a $\mathcal{F}$-trivial fibration,*

(ii) *f has the RLP with respect to the family $\mathcal{C}(\mathcal{F})$ of inclusions*

$$* \longrightarrow M_\lambda \ , \ \lambda \in \Lambda \ ,$$

$$\Sigma^k M_\lambda \longrightarrow C\Sigma^k M_\lambda \ , \ k \geq 0 \ , \ \lambda \in \Lambda \ .$$

Using the characterization of $\mathcal{F}$-trivial fibrations by the RLP with respect to a family of maps, one can prove following result, see [9] .

Theorem 2.2. *The category* **Top*** *together with the classes of $\mathcal{F}$-fibrations, $\mathcal{F}$-cofibrations and $\mathcal{F}$-weak equivalences, has the structure of a closed model category.*

Remark 2.2. *P.S. Hirschhorn a [11] and E. Dror-Farjoun [7] have been working with cellularization functors associated to a set A of objects in a closed model category . P.S. Hirschhorn proves that there is a closed model structure on* $\mathbf{Top}^*$ *taking as fibrations the usual Serre fibrations of* $\mathbf{Top}^*$ *, as weak equivalences they consider A-cellular equivalences and the A-cellular cofibrations are defined by the LLP with respect to all the maps which are both fibrations and A-cellular equivalences. Taking as set of objects $A = \{\bigvee_{\lambda\in\Lambda} M_\lambda\}$ if we consider the closed model structure given by P.S. Hirschhorn, we have that the class of $\mathcal{F}$-weak equivalences is exactly the class of A-cellular equivalences. To see this fact it is necessary to take into account that $\bigvee_{\lambda\in\Lambda} M_\lambda$ is a suspension space that induces nice properties in the corresponding function space. However, one has that in* $\mathbf{Top}^*$ *the class of $\mathcal{F}$-fibrations is larger than the class of fibrations. For example, since $0 \to I$ is not a Serre fibration (in* $\mathbf{Top}$*) we have that in* $\mathbf{Top}^*$ *the map $* + 0 \to * + I$ is an $\mathcal{F}$-fibration which is not a Serre fibration. Therefore the CMC structure given in this work is different to the CMC structure given in [11] . However, it is interesting to note that a space is $\mathcal{F}$-cofibrant if and only if it is connected and cofibrant in the closed model category given by Hirschhorn .*

We denote by $\mathcal{F}-\mathbf{Top}^*$ the closed model category $\mathbf{Top}^*$ with the distinguished families of $\mathcal{F}$-fibrations, $\mathcal{F}$-cofibrations and $\mathcal{F}$-weak equivalences and by $\mathbf{Ho}(\mathcal{F}-\mathbf{Top}^*)$ the category of fractions obtained from $\mathcal{F}-\mathbf{Top}^*$ by formal inversion of the family of $\mathcal{F}$-weak equivalences.

One of the basic tool of this paper will be the factorization technique given by the following generalization of the argument of the small object, see [13], [7], [11]. Let $f\colon X \longrightarrow Y$ be a map in $\mathbf{Top}^*$, then f can be factored in two ways:

(i) $f = pi$, where i is a $\mathcal{F}$-cofibration and p is a $\mathcal{F}$-trivial fibration,
(ii) $f = qj$, where j is an $\mathcal{F}$-weak equivalence having the LLP with respect to all $\mathcal{F}$-fibrations and q is a $\mathcal{F}$-fibration.

For instance, in order to obtain the first factorization, we choose a limit ordinal γ whose cardinality is greater than the cardinal of the set of cells of M_λ for every $\lambda \in \Lambda$.

First we can consider all maps of the form $v\colon M_\lambda \longrightarrow Y$, $\lambda \in \Lambda$ to construct the space $X^0 = X \bigvee (\bigvee_v M_{\lambda(v)})$ and the map $p^0 : X^0 \longrightarrow Y$ defined by the sum of f and all the maps v . This map $p^0\colon X^0 \longrightarrow Y$ has the RLP with respect to the maps $* \longrightarrow M_\lambda$. Now we construct the following γ-sequence, for any ordinal $\beta \le \gamma$

$$X^0 \to X^1 \to X^2 \to \cdots \to X^\beta \to \cdots$$

and compatible maps $p^\beta\colon X^\beta \longrightarrow Y$. For $\beta = 0$, we have the map $p^0 : X^0 \longrightarrow Y$. Given an ordinal β , suppose that we have $p^\alpha\colon X^\alpha \longrightarrow X$ for any $\alpha < \beta$. Now we consider two cases:

First case: β is the successor ordinal of α , then we take all commutative diagrams D of the form

$$\begin{array}{ccc} \Sigma^k M_\lambda & \xrightarrow{u^D} & X^\alpha \\ \downarrow & & \downarrow{\scriptstyle p^\alpha} \\ C\Sigma^k M_\lambda & \xrightarrow[v^D]{} & Y \end{array}$$

where $k \geq 0$ and $\lambda \in \Lambda$. Define $j^\beta \colon X^\alpha \longrightarrow X^\beta$, by the pushout

$$\begin{array}{ccc} \bigvee_D \Sigma^k M_\lambda & \longrightarrow & X^\alpha \\ \downarrow & & \downarrow{\scriptstyle j_\beta} \\ \bigvee_D C\Sigma^k M_\lambda & \longrightarrow & X^\beta \end{array}$$

and define $p^\beta \colon X^\beta \longrightarrow Y$ by the sum of p^α and all the v^D .

Second case: β is a limit ordinal. In this case we take

$$X^\beta = \mathrm{colim}_{\alpha<\beta} X^\alpha$$

$$p^\beta = \mathrm{colim}_{\alpha<\beta} p^\alpha$$

By transfinite induction we obtain an $\mathcal{F}$-cofibration $i \colon X \longrightarrow X^\gamma$ and a $\mathcal{F}$-trivial fibration $p \colon X^\gamma \longrightarrow Y$.

The other factorization $f = qj$ is similarly obtained. In this case, we also have that j has the LLP with respect to all $\mathcal{F}$-fibrations.

As consequence of the presence of the closed model structure one has the following version of the Whitehead Theorem:

Theorem 2.3. *Let $f \colon X \longrightarrow Y$ be a map in* $\mathbf{Top}^*$ *and suppose that X and Y are $\mathcal{F}$-cofibrant, then f is a pointed homotopy equivalence if and only*

$$[\Sigma^k M_\lambda, f] \colon [\Sigma^k M_\lambda, X] \longrightarrow [\Sigma^k M_\lambda, Y]$$

is an isomorphism for each $k \geq 0$ and $\lambda \in \Lambda$.

We note that the factorizations above are functorial. This will be interesting when we consider left-derived and right-derived functors. This also implies that we have functorial cylinders and cocylinders. Note that if $X = *$, using the construction above we obtain an $\mathcal{F}$-cofibrant space $Y^{\mathcal{F}}$ and an $\mathcal{F}$-trivial fibration $p \colon Y^{\mathcal{F}} \longrightarrow Y$. This construction induces a well defined functor $(-)^{\mathcal{F}} \colon \mathbf{Top}^* \longrightarrow \mathbf{Top}^*$, and a natural transformation $Y^{\mathcal{F}} \longrightarrow Y$.

Definition 2.2. *The $\mathcal{F}$-cofibrant space obtained through the factorization of $* \longrightarrow Y$ as the composite of an $\mathcal{F}$-cofibration and an $\mathcal{F}$-trivial fibration, will be called the $\mathcal{F}$-colocalization of Y . The $\mathcal{F}$-trivial fibration $Y^{\mathcal{F}} \to Y$ will be called the $\mathcal{F}$-colocalization map of Y.*

Since on a closed model category the hom-set from a cofibrant object to a fibrant object can be realized as a set of homotopy classes, we have

Theorem 2.4. *Let X be a $\mathcal{F}$-cofibrant space and let $Y^{\mathcal{F}} \to Y$ be the $\mathcal{F}$–colocalizacion map of a space Y , then*

$$\mathbf{Ho}(Q{-}\mathbf{Top}^*)(\mathrm{X}, \mathrm{Y}^{\mathcal{F}}) \longrightarrow \mathbf{Ho}(\mathrm{Q}{-}\mathbf{Top}^*)(\mathrm{X}, \mathrm{Y})$$

is an isomorphism. In particular, if Y is $\mathcal{F}$-weakly equivalent to a point, then $\mathbf{Ho}(Q{-}\mathbf{Top}^*)(\mathrm{X}, \mathrm{Y}) \cong *$.

Therefore the $\mathcal{F}$-colocalizacion map $Y^{\mathcal{F}} \to Y$ is finally universal in the homotopy category among the maps $X \to Y$ from an $\mathcal{F}$-cofibrant space X to Y . One also has that the map $Y^{\mathcal{F}} \to Y$ is initially universal in the homotopy category among the maps $X \to Y$ which are $\mathcal{F}$-weak equivalences.

2.2. **Some basic notions and properties of abelian groups.** We recall some basic notions that are quite useful for the category of abelian groups and that will be used in this paper.

Definition 2.3. *An abelian group A is said to be left orthogonal to B and B is said to be right orthogonal to A if* $\mathrm{Hom}(A, B) \cong 0$ *and* $\mathrm{Ext}(A, B) \cong 0$. *Given classes $\mathcal{A}$ and $\mathcal{B}$, if for every A of $\mathcal{A}$ and every B of $\mathcal{B}$, A is left orthogonal to B , the class $\mathcal{A}$ is said to be left orthogonal to $\mathcal{B}$ and $\mathcal{B}$ is said to be right orthogonal to $\mathcal{A}$. If* $\mathrm{Ext}(A, B) \cong 0$ *we use the terms left Ext-orthogonal and right Ext-orthogonal. If* $\mathrm{Hom}(A, B)$ *is trivial, we use the term Hom-orthogonal. This last terminology is also used for non abelian groups.*

Definition 2.4. *An abelian group A is said to be $\otimes$Tor-orthogonal if $A \otimes B \cong 0$ and* $\mathrm{Tor}(A, B) \cong 0$. *Given classes $\mathcal{A}$ and $\mathcal{B}$, if for every A of $\mathcal{A}$ and every B of $\mathcal{B}$, A is $\otimes$Tor-orthogonal to B , the class $\mathcal{A}$ is said to be $\otimes$Tor-orthogonal to $\mathcal{B}$. If* $\mathrm{Tor}(A, B) \cong 0$ *we use the term Tor-orthogonal and if $A \otimes B$ is trivial, we use the term $\otimes$-orthogonal.*

Given a closed multiplicative system S and an abelian group A, for each $s \in S$ one can consider the induced map $\tilde{s} \colon A \to A$ defined by $\tilde{s}a = sa$

Definition 2.5. . *An abelian group A is said to be S-uniquely divisible if for every $s \in S$ the map $\tilde{s} \colon A \to A$ is a bijection. A is said to be S-free-torsion if for every $s \in S$ the map $\tilde{s} \colon A \to A$ is an injection. A is said to be S-divisible if for every $s \in S$ the map $\tilde{s} \colon A \to A$ is a surjection. If S is generated by a prime p some times we write p-uniquely divisible, p-free-torsion or p-divisible.*

An abelian group which is right orthogonal to $\mathbb{Z}/s$ satisfies that the map $\tilde{s} \colon A \longrightarrow A$ is an isomorphism. Therefore one has:

(1) an abelian group is right orthogonal to the family $\{\mathbb{Z}/s | s \in S\}$ if and only if A is S-uniquely divisible,

(2) an abelian group A is right Hom-orthogonal to $\{\mathbb{Z}/s | s \in S\}$ if and only if A is S-torsion-free,

(3) an abelian group A is right Ext-orthogonal to the family $\{\mathbb{Z}/s | s \in S\}$ if and only if A is S-divisible.

If S is a multiplicative closed system of integers, recall that $\mathbb{Z}[S^{-1}]$ is the of the fractions of the form $\frac{z}{s}$ with z an integer and $s \in S$ and the quotient abelian group $\mathbb{Z}[S^{-1}]/\mathbb{Z}$ is denoted by $\mathbb{C}[S^{-1}]$. The following exact sequence will be frequently used

$$\begin{aligned} 0 \to \quad & \mathrm{Hom}(\mathbb{C}[S^{-1}], B) \to \mathrm{Hom}(\mathbb{Z}[S^{-1}], B) \to \\ & B \to \mathrm{Ext}(\mathbb{C}[S^{-1}], B) \to \mathrm{Ext}(\mathbb{Z}[S^{-1}], B) \to 0 \end{aligned}$$

Given an abelian group B, there exists a maximal S-divisible subgroup $D_S B$ which contains every S-divisible subgroup of B. We have the following properties:

(4) B is right orthogonal to $\mathbb{C}[S^{-1}]$ if and only if B is S-uniquely divisible,

(5) B is right Hom-orthogonal to $\mathbb{C}[S^{-1}]$ if and only if the maximal S-divisible subgroup $D_S(B)$ of B is S-uniquely divisible,

(6) B is right Ext-orthogonal to $\mathbb{C}[S^{-1}]$ if and only of B is S-divisible.

If G is a group and we assume that all the integers of S are positive, we can consider the progroup $\{Ker\tilde{s}|\ s \in S\}$ whose bounding maps are of the form $Kers_1 \to Kers_0$, $x \to tx$ if $s_1 = s_0 t$. Note G is right Hom-orthogonal to $\mathbb{C}[S^{-1}]$ if and only the pointed set $\lim Ker\tilde{s}$ is trivial, where $\tilde{s}\colon G \to G$ is the map $g \to g^s$.

Definition 2.6. *An abelian group B is said to be Ext-S-complete if the boundary morphism of the exact sequence above induces an isomorphism $B \cong \mathrm{Ext}(\mathbb{C}[S^{-1}], B)$. If $\mathrm{Hom}(\mathbb{Z}[S^{-1}], B) \cong 0$, B is said to be S-reduced and if $\mathrm{Ext}(\mathbb{Z}[S^{-1}], B) \cong 0$, B is said to be S-cotorsion. Note that B is S-reduced if and only if B has no (non trivial) S-divisible subgroups.*

(7) B is right orthogonal to $\mathbb{Z}[S^{-1}]$ if and only if B is Ext-S-complete,

(8) B is right Hom-orthogonal to $\mathbb{Z}[S^{-1}]$ if and only if B is S-reduced, or equivalently if $D_S(B) \cong 0$,

(9) B is right Ext-orthogonal to $\mathbb{Z}[S^{-1}]$ if and only of B is S-cotorsion.

An abelian group which is $\otimes$Tor-orthogonal to $\mathbb{Z}/s$ satisfies that the map $\tilde{s}\colon A \longrightarrow A$ is an isomorphism. Therefore we have:

(10) an abelian group is $\otimes$Tor-orthogonal to the family $\{\mathbb{Z}/s | s \in S\}$ if and only if A is S-uniquely divisible.

(11) an abelian group A is $\otimes$-orthogonal to the family $\{\mathbb{Z}/s | s \in S\}$ if and only if A is S-divisible,

(12) an abelian group A is Tor-orthogonal to the family $\{\mathbb{Z}/s | s \in S\}$ if and only if A is S-torsion-free.

Definition 2.7. *An abelian group A is said to be S-torsion for every $a \in A$ there exist $s \in S$ such that $sa = 0$. If S is generated by a prime p we write p-torsion. For an abelian group A , we denote by $T_S(A)$ the maximal S-torsion subgroup of A . An abelian group is said to be S-adjusted if $A/T_S(A)$ is S-uniquely divisible.*

If we consider the exact sequence

$$0 \to \mathrm{Tor}(A, \mathbb{C}[S^{-1}]) \to A \to A \otimes \mathbb{Z}[S^{-1}] \to A \otimes \mathbb{C}[S^{-1}] \to 0$$

one has that $T_S(A) \cong \mathrm{Tor}(A, \mathbb{C}[S^{-1}])$. Using this notation one has:

(13) an abelian group is $\otimes$Tor-orthogonal to $\mathbb{C}[S^{-1}]$ if and only if A is S-uniquely divisible.

(14) an abelian group A is $\otimes$-orthogonal to $\mathbb{C}[S^{-1}]$ if and only if A is S-adjusted or equivalently if $A/T_S(A)$ is S-uniquely divisible. If S is generated by a set P of primes, A is $\otimes$-orthogonal to $\mathbb{C}[S^{-1}]$ if and only for each $p \in P$ for all $a \in A$ and for all $i \geq 0$ there is $x_i \in A$ such that $a - p^i x_i$ is a p-torsion element.

(15) an abelian group A is Tor-orthogonal to $\mathbb{C}[S^{-1}]$ if and only if A is S-torsion-free.

(16) an abelian group is $\otimes$Tor-orthogonal to $\mathbb{Z}[S^{-1}]$ if and only if A is S-torsion,

(17) an abelian group A is $\otimes$-orthogonal to $\mathbb{Z}[S^{-1}]$ if and only if A is S-torsion,

(18) every abelian group A is Tor-orthogonal to $\mathbb{Z}[S^{-1}]$.

3. Some closed model categories associated to a set S of integers

In order to introduce model structures associated with a set of integers S and an integer $n > 0$, we recall briefly the definition of homotopy groups with coefficients. For a more complete description and properties we refer the reader to Hilton [10]. For $k \geq 1$ and an abelian group A , we have the canonical space $M(A; k)$ which is usually called the Moore space with coefficient group A and degree k . For a pointed space X , consider the set of pointed homotopy classes $\pi_k(A; X) = [M(A, k), X]$. This hom-set admits the structure of a group for $k \geq 2$ which abelian for $k \geq 3$. It is said that $\pi_k(A; X)$ is the k-th homotopy group of X with coefficients in A . We also refer the reader to Neisendorfer [15] for some properties of homotopy groups with coefficients.

We shall frequently use the following exact sequence for $k \geq 1$:

$$0 \to \mathrm{Ext}(A, \pi_{k+1}X) \to \pi_k(A; X) \to \mathrm{Hom}(A, \pi_k X) \to 0.$$

In the category of pointed topological spaces and continuous maps, **Top*** , for a set S of non-zero integers and $n > 0$, in [3] we have considered the family S_n of Moore spaces, which in this paper is denoted by $\mathcal{T}_n S$

$$S_n = \{M(\mathbb{Z}/s; n) | s \in S\} = \mathcal{T}_n S$$

and we have studied the associated closed model structure.

If S is multiplicative closed, we consider the ring $\mathbb{Z}[S^{-1}]$ of the fractions of the form $\frac{z}{s}$ with z an integer and $s \in S$. The quotient abelian group $\mathbb{Z}[S^{-1}]/\mathbb{Z}$ will be denoted by $\mathbb{C}[S^{-1}]$.

In [9] , we have considered the closed model structure induced by the family

$$\mathcal{D}_n S = \{M(\mathbb{Z}[S^{-1}]; n)\}.$$

In the present paper is devoted to study the closed model structure induced on pointed spaces by the family

$$\mathcal{T}_{\mathcal{D}_n} S = \{M(\mathbb{C}[S^{-1}]; n)\}$$

which only has one Moore space.

For the family $\mathcal{T}_n S$, a map $f : X \longrightarrow Y$ in $\mathbf{Top}^*$ is a $\mathcal{T}_n S$-weak equivalence if the induced map

$$\pi_l(\mathbb{Z}/s; f)\colon \pi_l(\mathbb{Z}/s; X) \longrightarrow \pi_l(\mathbb{Z}/s; Y)$$

is an isomorphism for each $l \geq n$ and $s \in S$.

For the family $\mathcal{D}_n S$, a map $f : X \longrightarrow Y$ in $\mathbf{Top}^*$ is a $\mathcal{D}_n S$- weak equivalence if the induced map

$$\pi_l(\mathbb{Z}[S^{-1}]; f)\colon \pi_l(\mathbb{Z}[S^{-1}]; X) \longrightarrow \pi_l(\mathbb{Z}[S^{-1}]; Y)$$

is an isomorphism for each $l \geq n$.

With respect to the family $\mathcal{T}_{\mathcal{D}_n} S$, f is a $\mathcal{T}_{\mathcal{D}_n} S$-weak equivalence if the induced map

$$\pi_l(\mathbb{C}[S^{-1}]; f)\colon \pi_l(\mathbb{C}[S^{-1}]; X) \longrightarrow \pi_l(\mathbb{C}[S^{-1}]; Y)$$

is an isomorphism for each $l \geq n$.

We note that the homotopy groups with coefficients only depend on the path component C of the given base point of X . Therefore the inclusion $C \longrightarrow X$ is always a weak equivalence for the model structures associated with the families $\mathcal{T}_n S$, $\mathcal{D}_n S$ and $\mathcal{T}_{\mathcal{D}_n} S$. It as also clear that all objects in $\mathbf{Top}^*$ are fibrant in the corresponding structures.

We denote by $\mathcal{T}_{\mathcal{D}_n} S{-}\mathbf{Top}^*$ the closed model category $\mathbf{Top}^*$ with the distinguished classes of fibrations, $\mathcal{T}_{\mathcal{D}_n} S$-cofibrations and $\mathcal{T}_{\mathcal{D}_n} S$-weak equivalences and by $\mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n} S{-}\mathbf{Top}^*)$ the category of fractions obtained from $\mathcal{T}_{\mathcal{D}_n} S{-}\mathbf{Top}^*$ by formal inversion of the family of $\mathcal{T}_{\mathcal{D}_n} S$-weak equivalences. Similar notation will be used for $\mathcal{D}_n S$ or for $\mathcal{T}_n S$.

In these closed model categories it is very interesting to determine the classes of cofibrant spaces. If S is multiplicative closed and $n > 1$ one has, see [3], that a space X is weakly equivalent to a $\mathcal{T}_n S$-cofibrant space if and only if X is $(n-1)$-connected and for $k \geq n$ the homotopy groups of X are S-torsion abelian groups. In [9], we have shown that for $n > 1$, a space X is weakly equivalent to a $\mathcal{D}_n S$-cofibrant space if and only if X is $(n-1)$-connected and for $k \geq n$ the homotopy groups of X are S-uniquely divisible abelian groups.

In this paper, we shall give a characterization of $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant spaces for $n > 1$. However, it remains to study these kind of $\mathcal{F}$- structures and the corresponding characterizations of "cofibrant spaces" for $n = 1$.

For $n > 1$, we note that a space Y is $\mathcal{T}_nS$-weakly equivalent to a point if π_nY is right Hom-orthogonal to S-torsion abelian groups (S-torsion-free) and for $k > n$, π_kY is right orthogonal to S-torsion abelian groups (S-uniquely divisible). As a consequence of Theorem 2.4 one has that if X is an $\mathcal{T}_nS$-cofibrant space with $n > 1$ and B is an abelian group which is right orthogonal to $\mathbb{Z}/s$ for every $s \in S$, then the reduced cohomology of X with coefficients in B is trivial. Moreover, if B is S-torsion-free, then $H^n(X; B)$ is trivial.

With respect the $\mathcal{D}_nS$-structure, if $n > 1$, a space Y is $\mathcal{D}_nS$-weakly equivalent to a point if π_nY is S-reduced and for $k > n$, π_kY is S-complete. As a consequence of Theorem 2.4 one has that if X is an $\mathcal{D}_nS$-cofibrant space with $n > 1$ and B is an S-complete abelian group, then the reduced cohomology of X with coefficients in B is trivial. Moreover, if B is S-reduced, then $H^n(X; B)$ is trivial.

For $n > 1$, one has that a space Y is $\mathcal{T}_{\mathcal{D}_n}S$-weakly equivalent to a point if the maximal S-divisible subgroup of π_nY is S-uniquely divisible and and for $k > n$, π_kY is S-uniquely divisible. Given an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space X , if B is an abelian group whose maximal S-divisible subgroup is S-uniquely divisible, then $H^n(X; B)$ is trivial and if B is a S-uniquely divisible group, then for $k \geq n$, $H^k(X; B)$ is trivial.

Remark 3.1. *In order to give the factorizations of axiom $CM5$, we have chosen a determined limit ordinal. Since the standard Moore space $M(\mathbb{Z}/s, n)$ has a finite number of cells, then for the case of the $\mathcal{T}_nS$-structure we can choose the countable limit ordinal $\aleph_0$. Since the standard Moore space $M(\mathbb{C}[S^{-1}], n)$ has a countable number of cells, then for the $\mathcal{T}_{\mathcal{D}_n}S$-structure we have to choose the continuum limit ordinal $\aleph_1$.*

4. $\mathcal{T}_{\mathcal{D}_n}S$-COFIBRANT SPACES FOR $n > 1$

In this section, we suppose that $n > 1$. We also consider a multiplicative system S generated by a set P of positive primes.

We note that an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space is $(n-1)$-connected. We also observe that n^{th} singular homology group of an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space is an S-torsion divisible abelian group and for $q > n$ we shall prove that the q^{th} singular homology group is an S-torsion abelian group.

This properties of the homology groups will imply that the homotopy groups of an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space satisfy similar properties in dimension n and $> n$, respectively. In this section, we show that these properties give up to weak equivalence a characterization of the class of $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant spaces.

Lemma 4.1. *If X is an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space, then X is an $(n-1)$-connected space.*

Proof. For any ordinal $\beta \leq \aleph_1$, consider the $\aleph_1$-sequence given in §2:

$$X^0 \to X^1 \to X^2 \to \cdots \to X^\beta \to \cdots$$

where $X^0 = \bigvee_f M(\mathbb{C}[S^{-1}]; n)_f$ for all maps $f\colon M(\mathbb{C}[S^{-1}]; n) \longrightarrow X$. For X^β we have two cases:

If β is the successor ordinal of α , then X^β has the homotopy type of the homotopy cofibre of a map of the form $\bigvee_D M(\mathbb{C}[S^{-1}]; m_D) \longrightarrow X^\alpha$, $m_D \geq n$.

If β is a limit ordinal. We have that

$$X^\beta = \mathrm{colim}_{\alpha<\beta} X^\alpha$$

By transfinite induction we obtain an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space $X^{\aleph_1}$ and an $\mathcal{T}_{\mathcal{D}_n}S$-trivial fibration $p\colon X^{\aleph_1} \longrightarrow X$.

It is clear that X^0 is an $(n-1)$-connected space. For the first case, using the excision theorem for homotopy groups, it follows that if X^α is $(n-1)$-connected, then X^β is also $(n-1)$-connected. For the second case, one has that the homotopy groups commute with homotopy colimits. Then by transfinite induction we have that $X^{\aleph_1}$ is an $(n-1)$-connected space. Since X is a cofibrant space we have that the $\mathcal{T}_{\mathcal{D}_n}S$-trivial fibration $p\colon X^{\aleph_1} \to X$ is a weak equivalence, hence X is also an $(n-1)$-connected space. □

The following result gives up to weak equivalence some algebraic characterizations of $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant spaces.

Theorem 4.1. *Let X be a pointed space, then the following statements are equivalent*
(i) *X is weakly equivalent to an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space,*
(ii) *X is a $(n-1)$-connected space, for every S-uniquely divisible abelian group B the reduced singular cohomology groups $\tilde{H}^q(X; B)$ are trivial and for any abelian group C with no S-divisible (nontrivial) subgroups the singular cohomology group $H^n(X; C)$ is trivial,*
(iii) *X is an $(n-1)$-connected space, for every $s \in S$ the singular homology groups $H_n(X; \mathbb{Z}/s)$ are trivial and for $q \geq n$ $H_q(X; \mathbb{Z}[S^{-1}]) \cong 0$.*
(iv) *X is a $(n-1)$-connected space, $H_n X$ is an S-torsion divisible group and for $q > n$, $H_q X$ is an S-torsion group.*
(v) *X is a $(n-1)$-connected space, $\pi_n X$ is an S-torsion divisible group and for $q > n$, $\pi_q X$ is an S-torsion group.*

Proof. (i) => (ii). Lemma 4.1 and the cohomological results given at the end of §3 (before Remark 3.1 .)

(ii) => (iii). Note that if $s \in S$ then any $\mathbb{Z}/s$-module M is right Hom-orthogonal to $\mathbb{C}[S^{-1}]$. Therefore the reduced n^{th} cohomology group of X with coefficients in a $\mathbb{Z}/s$-module M vanishes if $s \in S$. By the universal coefficient theorem for $\mathbb{Z}/s$-module chain complexes we have that $\mathrm{Hom}(H_n(X; \mathbb{Z}/s), M) \cong 0$. In particular one has that

$$\mathrm{Hom}(H_n(X; \mathbb{Z}/s), H_n(X; \mathbb{Z}/s)) \cong 0 .$$

This implies that $H_n(X;\mathbb{Z}/s)\cong 0$. We can repeat the argument for $\mathbb{Z}[S^{-1}]$-modules to obtain that for $q\geq n$ $H_q(X;\mathbb{Z}[S^{-1}])\cong 0$.

(iii) <=> (iv). This is obvious from the universal coefficient theorem and the properties (11) and (16) of §2 .

(iv) <=> (v). It follows from Serre mod $\mathcal{C}$ theory, see [19] . A 1-connected space has uniquely S-divisible homology groups if and only if it has uniquely S-divisible homotopy groups.

(v) => (i). Assume that X satisfies (v). Take the $\mathcal{T}_{\mathcal{D}_n}S$-trivial fibration $p\colon X^{\mathcal{T}_{\mathcal{D}_n}S}\to X$ with fibre F . If we consider the exact sequence of homotopy groups of the fibration p :

$$\cdots\to\pi_{l+1}X^{\mathcal{T}_{\mathcal{D}_n}S}\to\pi_{l+1}X\to\pi_lF\to\pi_lX^{\mathcal{T}_{\mathcal{D}_n}S}\to\pi_lX\to\cdots$$

we obtain that π_lF is an S-torsion for $l\geq n-1$, $\pi_{n-1}F$ is a divisible group, and F is an $(n-2)$-connected space.

On the other hand, because p is a fibration we also have the exact sequence
$\cdots\to\pi_k(\mathbb{C}[S^{-1}];X^{\mathcal{T}_{\mathcal{D}_n}S})\to\pi_k(\mathbb{C}[S^{-1}];X)\to\pi_{k-1}(\mathbb{C}[S^{-1}];F)$
$\to\pi_{k-1}(\mathbb{C}[S^{-1}];X^{\mathcal{T}_{\mathcal{D}_n}S})\to\pi_{k-1}(\mathbb{C}[S^{-1}];X)\to\cdots$

Since p is an $\mathcal{T}_{\mathcal{D}_n}S$-trivial fibration and

$$\pi_{n-1}(\mathbb{C}[S^{-1}];X^{\mathcal{T}_{\mathcal{D}_n}S})\cong\mathrm{Ext}(\mathbb{C}[S^{-1}];\pi_nX^{\mathcal{T}_{\mathcal{D}_n}S})\cong 0$$

because $\pi_nX^{\mathcal{T}_{\mathcal{D}_n}S}$ is divisible, it follows that $\pi_k(\mathbb{C}[S^{-1}];F)\cong 0$ for $k\geq n-1$.

Because $\pi_{n-1}F$ is an S-divisible group, one has that its maximal S-divisible subgroup is $\pi_{n-1}F$. Because $\pi_{n-1}F$ is right Hom-orthogonal to $\mathbb{C}[S^{-1}]$, this maximal subgroup $\pi_{n-1}F$ is S-uniquely divisible. However we also have that $\pi_{n-1}F$ is an S-torsion group. Then one has that $\pi_{n-1}F\cong 0$. For $q\geq n$, one also has that π_qF is an S-torsion S-uniquely divisible group, hence $\pi_qF\cong 0$. Therefore the map $p\colon X^{\mathcal{T}_{\mathcal{D}_n}S}\to X$ is a weak equivalence.

□

Now we study the homotopy groups with coefficients in $\mathbb{C}[S^{-1}]$ of the $\mathcal{T}_{\mathcal{D}_n}S$-colocalization of a space X and in particular the homotopy groups with coefficients of an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space.

Proposition 4.1. *Let $X^{\mathcal{T}_{\mathcal{D}_n}S}$ be the $\mathcal{T}_{\mathcal{D}_n}S$-colocalization of a pointed space X . Then for $q\geq n$ the following sequence is exact*

$$0\to\mathrm{Ext}(\mathbb{C}[S^{-1}],\pi_{q+1}X)\to\pi_q(\mathbb{C}[S^{-1}];X^{\mathcal{T}_{\mathcal{D}_n}S})\to\mathrm{Hom}(\mathbb{C}[S^{-1}],\pi_qX)\to 0$$

In particular, if the maximal S-divisible subgroup of π_qX is S-uniquely divisible, one has

$$\pi_q(\mathbb{C}[S^{-1}];X^{\mathcal{T}_{\mathcal{D}_n}S})\cong\mathrm{Ext}(\mathbb{C}[S^{-1}],\pi_{q+1}X)$$

with the additional condition that $\pi_{q+1}X$ is an S-cotorsion group, then

$$\pi_q(\mathbb{C}[S^{-1}];X^{\mathcal{T}_{\mathcal{D}_n}S})\cong\pi_{q+1}X/D_S(\pi_{q+1}X)$$

where $D_S(\pi_{q+1}X)$ is the maximal S-divisible subgroup of $\pi_{q+1}X$ and if $\pi_{q+1}X$ is an S-cotorsion S-reduced group

$$\pi_q(\mathbb{C}[S^{-1}]; X^{\mathcal{T}_{\mathcal{D}_n}S}) \cong \pi_{q+1}X.$$

On the other hand, if $\pi_{q+1}X$ is S-cotorsion and S-divisible then

$$\pi_q(\mathbb{C}[S^{-1}]; X^{\mathcal{T}_{\mathcal{D}_n}S}) \cong \mathrm{Hom}(\mathbb{C}[S^{-1}], \pi_q X).$$

Corollary 4.1. *Suppose that B is an abelian group and $K(B,q)$ the Eilenberg-Mac Lane space at dimension q . Then for $m > n$, $K(B,m)^{\mathcal{T}_{\mathcal{D}_n}S}$ has two possible non trivial homotopy groups with coefficients in $\mathbb{C}[S^{-1}]$*

$$\pi_{m-1}(\mathbb{C}[S^{-1}]; K(B,m)^{\mathcal{T}_{\mathcal{D}_n}S}) \cong \mathrm{Ext}(\mathbb{C}[S^{-1}], B),$$

$$\pi_m(\mathbb{C}[S^{-1}]; K(B,m)^{\mathcal{T}_{\mathcal{D}_n}S}) \cong \mathrm{Hom}(\mathbb{C}[S^{-1}], B).$$

If the maximal S-divisible subgroup of B is S-uniquely divisible, then the space $K(B,m)^{\mathcal{T}_{\mathcal{D}_n}S}$ has only one non trivial homotopy group with coefficients in $\mathbb{C}[S^{-1}]$

$$\pi_{m-1}(\mathbb{C}[S^{-1}]; K(B,m)^{\mathcal{T}_{\mathcal{D}_n}S}) \cong \mathrm{Ext}(\mathbb{C}[S^{-1}], B).$$

If B is S-cotorsion S-divisible, then $K(B,m)^{\mathcal{T}_{\mathcal{D}_n}S}$ has only one non trivial homotopy group with coefficients in $\mathbb{C}[S^{-1}]$

$$\pi_m(\mathbb{C}[S^{-1}]; K(B,m)^{\mathcal{T}_{\mathcal{D}_n}S}) \cong \mathrm{Hom}(\mathbb{C}[S^{-1}], B).$$

Remark 4.1. *We can also compute the homotopy group $\pi_{n-1}(X^{\mathcal{T}_{\mathcal{D}_n}S}; \mathbb{C}[S^{-1}])$ if we take into account Theorem 5.3 of the following section. In particular, one has that*

$$\pi_{n-1}(\mathbb{C}[S^{-1}]; K(B,n)^{\mathcal{T}_{\mathcal{D}_n}S}) \cong \mathrm{Ext}(\mathbb{C}[S^{-1}], D_S T_S B).$$

5. $\mathcal{T}_n S$-colocalizations, $\mathcal{T}_{\mathcal{D}_n}S$-colocalizations and S-cocompletions

Trough all this section we assume that $n > 1$ and that S is a closed multiplicative system.

We consider the Sullivan-Quillen localization for 1-connected spaces, then for a 1-connected space X we have the localization $l\colon X \longrightarrow X[S^{-1}]$. The homotopy fibre FX of the localization map is called the S-cocomplection of X . In this section, we compare the S-cocomplection FX with the $\mathcal{T}_n S$-colocalization $X^{\mathcal{T}_n S}$ and the $\mathcal{T}_{\mathcal{D}_n}S$-colocalization $X^{\mathcal{T}_{\mathcal{D}_n}S}$. We denote by Y^n the n-connective covering of Y .

Theorem 5.1. *Let X be a 1-connected space. Then we have:*
(i)*If $\pi_q X$ is S-uniquely divisible for $q \le n-1$ and $\pi_n X$ is S-adjusted, then FX is weakly equivalent to $X^{\mathcal{T}_n S}$.*
(ii) *If $\pi_q X$ is S-uniquely divisible for $q \le n-1$, and $\pi_n X$ is S-divisible, then FX is weakly equivalent to $X^{\mathcal{T}_{\mathcal{D}_n}S}$.*

Proof. For $k \geq 0$ we have the following exact sequence

$$\cdots \to \pi_{k+1}X \to \pi_{k+1}X[S^{-1}] \to \pi_k FX \to \pi_k X \to \pi_k X[S^{-1}] \ ,$$

Then one has for $k \geq 1$ the following exact sequence

$$0 \to \pi_{k+1}X \otimes \mathbb{C}[S^{-1}] \to \pi_k FX \to \mathrm{Tor}(\pi_k X, \mathbb{C}[S^{-1}]) \to 0.$$

In case (i), because X is 1-connected, $\pi_q X$ is S-uniquely divisible for $q \leq n-1$ and $\pi_n X$ is S-adjusted, by (13), (14) of §2 it follows that FX is $(n-1)$-connected. In case (ii), one obtains that FX is $(n-1)$-connected and $\pi_n FX \cong \pi_{n+1}X \otimes \mathbb{C}[S^{-1}]$ is a divisible abelian group. In both cases we have that for $q \geq n$, $\pi_q FX$, is an S-torsion abelian group.

As a consequence of Theorem 2.4, for a $(n-1)$-connected space FX with S-torsion homotopy groups, we have the bijection

$$p_* \colon \mathbf{Ho}(Q\text{--}\mathbf{Top}^*)(FX, X^{\mathcal{T}_n S}) \longrightarrow \mathbf{Ho}(Q\text{--}\mathbf{Top}^*)(FX, X)$$

where $p \colon X^{\mathcal{T}_n S} \to X$ is the colocalization map.

With the additional condition that $\pi_n FX$ is divisible we also have the bijection

$$\mathbf{Ho}(Q\text{--}\mathbf{Top}^*)(FX, X^{\mathcal{T}_{\mathcal{D}_n} S}) \cong \mathbf{Ho}(Q\text{--}\mathbf{Top}^*)(FX, X)$$

Therefore, for the maps $i \colon FX \longrightarrow X$, there exists a map $i' \colon FX \longrightarrow X^{\mathcal{T}_n S}$, such that $i'p = i$ in $\mathbf{Ho}(Q\text{--}\mathbf{Top}^*)$.

On the other hand, because $X[S^{-1}]$ is S-uniquely divisible space, it follows that $X[S^{-1}]$ is $\mathcal{T}_n S$-weakly equivalent to a point. Then

$$\mathbf{Ho}(Q\text{--}\mathbf{Top}^*)(X^{\mathcal{T}_n S}, X[S^{-1}]) \cong \mathbf{Ho}(\mathcal{T}_n S\text{--}\mathbf{Top}^*)(X^{\mathcal{T}_n S}, X[S^{-1}]) \cong 0.$$

By the same reason,

$$\mathbf{Ho}(Q\text{--}\mathbf{Top}^*)(X^{\mathcal{T}_n S}, \Omega X[S^{-1}]) \cong 0.$$

Because $FX \longrightarrow X \longrightarrow X[S^{-1}]$ is a fibration sequence, it follows that

$$i_* \colon \mathbf{Ho}(Q\text{--}\mathbf{Top}^*)(X^{\mathcal{T}_n S}, FX) \longrightarrow \mathbf{Ho}(Q\text{--}\mathbf{Top}^*)(X^{\mathcal{T}_n S}, X)$$

is a bijection. Therefore there exists a map $p' \colon X^{\mathcal{T}_n S} \longrightarrow FX$ such that $ip' = p$.

Finally it is easy to check that $p'i' = id$, $i'p' = id$. Therefore $X^{\mathcal{T}_n S}$ is weakly equivalent to FX . Similarly, under condition (ii) we also have that $X^{\mathcal{T}_{\mathcal{D}_n} S}$ is weakly equivalent to FX . □

The following result proves that for any space X, the colocalizations can be expressed in terms of cocompletions, connective coverings and amplification constructions induced by certain cohomological elements.

We note that for $n > 1$, given any space X the n-connective covering $X^n \to X$ induces a weak equivalence $(X^n)^{\mathcal{T}_n S} \to X^{\mathcal{T}_n S}$ where X^n is a 1-connected space.

Theorem 5.2. *The space $X^{\mathcal{T}_n S}$ is weakly equivalent to $(FX^n)^n$. Moreover, for $k \geq n$ the following sequence is exact*

$$0 \to \pi_{k+1}X \otimes \mathbb{C}[S^{-1}] \to \pi_k(X^{\mathcal{T}_n S}) \to \mathrm{Tor}(\pi_k X, \mathbb{C}[S^{-1}]) \to 0.$$

Proof. It similar to the proof above. □

Theorem 5.3. *Consider the homomorphism of abelian groups*

$$a\colon \pi_n(X^{\mathcal{T}_n S}) \to \mathrm{Tor}(\pi_n X, \mathbb{C}[S^{-1}]) = T_S(\pi_n X) \to T_S(\pi_n X)/D_S(T_S(\pi_n X)),$$

the corresponding cohomological element

$$A\colon (FX^n)^n \to K(T_S(\pi_n X)/D_S T_S(\pi_n X), n)$$

and denote by $(\bar{F}X^n)^n$ the homotopy fibre of A . Then $X^{\mathcal{T}_{\mathcal{D}_n} S}$ is weakly equivalent to $(\bar{F}X^n)^n$. Moreover, for $k \geq n+1$ the following sequence is exact

$$0 \to \pi_{k+1}X \otimes \mathbb{C}[S^{-1}] \to \pi_k(X^{\mathcal{T}_{\mathcal{D}_n} S}) \to \mathrm{Tor}(\pi_k X, \mathbb{C}[S^{-1}]) \to 0,$$

and for $k = n$, we have the exact sequence

$$0 \to \pi_{n+1}X \otimes \mathbb{C}[S^{-1}] \to \pi_n(X^{\mathcal{T}_{\mathcal{D}_n} S}) \to D_S T_S(\pi_n X) \to 0.$$

Corollary 5.1. *Suppose that B is an abelian group and $K(B,q)$ the Eilenberg-Mac Lane space at dimension $q > 1$. Then $K(B,n)^{\mathcal{T}_n S}$ is an Eilenberg-Mac Lane space such that $\pi_n(K(B,n)^{\mathcal{T}_n S}) \cong \mathrm{Tor}(\pi_n X, \mathbb{C}[S^{-1}])$. For $m > n$, $K(B,m)^{\mathcal{T}_n S}$ has two possible non trivial homotopy groups*

$$\pi_{m-1}(K(B,m)^{\mathcal{T}_n S}) \cong B \otimes \mathbb{C}[S^{-1}],$$

$$\pi_m(K(B,m)^{\mathcal{T}_n S}) \cong \mathrm{Tor}(B, \mathbb{C}[S^{-1}]).$$

Corollary 5.2. *Suppose that B is an abelian group and $K(B,q)$ the Eilenberg-Mac Lane space at dimension $q > 1$. Then $K(B,n)^{\mathcal{T}_{\mathcal{D}_n} S}$ is an Eilenberg-Mac Lane space such that $\pi_n(K(B,n)^{\mathcal{T}_{\mathcal{D}_n} S}) \cong D_S T_S(B)$. For $m > n$, $K(B,m)^{\mathcal{T}_{\mathcal{D}_n} S}$ has two possible non trivial homotopy groups*

$$\pi_{m-1}(K(B,m)^{\mathcal{T}_{\mathcal{D}_n} S}) \cong B \otimes \mathbb{C}[S^{-1}],$$

$$\pi_m(K(B,m)^{\mathcal{T}_{\mathcal{D}_n} S}) \cong \mathrm{Tor}(B, \mathbb{C}[S^{-1}]).$$

6. Homology of $\mathcal{T}_n S$-colocalizations and $\mathcal{T}_{\mathcal{D}_n} S$-colocalizations

In all this section we suppose that $n > 1$ and S will be a closed multiplicative system. We shall consider the Serre spectral sequence of a fibre map in order to study some properties of the homology of $\mathcal{T}_n S$-colocalizations and $\mathcal{T}_{\mathcal{D}_n} S$-colocalizations.

Proposition 6.1. *Let $X^{\mathcal{T}_n S}$, $X^{\mathcal{T}_{\mathcal{D}_n} S}$ be the corresponding colocalizations of a space X , then*

(i) *For every integer q we have*

$$\tilde{H}_q(X^{\mathcal{T}_n S};\mathbb{Z}) \cong \tilde{H}_{q+1}(X^{\mathcal{T}_n S};\mathbb{C}[S^{-1}])$$
$$\tilde{H}_q(X^{\mathcal{T}_{\mathcal{D}_n} S};\mathbb{Z}) \cong \tilde{H}_{q+1}(X^{\mathcal{T}_{\mathcal{D}_n} S};\mathbb{C}[S^{-1}])$$

(ii) *For every $s \in S$ one has*

$$H_n(X^{\mathcal{T}_n S};\mathbb{Z}/s) \cong T_S(\pi_n X) \otimes \mathbb{Z}/s,$$

and for any S-torsion abelian group T :

$$\tilde{H}_q(X^{\mathcal{T}_{\mathcal{D}_n} S};T) \cong 0 \ for \ q \leq n.$$

Proof. (i) follows from the exact sequence of homology groups induced by the short exact sequence of coefficients $0 \to \mathbb{Z} \to \mathbb{Z}[S^{-1}] \to \mathbb{C}[S^{-1}] \to 0$.

For (ii), we can apply the Hurewicz Theorem and Theorem 5.2. For the second isomorphism, recall that by Theorem 4.1 we have that for any $s \in S$, $H_n(X^{\mathcal{T}_{\mathcal{D}_n} S};\mathbb{Z}/s) \cong 0$. For a given S-torsion group T we have a short exact sequence $0 \to A \to B \to T \to 0$ where B is a sum of groups of the form $\mathbb{Z}/s$. By [14], a subgroup of a sum of cyclic groups is also a sum of cyclic groups, then one has A is a sum of cyclic groups and because B is S-torsion A is also a sum of groups of the form $\mathbb{Z}/s$. From the exact sequence induced by the short exact sequence of coefficients above we also have that for any S-torsion group T , $H_n(X^{\mathcal{T}_n S};T) \cong 0$.

□

Proposition 6.2. *Suppose that X is a 1-connected space. Then we have:*
(i) *If $\pi_k X$ is S-uniquely divisible for $k \leq n-1$ and $\pi_n X$ is S-adjusted, then $H_q(X^{\mathcal{T}_n S};T) \cong H_q(X;T)$ for any S-torsion abelian group T and for $q > 0$ the following sequence is exact*

$$0 \to H_{q+1}X \otimes \mathbb{C}[S^{-1}] \to H_q(X^{\mathcal{T}_n S}) \to \mathrm{Tor}(H_q X, \mathbb{C}[S^{-1}]) \to 0 \ .$$

In particular, if $H_q X$ is S-torsion-free, then $H_q(X^{\mathcal{T}_n S}) \cong H_{q+1}X \otimes \mathbb{C}[S^{-1}]$, and if $H_{q+1}X$ is S-adjusted, then $H_q(X^{\mathcal{T}_n S}) \cong \mathrm{Tor}(H_q X, \mathbb{C}[S^{-1}])$.
(ii) *If $\pi_k X$ is S-uniquely divisible for $k \leq n-1$ and $\pi_n X$ is S-divisible, then $H_q(X^{\mathcal{T}_{\mathcal{D}_n} S};T) \cong H_q(X;T)$ for any S-torsion abelian group T and for $q > 0$ he following sequence is exact*

$$0 \to H_{q+1}X \otimes \mathbb{C}[S^{-1}] \to H_q(X^{\mathcal{T}_{\mathcal{D}_n} S}) \to \mathrm{Tor}(H_q X, \mathbb{C}[S^{-1}]) \to 0 \ .$$

Therefore, if $H_q X$ is S-torsion-free, then $H_q(X^{\mathcal{T}_{\mathcal{D}_n} S}) \cong H_{q+1}X \otimes \mathbb{C}[S^{-1}]$, and if $H_{q+1}X$ is S-adjusted, then $H_q(X^{\mathcal{T}_{\mathcal{D}_n} S}) \cong \mathrm{Tor}(H_q X, \mathbb{C}[S^{-1}])$.

Proof. For each space X we have the fibre sequence $\Omega(X[S^{-1}]) \to FX \to X$. Since X is 1-connected, the homotopy groups and the reduced homology groups of $\Omega(X[S^{-1}])$ are S-uniquely divisible, then for any S-torsion abelian group T, using the spectral sequence of a fibre map

$$E^2_{pq} = H_p(X; H_q(\Omega(X[S^{-1}]);T))$$

we have that $H_k(FX;T) \cong H_k(X;H_0(\Omega X[S^{-1}];T)) \cong H_k(X;T)$ for all k . Under the conditions of (i), one has that $FX \cong X^{\mathcal{T}_nS}$ and under conditions of (ii) $FX \cong X^{\mathcal{T}_{\mathcal{D}_n}S}$. If we take $T = \mathbb{C}[S^{-1}]$ by Proposition 6.1 and the formula of universal coefficients we obtain the short exact sequences of (i) and (ii). □

Proposition 6.3. *Let $f\colon X \to Y$ be a map between 1-connected spaces, then we have:*

(i) *Suppose that $\pi_k X$, $\pi_k Y$ for $k \leq n-1$ are S-uniquely divisible and $\pi_n X$, $\pi_n Y$ are S-adjusted. If $H_q(f, \mathbb{Z}/s)$ is an isomorphism for every $q \geq n$ and $s \in S$, then f is a $\mathcal{T}_nS$-weak equivalence.*

(ii) *Suppose that $\pi_k X$, $\pi_k Y$ are S-uniquely divisible for $k \leq n-1$, and $\pi_n X$, $\pi_n Y$ are S-divisible. If $H_q(f, \mathbb{C}[S^{-1}])$ is an isomorphism for every $q \geq n+1$, then f is a $\mathcal{T}_{\mathcal{D}_n}S$-weak equivalence.*

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} X^{\mathcal{T}_nS} & \xrightarrow{p^X} & X \\ {\scriptstyle f^{\mathcal{T}_nS}}\downarrow & & \downarrow{\scriptstyle f} \\ Y^{\mathcal{T}_nS} & \xrightarrow{p^Y} & Y \end{array}$$

By Proposition 6.2 , p^X and p^Y induce isomorphism on the homology groups with coefficients in any S-torsion group. Because f also induce isomorphism $H_q(f, \mathbb{Z}/s)$ for every $q \geq n$ and $s \in S$, it follows that $f^{\mathcal{T}_nS}$ induces isomorphism $H_q(f^{\mathcal{T}_nS}, \mathbb{Z}/s)$ for every $q \geq n$ and $s \in S$. Therefore $f^{\mathcal{T}_nS}$ induces isomorphism $H_q(f^{\mathcal{T}_nS}, \mathbb{C}[S^{-1}])$ for every $q \geq n+1$. By Proposition 6.1 , we have that $f^{\mathcal{T}_nS}$ induces isomorphism on homology with coefficients in $\mathbb{Z}$. Since $X^{\mathcal{T}_nS}$, $Z^{\mathcal{T}_nS}$ are 1-connected space we have that $f^{\mathcal{T}_nS}$ is a homotopy equivalence. Taking into account that p^X , p^Y and $f^{\mathcal{T}_nS}$ are $\mathcal{T}_nS$-weak equivalences, one has that f is also a $\mathcal{T}_nS$-weak equivalence. For the case (ii) the proof is similar.

□

Corollary 6.1. *Suppose that B is an abelian group and $M(B,q)$ the Moore space at degree $q>1$. Then for $m>n$, $M(B,m)^{\mathcal{T}_nS}$ has two possible non trivial reduced homology groups*

$$H_{m-1}(M(B,m)^{\mathcal{T}_nS}) \cong B \otimes \mathbb{C}[S^{-1}],$$
$$H_m(M(B,m)^{\mathcal{T}_nS}) \cong \mathrm{Tor}(B, \mathbb{C}[S^{-1}]).$$

7. The categories $\mathbf{Ho}(\mathcal{T}_nS\mathbf{-Top}^*)$ and $\mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S\mathbf{-Top}^*)$

In this section, we compare the closed model categories induced by the families $\mathcal{T}_nS$, $\mathcal{T}_{\mathcal{D}_n}S$ and the standard closed model category of pointed spaces. Notice that there is no problem if we assume that S is generated by a set of primes. As usual we suppose that in this section $n>1$.

It is interesting to note the existence of short exact sequences

$$0 \to A \to B \to \mathbb{C}[S^{-1}] \to 0$$

$$0 \to \mathbb{Z}/s \to \mathbb{C}[S^{-1}] \to \mathbb{C}[S^{-1}] \to 0$$

where $s \in S$ and A , B are direct sums of subgroups of the form $\mathbb{Z}/s$ with $s \in S$.

From the exact sequences of the homotopy groups with coefficients in the abelian groups in sequences above, it follows that the weak equivalences of the corresponding closed model categories satisfy the following relations:

$\mathcal{T}_{\mathcal{D}_n}S$-w.e. $\supset$ $\mathcal{T}_{n+1}S$-w.e. $\supset$ $\mathcal{T}_{\mathcal{D}_n}S$-w.e. $\supset$ $\mathcal{T}_nS$-w.e. $\supset$ w.e. ,

and for the classes of cofibrations (and cofibrant spaces) one has:

$\mathcal{T}_{\mathcal{D}_{n+1}}S$-cof. $\subset$ $\mathcal{T}_{n+1}S$-cof. $\subset$ $\mathcal{T}_{\mathcal{D}_n}S$-cof. $\subset$ $\mathcal{T}_nS$-c. $\subset$ cof. .

Now using the functors $(-)^{\mathcal{T}_nS}$, $(-)^{\mathcal{T}_{\mathcal{D}_n}S}\colon \mathbf{Top}^* \longrightarrow \mathbf{Top}^*$ given in Definition 1.2, we have:

Theorem 7.1. (i) *There exist the following pairs of adjoint functors*

$$\mathbf{Ho}(\mathcal{T}_nS{-}\mathbf{Top}^*) \underset{Id}{\overset{(\ )^{\mathcal{T}_nS}}{\rightleftarrows}} \mathbf{Ho}(\mathrm{Q}{-}\mathbf{Top}^*)$$

$$\mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*) \underset{Id}{\overset{(\ )^{\mathcal{T}_{\mathcal{D}_n}S}}{\rightleftarrows}} \mathbf{Ho}(\mathrm{Q}{-}\mathbf{Top}^*)$$

$$\mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*) \underset{Id}{\overset{(\ )^{\mathcal{T}_{\mathcal{D}_n}S}}{\rightleftarrows}} \mathbf{Ho}(\mathcal{T}_nS{-}\mathbf{Top}^*)$$

$$\mathbf{Ho}(\mathcal{T}_{n+1}S{-}\mathbf{Top}^*) \underset{Id}{\overset{(\ )^{\mathcal{T}_{n+1}S}}{\rightleftarrows}} \mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*)$$

$$\mathbf{Ho}(\mathcal{T}_{n+1}S{-}\mathbf{Top}^*) \underset{Id}{\overset{(\ )^{\mathcal{T}_{n+1}S}}{\rightleftarrows}} \mathbf{Ho}(\mathcal{T}_nS{-}\mathbf{Top}^*)$$

$$\mathbf{Ho}(\mathcal{T}_{\mathcal{D}_{n+1}}S{-}\mathbf{Top}^*) \underset{Id}{\overset{(\ )^{\mathcal{T}_{\mathcal{D}_{n+1}}S}}{\rightleftarrows}} \mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}\mathrm{S}{-}\mathbf{Top}^*)$$

where the upper arrows are always left adjoint functors.

(ii) *The following restrictions*

$$\mathbf{Ho}(\mathcal{T}_nS{-}\mathbf{Top}^*) \underset{Id}{\overset{(\ )^{\mathcal{T}_nS}}{\rightleftarrows}} \mathbf{Ho}(Q{-}\mathbf{Top}^*)|_{\mathcal{T}_nS-cof}$$

$$\mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*) \underset{Id}{\overset{(\ )^{\mathcal{T}_{\mathcal{D}_n}S}}{\rightleftarrows}} \mathbf{Ho}(Q{-}\mathbf{Top}^*)|_{\mathcal{T}_{\mathcal{D}_n}S-cof}$$

are equivalence of categories, where $\mathbf{Ho}(Q{-}\mathbf{Top}^*)|_{\mathcal{T}_nS-cof}$, $\mathbf{Ho}(Q{-}\mathbf{Top}^*)|_{\mathcal{T}_{\mathcal{D}_n}S-cof}$ *are the full subcategories determined by the corresponding cofibrant spaces.*

Proof. It suffices to check that the units and the counits of the adjunctions are isomorphism. □

Remark 7.1. *(i) The family of functors* $(\ )^{\mathcal{T}_n S}\colon \mathbf{Ho}(\mathcal{T}_n S{-}\mathbf{Top}^*) \longrightarrow \mathbf{Ho}(\mathrm{Q}{-}\mathbf{Top}^*)$ *give for each space* X *a tower of fibrations:*

$$\cdots \to ((X^{\mathcal{T}_1 S})^{\mathcal{T}_2 S})^{\mathcal{T}_3 S} \to (X^{\mathcal{T}_1 S})^{\mathcal{T}_2 S} \to X^{\mathcal{T}_1 S} \to X$$

We note that $(X^{\mathcal{T}_1 S})^{\mathcal{T}_2 S}$ *is isomorphic to* $X^{\mathcal{T}_2 S}$ *in* $\mathbf{Ho}(Q{-}\mathbf{Top}^*)$ *. Therefore the tower of fibrations above will be written as* $\cdots \to X^{\mathcal{T}_3 S} \to X^{\mathcal{T}_2 S} \to X^{\mathcal{T}_1 S} \to X$ *. We have that for* $q \geq 3$ *the fibre of* $X^{\mathcal{T}_{q+1} S} \to X^{\mathcal{T}_q S}$ *has only possible non trivial homotopy groups with coefficients in* $\mathbb{Z}/s$ *in degrees* q *,* $q-1$ *and* $q-2$ *, for all* $s \in S$ *.*

(ii) For family of functors $(\ )^{\mathcal{T}_{\mathcal{D}_n} S}$ *we have a similar tower of fibrations:*

$$\cdots \to ((X^{\mathcal{T}_{\mathcal{D}_1} S})^{\mathcal{T}_{\mathcal{D}_2} S})^{\mathcal{T}_{\mathcal{D}_3} S} \to (X^{\mathcal{T}_{\mathcal{D}_1} S})^{\mathcal{T}_{\mathcal{D}_2} S} \to X^{\mathcal{T}_{\mathcal{D}_1} S} \to X$$

that as above will be denoted by $\cdots \to X^{\mathcal{T}_{\mathcal{D}_3} S} \to X^{\mathcal{T}_{\mathcal{D}_2} S} \to X^{\mathcal{T}_{\mathcal{D}_1} S} \to X$ *. The fibres have a similar property for homotopy groups with coefficients in* $\mathbb{C}[S^{-1}]$ *.*

(iii) We can combine both kind of functor to obtain a tower of fibrations of the form

$$\cdots \to X^{\mathcal{T}_{\mathcal{D}_3} S} \to X^{\mathcal{T}_3 S} \to X^{\mathcal{T}_{\mathcal{D}_2} S} \to X^{\mathcal{T}_2 S} \to X^{\mathcal{T}_{\mathcal{D}_1} S} \to X^{\mathcal{T}_1 S} \to X$$

In this case, we can see that for higher degrees the fibre of $X^{\mathcal{T}_{q+1} S} \to X^{\mathcal{T}_{\mathcal{D}_q} S}$ *and the fibre of* $X^{\mathcal{T}_{\mathcal{D}_q} S} \to X^{\mathcal{T}_q S}$ *reduces the number of possible non trivial homotopy groups with coefficients.*

Proposition 7.1. *Let* S *be a closed multiplicative system and suppose that* $n > 1$*, then*
(i) *the homotopy fibre of the canonical map* $X^{\mathcal{T}_{\mathcal{D}_n} S} \to X^{\mathcal{T}_n S}$ *is an Eilenberg Mac Lane space of type* $K(T_S(\pi_n X)/D_S T_S(\pi_n X), n-1)$*,*
(ii) *the homotopy fibre of the canonical map* $X^{\mathcal{T}_{n+1} S} \to X^{\mathcal{T}_{\mathcal{D}_n} S}$ *is an Eilenberg Mac Lane space of type* $K(\pi_n X^{\mathcal{T}_{\mathcal{D}_n} S}, n-1)$*.*

Proof. It follows from the formulas given in Theorem 5.2 and Theorem 5.3 . □

Remark 7.2. *(i) Note that for* $n \geq 3$ *the homotopy fibre* F *of* $X^{\mathcal{T}_{\mathcal{D}_n} S} \to X^{\mathcal{T}_n S}$ *has only one non trivial homotopy group with coefficients in* $\mathbb{C}[S^{-1}]$ *:*

$$\pi_{n-2}(\mathbb{C}[S^{-1}]; F) \cong \mathrm{Ext}(\mathbb{C}[S^{-1}], T_S D_S(\pi_n X))$$

(ii) For $n \geq 2$ *and* $s \in S$ *, the homotopy fibre* F' *of* $X^{\mathcal{T}_{n+1} S} \to X^{\mathcal{T}_{\mathcal{D}_n} S}$ *has one non trivial homotopy group:*

$$\pi_{n-1}(\mathbb{Z}/s; F') \cong \mathrm{Hom}(\mathbb{Z}/s, \pi_{n-1} F')$$

and only one non trivial homotopy group with coefficients in $\mathbb{C}[S^{-1}]$ *:*

$$\pi_{n-1}(\mathbb{C}[S^{-1}]; F') \cong \mathrm{Hom}(\mathbb{C}[S^{-1}], \pi_{n-1} F')$$

8. The categories $\mathbf{Ho}(\mathcal{T}_nS{-}\mathbf{Top}^*)$ and $\mathbf{Ho}(\mathcal{T}_nS^p{-}\mathbf{Top}^*)$

In this section, we suppose that S is generated by a set of positive primes P and S^p is the multiplicative closed system generated by a positive prime $p \in P$. For $n > 1$ we analyze some relations between the closed model categories induced by the families $\mathcal{T}_nS$, $\mathcal{T}_nS^p$.

We note that the classes of distinguished maps satisfy the following relations:

$\mathcal{T}_nS$-w.e. $\supset \mathcal{T}_nS^p$-w.e.,

and for the classes of cofibrations (and cofibrant spaces) one has:

$\mathcal{T}_nS^p$-cof. $\subset \mathcal{T}_nS$-cof.

Then, one has the following pair of adjoint functors

$$\mathbf{Ho}(\mathcal{T}_nS^p{-}\mathbf{Top}^*)\underset{Id}{\overset{(\ )^{\mathcal{T}_nS^p}}{\rightleftarrows}}\mathbf{Ho}(\mathcal{T}_nS{-}\mathbf{Top}^*)$$

On the other hand if we consider the family $\mathcal{T}_nS^p{-}\mathbf{Top}^*$, $p \in P$, of closed models categories, we can take the product of these closed model structures $\prod_{p\in P}\mathcal{T}_nS^p{-}\mathbf{Top}^*$ and the localized category $\mathbf{Ho}(\prod_{p\in P}\mathcal{T}_nS^p{-}\mathbf{Top}^*)$ which is equivalent to $\prod_{p\in P}\mathbf{Ho}(\mathcal{T}_nS^p{-}\mathbf{Top}^*)$. The functor

$$\Delta\colon \mathcal{T}_nS{-}\mathbf{Top}^* \longrightarrow \prod_{p\in P}\mathcal{T}_nS^p{-}\mathbf{Top}^* ,$$

given by $\Delta X = (X)_{p\in P}$, is right adjoint to

$$W\colon \prod_{p\in P}\mathcal{T}_nS^p{-}\mathbf{Top}^* \longrightarrow \mathcal{T}_nS{-}\mathbf{Top}^* ,$$

$W(Y_p)_{p\in P} = \bigvee_{p\in P} Y_p$.

It is easy to check that Δ preserves weak equivalences and fibrations. To check that W carries weak equivalences between cofibrant objects into weak equivalences, suppose that $f_p\colon X_p \to Y_p$ is a weak equivalence in $(\mathcal{T}_nS^p{-}\mathbf{Top}^*)_{cof}$ for each $p \in P$. By Theorem 2.3 we have that for every $p \in P$, f_p is a pointed homotopy equivalence. Then $\bigvee_p f_p$ is also a pointed homotopy equivalence and it follows that $\bigvee_p f_p$ is a $\mathcal{T}_nS$-weak equivalence. Thus one has an induced adjunction (equivalence) on the localized categories:

Theorem 8.1. *For $n > 1$, the induced adjunction*

$$\prod_{p\in P}\mathbf{Ho}(\mathcal{T}_nS^p{-}\mathbf{Top}^*)\underset{\Delta}{\overset{W^L}{\rightleftarrows}}\mathbf{Ho}(\mathcal{T}_nS{-}\mathbf{Top}^*)$$

$$W^L(X_p)_{p\in P} = \bigvee_{p\in P} X_p^{\mathcal{T}_nS^p}$$

gives an equivalence of categories.

Proof. Using the functors $\tilde{H}_*(-;T)$ where T is either an S-torsion group or a p-torsion group with $p \in P$, one can check that the unit and the counit of the adjunction are weak equivalences. □

Corollary 8.1. *The homotopy category of torsion* 1*-connected* CW*-complexes is equivalent to the product of the homotopy categories of* 1*-connected* p*-torsion* CW*-complexes where* p *ranges on the set of positive primes.*

Corollary 8.2. *A torsion* 1*-connected space* X *is weakly equivalent to the wedge* $\bigvee_p X^{\mathcal{T}_2 S^p}$ *.*

On the other hand, the functor Δ , is left adjoint to

$$P\colon \prod_{p\in P} \mathcal{T}_n S^p - \mathbf{Top}^* \longrightarrow \mathcal{T}_n S - \mathbf{Top}^* ,$$

$P(Y_p)_{p\in P} = \prod_{p\in P} Y_p$.

We can check that P carries weak equivalences between cofibrant objects into weak equivalences as follows: Suppose that $f_p\colon X_p \to Y_p$ is a weak equivalence in $(\mathcal{T}_n S^p - \mathbf{Top}^*)_{\text{cof}}$ for each $p \in P$. By Theorem 2.3 , we have that each f_p is a pointed homotopy equivalence in $\mathbf{Top}^*$. Then $\prod_p f_p$ is a pointed homotopy equivalence. Therefore $P(f_p)$ is an $\mathcal{T}_n S$-weak equivalence. Thus one has an induced functor $P^L\colon \prod_{p\in P} \mathcal{T}_n S^p - \mathbf{Top}^* \longrightarrow \mathcal{T}_n S - \mathbf{Top}^*$ and one has:

Proposition 8.1. *For* $n > 1$ *, the pair of functors*

$$\prod_{p\in P} \mathbf{Ho}(\mathcal{T}_n S^p - \mathbf{Top}^*) \underset{\Delta}{\overset{P^L}{\rightleftarrows}} \mathbf{Ho}(\mathcal{T}_n S - \mathbf{Top}^*)$$

$$P^L(X_p)_{p\in P} = \prod_{p\in P} X_p^{\mathcal{T}_n S^p}$$

gives an equivalence of categories.

Proof. Using the universal property of the map $X^{\mathcal{T}_n S} \to X$ one has induced maps $X^{\mathcal{T}_n S} \to X^{\mathcal{T}_n S^p}$, $X^{\mathcal{T}_n S} \to \prod_{p\in P} X^{\mathcal{T}_n S^p}$. Note that the maps $X^{\mathcal{T}_n S} \to X$, $X^{\mathcal{T}_n S} \to \prod_{p\in P} X^{\mathcal{T}_n S^p}$ are $\mathcal{T}_n S$-weak equivalences, then $P^L\Delta$ is isomorphic to the identity functor. On the other hand, each projection $\prod_{p\in P} X_p^{\mathcal{T}_n S^p} \to X_p^{\mathcal{T}_n S^p}$ is a $\mathcal{T}_n S^p$-weak equivalence. Therefore ΔP^L is also isomorphic to the identity functor and we have an equivalence of categories.

□

Corollary 8.3. *For any space* X *and* $n > 1$ *, the inclusion* $\bigvee_{p\in P} X_p^{\mathcal{T}_n S^p} \to \prod_{p\in P} X_p^{\mathcal{T}_n S^p}$ *is an* $\mathcal{T}_n S$*-weak equivalence. Moreover, if for each* $k \geq n$ *,* $\pi_k X$ *has finitely many non trivial torsion components, and for each* $k \geq n+1$ $\pi_k X$ *is* p*-divisible except for finitely many primes* p *, then the inclusion above is a weak equivalence.*

Proof. For $k \geq n$ we can use the formula

$$0 \to \pi_{k+1}X \otimes \mathbb{C}[(S^p)^{-1}] \to \pi_k(X^{\mathcal{T}_n S^p}) \to \mathrm{Tor}(\pi_k X, \mathbb{C}[(S^p)^{-1}]) \to 0$$

to prove that under that conditions on the homotopy groups of X one has that $\prod_{p \in P} X_p^{\mathcal{T}_n S^p}$ has S-torsion homotopy groups. Now the result follows from Theorem 2.3 . □

Remark 8.1. *The results given in Corollary 8.2 or in Corollary 8.3 for the case of of X a 1-connected CW-complex with finitely generated torsion homotopy groups can be obtained from the fracture lemma, see 6.3 of ch V in* [1] *, or from the Pullback Theorem given in* [16] *.*

9. S-torsion and Ext-S-complete spaces

In this section, for a space X, we consider the ring $R = \mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_r$, where $p_1, \cdots p_r$ are primes, and the R-localization $X \to R_\infty X$ given by Bousfield-Kan [1], see also [2] . Through all this section we assume that $n > 1$ and the closed multiplicative system S is generated by a finite set of primes $p_1, \cdots p_r$.

Recall that an abelian group B is said to be Ext-S-complete if the extension group $\mathrm{Ext}(\mathbb{C}[S^{-1}], B) \cong B$. Note that

$$\mathrm{Ext}(\mathbb{C}[S^{-1}], B) \cong \mathrm{Ext}(\mathbb{C}[\frac{1}{p_1}], B) \times \cdots \times \mathrm{Ext}(\mathbb{C}[\frac{1}{p_r}], B) .$$

Definition 9.1. *A 1-connected space Y is said to be Ext-S-complete if its homotopy groups are Ext-S-complete.*

Applying the universal properties of the constructions $R_\infty X$ and $(-)^{\mathcal{T}_{\mathcal{D}_n} S}$, one has:

Theorem 9.1. *The left derived functor*
$R_\infty^L \colon \mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n} S{-}\mathbf{Top}^*) \longrightarrow \mathbf{Ho}$*(Ext-$S$-complete n-connected spaces)*
is left adjoint to
$(\ )^{\mathcal{T}_{\mathcal{D}_n} S} \colon \mathbf{Ho}$*(Ext-$S$-complete n-connected spaces)* $\longrightarrow \mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n} S{-}\mathbf{Top}^*)$.
Moreover, the pair of functors above give an equivalence categories.

Proof. For the case of one prime p , the homotopy groups of $(\mathbb{Z}/p)_\infty X$ are given by the exact sequence:

$$0 \to \mathrm{Ext}(\mathbb{C}[\frac{1}{p}], \pi_k X) \to \pi_k(\mathbb{Z}/p)_\infty X) \to \mathrm{Hom}(\mathbb{C}[\frac{1}{p}], \pi_{k-1} X) \to 0$$

If S is generated by finite primes $p_1, \cdots p_r$ using the formulas
$(\mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_r)_\infty X \cong (\mathbb{Z}/p_1)_\infty X \times \cdots \times (\mathbb{Z}/p_r)_\infty X$
$\pi_k(R)_\infty X \cong \pi_k((\mathbb{Z}/p_1)_\infty X) \times \cdots \times \pi_k((\mathbb{Z}/p_r)_\infty X)$
$\mathrm{Ext}(\mathbb{C}[S^{-1}], \pi) \cong \mathrm{Ext}(\mathbb{C}[\frac{1}{p_1}], \pi) \times \cdots \times \mathrm{Ext}(\mathbb{C}[\frac{1}{p_r}], \pi)$
$\mathrm{Hom}(\mathbb{C}[S^{-1}], \pi) \cong \mathrm{Hom}(\mathbb{C}[\frac{1}{p_1}], \pi) \times \cdots \times \mathrm{Hom}(\mathbb{C}[\frac{1}{p_r}], \pi)$
we have a similar formula for $R_\infty X$

$$0 \to \mathrm{Ext}(\mathbb{C}[S^{-1}], \pi_k X) \to \pi_k R_\infty X \to \mathrm{Hom}(\mathbb{C}[S^{-1}], \pi_{k-1} X) \to 0$$

If X is an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space, we have that $\pi_n X$ is S-divisible, then $\pi_n R_\infty X \cong \mathrm{Ext}(\mathbb{C}[S^{-1}], \pi_n X) \cong 0$ and $R_\infty X$ is an Ext-S-complete n-connected space. On the other hand, if Y is an Ext-S-complete n-connected space, then $Y^{\mathcal{T}_{\mathcal{D}_n}S}$ is an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space.

By the universal properties of the constructions R^L_∞ and $(-)^{\mathcal{T}_{\mathcal{D}_n}S}$ we have that on the homotopy categories R^L_∞ is left adjoint to $(-)^{\mathcal{T}_{\mathcal{D}_n}S}$.

The unit of the adjunction in contained in the commutative diagram

$$\begin{array}{ccc} & & (R_\infty X)^{\mathcal{T}_{\mathcal{D}_n}S} \\ & \nearrow & \downarrow \\ X & \longrightarrow & R_\infty X \end{array}$$

where we have supposed that X is $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant. The localization $X \to R_\infty X$ induces isomorphism on singular homology with coefficients in $\mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_r$. Since $R_\infty X$ is n-connected we apply Proposition 6.2 to obtain that $(R_\infty X)^{\mathcal{T}_{\mathcal{D}_n}S} \to R_\infty X$ induces isomorphism on singular homology with coefficients in $\mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_r$. Therefore the unit $X \to (R_\infty X)^{\mathcal{T}_{\mathcal{D}_n}S}$ induces isomorphism on homology with coefficients in $\mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_r$. From this fact we also have that the unit induce isomorphism on homology with coefficients in every S-torsion group. By Proposition 6.3 , because X and $(R_\infty X)^{\mathcal{T}_{\mathcal{D}_n}S}$ are 1-connected spaces, one has that the unit is a homotopy equivalence.

On the other hand, for the counit of the adjunction we have the commutative diagram

$$\begin{array}{ccc} Y^{\mathcal{T}_{\mathcal{D}_n}S} & \longrightarrow & Y \\ \downarrow & \nearrow & \\ R_\infty(Y^{\mathcal{T}_{\mathcal{D}_n}S}) & & \end{array}$$

where Y is an Ext-S-complete n-connected space.

The localization $Y^{\mathcal{T}_{\mathcal{D}_n}S} \to R_\infty(Y^{\mathcal{T}_{\mathcal{D}_n}S})$ induces isomorphism on singular homology with coefficients in $\mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_r$. Since Y is n-connected , by Proposition 6.2 we have that $Y^{\mathcal{T}_{\mathcal{D}_n}S} \to Y$ induces isomorphism on singular homology with coefficients in $\mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_r$. Therefore the counit $R_\infty(Y^{\mathcal{T}_{\mathcal{D}_n}S}) \to Y$ induces isomorphism on homology with coefficients in $\mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_r$. Taking into account that $R_\infty(Y^{\mathcal{T}_{\mathcal{D}_n}S})$ and Y are Ext-S-complete spaces, it follows that the counit is a weak equivalence. □

Theorem 9.2. *Let X be an $(n-1)$-connected space with $\pi_n X$ an S-divisible group, then for $k > n$*

$$\pi_k R_\infty X \cong \pi_{k-1}(\mathbb{C}[S^{-1}]; X)$$

Moreover, if we also assume that for $k \geq n$ $\pi_k X$ an S-torsion group then one has the following exact sequence for $k > n$

$$0 \to \pi_k(\mathbb{C}[S^{-1}]; X) \otimes \mathbb{C}[S^{-1}] \to \pi_k X \to \mathrm{Tor}(\pi_{k-1}(\mathbb{C}[S^{-1}]; X), \mathbb{C}[S^{-1}]) \to 0$$

and for $k = n$ one has

$$\pi_n X \cong \pi_n(\mathbb{C}[S^{-1}]; X) \otimes \mathbb{C}[S^{-1}]$$

Proof. Notice that the maps $X \to R_\infty X$ and $S^k \to R_\infty S^k$ for $k > n$ induce isomorphism on singular homology with coefficients in every S-torsion abelian group. By Proposition 6.3 (ii) these maps are $\mathcal{T}_{\mathcal{D}_n}S$-weak equivalence, hence $X^{\mathcal{T}_{\mathcal{D}_n}S} \to (R_\infty X)^{\mathcal{T}_{\mathcal{D}_n}S}$, $(S^k)^{\mathcal{T}_{\mathcal{D}_n}S} \to (R_\infty S^k)^{\mathcal{T}_{\mathcal{D}_n}S}$ are homotopy equivalences. Using Proposition 6.2 (ii) it follows that $(S^k)^{\mathcal{T}_{\mathcal{D}_n}S}$ is a Moore space of type $M(\mathbb{C}[S^{-1}], k-1)$. Now one has the following isomorphism

$$\begin{aligned}
\pi_k R_\infty X &\cong \mathbf{Ho}(Q{-}\mathbf{Top}^*)(R_\infty S^k, R_\infty X)\\
&\cong \mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*)((R_\infty S^k)^{\mathcal{T}_{\mathcal{D}_n}S}, (R_\infty X)^{\mathcal{T}_{\mathcal{D}_n}S})\\
&\cong \mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*)(M(\mathbb{C}[S^{-1}], k-1), X^{\mathcal{T}_{\mathcal{D}_n}S})\\
&\cong \mathbf{Ho}(Q{-}\mathbf{Top}^*)(M(\mathbb{C}[S^{-1}], k-1), X^{\mathcal{T}_{\mathcal{D}_n}S})\\
&\cong \pi_{k-1}(\mathbb{C}[S^{-1}]; X^{\mathcal{T}_{\mathcal{D}_n}S})\\
&\cong \pi_{k-1}(\mathbb{C}[S^{-1}]; X)
\end{aligned}$$

where we have used the universal property of the localization maps, the equivalence of categories given in Theorem 9.1, the fact that $M(\mathbb{C}[S^{-1}], k-1)$ is $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant and $X^{\mathcal{T}_{\mathcal{D}_n}S}$ is $\mathcal{T}_{\mathcal{D}_n}S$-fibrant and finally that $X^{\mathcal{T}_{\mathcal{D}_n}S} \to X$ is an $\mathcal{T}_{\mathcal{D}_n}S$-weak equivalence.

For the second part of the theorem, we note that by by Theorem 4.1 X is weak equivalent to an $\mathcal{T}_{\mathcal{D}_n}S$-cofibrant space, then one has that $(R_\infty X)^{\mathcal{T}_{\mathcal{D}_n}S} \cong X$. Now the result follows from the exact sequences given in Theorem 5.3 . □

Corollary 9.1. *Let A be an S-torsion group,then then we have a splittable short sequence*

$$0 \to \mathrm{Hom}(\mathbb{C}[S^{-1}], A) \otimes \mathbb{C}[S^{-1}] \to A \to \mathrm{Tor}(\mathrm{Ext}(\mathbb{C}[S^{-1}], A), \mathbb{C}[S^{-1}]) \to 0$$

where $\mathrm{Hom}(\mathbb{C}[S^{-1}], A) \otimes \mathbb{C}[S^{-1}]$ *is isomorphic to $D_S A$ the maximal S-divisible subgroup of A and* $\mathrm{Tor}(\mathrm{Ext}(\mathbb{C}[S^{-1}], A), \mathbb{C}[S^{-1}])$ *has no S-divisible (non trivial) subgroups.*

On the other hand, if B is an Ext-S-complete group, then one has a splittable short sequence

$$0 \to \mathrm{Ext}(\mathbb{C}[S^{-1}], \mathrm{Tor}(B, \mathbb{C}[S^{-1}])) \to B \to \mathrm{Hom}(\mathbb{C}[S^{-1}], B \otimes \mathbb{C}[S^{-1}]) \to 0$$

where $\mathrm{Ext}(\mathbb{C}[S^{-1}], \mathrm{Tor}(B, \mathbb{C}[S^{-1}]))$ *is S-adjusted and S-complete, and* $\mathrm{Hom}(\mathbb{C}[S^{-1}], B \otimes \mathbb{C}[S^{-1}])$ *is S-torsion-free and S-complete.*

Proof. It suffices to consider the Eilenberg Mac Lane space $K(A, m)$ for an S-torsion group, with $m > n$ and compute the homotopy groups of $R_\infty K(A, m)$ and $(R_\infty K(A, m))^{\mathcal{T}_{\mathcal{D}_n}S}$.

For B an S-complete group we compute the homotopy groups of $K(A,m)^{\mathcal{T}_{\mathcal{D}_n}S}$ and $R_\infty(K(A,m)^{\mathcal{T}_{\mathcal{D}_n}S})$. □

Remark 9.1. *The formulas of Corollary above are well known. We refer the reader to [BK, ch VI] and Harrison [Ha]. Note that we consider the case that S is generated by a finite number of primes.*

Using the equivalence of categories

$$\mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S{-}\mathbf{Top}^*) \simeq \prod_{i=1}^{r}\mathbf{Ho}(\mathcal{T}_{\mathcal{D}_n}S^{p_i}{-}\mathbf{Top}^*)$$

and Theorem 9.1 , one can prove the following results:

Corollary 9.2. *The homotopy category of n-connected Ext-S-complete spaces is equivalent to the finite product of the homotopy categories of n-connected Ext-p_i-complete spaces for $i = 1\cdots r$.*

Corollary 9.3. *Let Y be an n-connected Ext-S-complete space, then*
(i) *Y is weakly equivalent to $(\mathbb{Z}/p_1)_\infty Y \times \cdots \times (\mathbb{Z}/p_r)_\infty Y$,*
(ii) *The inclusion*

$$(\mathbb{Z}/p_1)_\infty Y \bigvee \cdots \bigvee (\mathbb{Z}/p_r)_\infty Y \to (\mathbb{Z}/p_1)_\infty Y \times \cdots \times (\mathbb{Z}/p_r)_\infty Y \ ,$$

is a weak equivalence.

Remark 9.2. *(i) Note that the the category $\mathbf{Ho}(\mathcal{T}_nS{-}\mathbf{Top}^*)$ is equivalent to the homotopy category of $(n-1)$-connected Ext-S-complete spaces whose n^{th} homotopy group is S-adjusted.*

(ii) Given an Ext-S-complete abelian group B , there exist a unique Ext-S-complete space $C = C(\mathbb{C}[S^{-1}]; B, m)$ up to weak equivalence such that $\pi_m(\mathbb{C}[S^{-1}]; C) \cong B$ and for $k \neq m$, C has trivial homotopy groups with coefficients in $\mathbb{C}[S^{-1}]$. On the other hand, there exists a unique S-torsion space $T = T(\mathbb{C}[S^{-1}]; B, m)$ up to weak equivalence, such that with respect to homotopy groups with coefficients in $\mathbb{C}[S^{-1}]$, T is an Eilenberg Mac Lane space.

(iii) Recall that the standard homotopy groups are related with the homotopy groups with coefficients in $\mathbb{C}[S^{-1}]$ or in $\mathbb{Z}/s$ by the formulas:

$$0 \to \mathrm{Ext}(\mathbb{C}[S^{-1}], \pi_{k+1}X) \to \pi_k(\mathbb{C}[S^{-1}]; X) \to \mathrm{Hom}(\mathbb{C}[S^{-1}], \pi_k X) \to 0$$

$$0 \to \mathrm{Ext}(\mathbb{Z}/s, \pi_{k+1}X) \to \pi_k(\mathbb{Z}/s; X) \to \mathrm{Hom}(\mathbb{Z}/s, \pi_k X) \to 0$$

It is interesting to note that the homotopy groups with coefficients in $\mathbb{C}[\frac{1}{p}]$ and in $\mathbb{Z}/s$, with $s = p^l$ are related by the formulas

$$0 \to \lim_l{}^1 \pi_{k+1}(\mathbb{Z}/p^l; X) \to \pi_k(\mathbb{C}[\frac{1}{p}]; X) \to \lim_l \pi_k(\mathbb{Z}/p^l; X) \to 0$$

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^l, \pi_{k+1}(\mathbb{C}[\frac{1}{p}]; X)) \to \pi_k(\mathbb{Z}/p^l; X) \to \mathrm{Hom}(\mathbb{Z}/p^l, \pi_k(\mathbb{C}[\frac{1}{p}]; X)) \to 0.$$

If X is an 1-connected Ext-S-complete space one has

$$\pi_k X \cong \pi_{k-1}(\mathbb{C}[S^{-1}]; X).$$

Finally, if X is an 1-connected space and there exists a multiplicative system S generated by a finite set of primes such that $\pi_2 X$ is S-divisible and for $k \geq 2$ $\pi_k X$ is S-torsion, then the standard homotopy groups and the homotopy groups with coefficients in $\mathbb{C}[S^{-1}]$ are related by the short exact sequence for $k \geq 2$:

$$0 \to \pi_k(\mathbb{C}[S^{-1}]; X) \otimes \mathbb{C}[S^{-1}] \to \pi_k X \to \mathrm{Tor}(\pi_{k-1}(\mathbb{C}[S^{-1}]; X), \mathbb{C}[S^{-1}]) \to 0.$$

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