Applications of simplicial M-sets to proper and strong shape theories

by

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Abstract: In this paper we have tried to reduce the classical classification problems for spaces and maps of the proper category and of the strong shape category to similar problems in the homotopy category of simplicial sets or in the homotopy category of simplicial M-sets, where M is the monoid of proper selfmaps of the discrete space **N** of non negative integers.

Given a prospace (pro-simplicial set) Y, we have constructed a simplicial set $\bar{\mathcal{P}}^R Y$ such that the Hurewicz homotopy groups of $\bar{\mathcal{P}}^R Y$ are the Grossman homotopy groups of Y. For the case of the end prospace $Y = \varepsilon X$ of a space X, we obtain Brown's proper homotopy groups and for the Vietoris prospace $Y = VX$ (introduced by Porter) of a compact metrisable space X, we have Quigley's inward groups. The simplicial subset $\bar{\mathcal{P}}^R Y$ of a tower Y contains, as a simplicial subset, the homotopy limit $\lim^R Y$. The inclusion $\lim^R Y \to \bar{\mathcal{P}}^R Y$ induces many relations between the homotopy and (co)homology invariants of the prospace Y.

Using the functor $\bar{\mathcal{P}}^R$ we prove Whitehead Theorems for proper homotopy, prohomotopy and strong shape theories as a particular case of the standard Whitehead Theorem. The algebraic condition is given in terms of Brown's proper groups, Grossman's homotopy groups and Quigley's inward groups, respectively. In all these cases an equivalent cohomological condition can be given by taking twisted coefficients.

The "singular" homology groups of $\bar{\mathcal{P}}^R Y$ provide homology theories for the Brown, Grossman and Quigley homotopy groups that satisfy Hurewicz Theorems in the corresponding settings. However, there are other homology theories for the homotopy groups above satisfying other Hurewicz Theorems.

We also analyse the notion of $\bar{\mathcal{P}}$-movable prospace. For a $\bar{\mathcal{P}}$-movable tower we prove easily (without $\lim^1$ functors) that the strong homotopy groups agree with the Čech homotopy groups and the Grossman homotopy groups are determined by the Čech (or strong) groups by the formula ${}^G\pi_q = \bar{\mathcal{P}}\check{\pi}_q$. This implies that the algebraic condition of the Whitehead Theorem can be given in terms of strong (Čech) groups when the condition of $\bar{\mathcal{P}}$-movability is included.

We also study homology theories for the strong (Steenrod) homotopy groups which satisfy Hurewicz Theorems but in general do not agree with the corresponding Steenrod-Sitnikov homology theories.

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0. Introduction.

The main purpose of this paper is to reduce as far as possible the study of proper homotopy theory and strong shape theory to notions and problems of standard homotopy theory. To reach this aim we will use the following tools:

1) The notion of closed simplicial model category, introduced by Quillen [Q.1], will be used to work with categories obtained when one divides by homotopy relations or when one inverts weak equivalences.

2) We consider the category SS of simplicial sets and the category $S(S_M)$ of simplicial M–sets. An M–set is a set together with the action of a monoid M. Both categories SS, $S(S_M)$ are provided with structures of closed simplicial model categories.

3) One of the most useful categories to study proper and strong shape theory is the closed simplicial model category of prospaces.

For the proper category at infinity, the Edwards–Hastings functor ε gives a full embedding of the proper homotopy category at infinity of T_2 locally compact, σ–compact spaces into the localized category of prospaces obtained by inverting the weak equivalences considered by Edwards–Hastings in [E–H]. There is a similar version for global proper maps and homotopies by taking global (or augmented) prospaces. For strong shape theory, one can use the Vietoris functor V, introduced by Porter [P.1, P.5], that gives a full embedding of the strong shape category into the "homotopy category" of prospaces.

4) We define "singular" functors from proper categories and procategories to simplicial M–sets and realization functors from simplicial M–sets to the category of prospaces and from adequate full subcategories of simplicial M–sets to the category of spaces and proper maps.

One of the main results of this paper is the construction of a simplicial set $\bar{\mathcal{P}}^R X$, associated with a pro-simplicial set X, that retains many homotopy properties of X. The simplicial set $\bar{\mathcal{P}}^R X$ contains as a simplicial subset the homotopy limit, $\lim^R X$, of X.

The Hurewicz homotopy groups of $\lim^R Y$ are the strong homotopy groups of the prosimplicial set Y. For the case $Y = \varepsilon X$, one gets the strong groups in the proper setting and for the case $Y = VX$ one has the approaching groups of Quigley [Quig, P.6].

In this paper, we prove that the Hurewicz homotopy groups of $\bar{\mathcal{P}}^R Y$ are the Grossman homotopy groups of the prospace Y. For $Y = \varepsilon X$ one has Brown's proper homotopy groups and for $Y = VX$ we have the "inward" groups of Quigley [Quig, P.6].

In the proper setting, the simplicial set $\bar{\mathcal{P}}^R \varepsilon X$ can be interpreted as the simplicial set of sequences of singular q–simplexes converging to infinity. We can also look at $\bar{\mathcal{P}}^R \varepsilon X$ as the mapping space of sequences of points of X converging to infinity provided with an adequate topology. If X is a compact metrisable space, it can be considered up to homeomorphism as a subspace of the pseudointerior of the Hilbert cube Q. We can then interpret $\bar{\mathcal{P}}^R VX$ as the simplicial set of sequences of singular q–simplexes of Q converging to X. Nevertheless, it is also possible to interpret $\bar{\mathcal{P}}^R VX$ as the simplicial set of singular q–simplexes of $Q - X$ converging to X, which is the same as the simplicial set of singular q–simplexes of $Q - X$ converging to infinity. That is, $\bar{\mathcal{P}}^R VX$ is isomorphic to $\bar{\mathcal{P}}^R \varepsilon(Q - X)$ in the homotopy category.

This paper has been divided into 10 sections. The first part, sections 1 through 6, is devoted to developing the technical tools. Section 7 establishes the relationships between simplicial M–sets and simplicial complexes, where M is the monoid of proper selfmaps of the discrete space of natural numbers. The last part, sections 8, 9 and 10, contains applications to proper homotopy theory, prohomotopy theory and strong shape theory.

Section 2 is devoted to analysing the closed simplicial model structure of simplicial M–sets that will be used in this paper. In section 3, we analyse realization and singular functors for simplicial M–sets. Let Δ denote the standard category whose objects are finite ordered sets of the form $[q] = \{0 < 1 < \cdots < q\}$ and the morphisms are monotone maps. Let ${}_M\mathcal{C}$ be the category of left M–objects in $\mathcal{C}$. Associated with a functor $\chi\colon \Delta \longrightarrow {}_M\mathcal{C}$, we consider a realization functor $R_\chi\colon S(S_M) \longrightarrow \mathcal{C}$ and a singular functor $S_\chi\colon \mathcal{C} \longrightarrow S(S_M)$. The construction of the realization functor depends of the existence of colimits in $\mathcal{C}$. One of the categories $\mathcal{C}$ that we consider is the category Pro of spaces and proper maps. The category Pro has only some colimits and for this reason, in section 3 we include some lemmas about the existence of colimits.

If we consider the monoid $M = Pro(\mathbf{N}, \mathbf{N})$, we take as a "proper" q–simplex a space of the form $\mathbf{N} \times |\Delta[q]|$. Given a proper map $\varphi\colon \mathbf{N} \longrightarrow \mathbf{N}$, we can attach $\mathbf{N} \times |\partial_i \Delta[q]|$ to $\mathbf{N} \times |\Delta[q-1]|$ in such a way that $\{n\} \times |\partial_i \Delta[q]|$ is identified with $\{\varphi(n)\} \times |\Delta[q-1]|$. Therefore if we have a simplicial M–set N, whose monoid structure is freely generated by a finite number of simplexes, we can construct a space $R_p N$ taking a "proper" simplex associated with each generator of N and gluing the different "proper" simplexes in the way indicated above. The full subcategory of this kind of simplicial M–sets is denoted by $S(S_M/ff)/fd$ $\quad (ff =$ freely generated by a finite set, $fd =$ finite dimension). In general the realization of a simplicial map of $S(S_M)$ is not proper, but the realization of a simplicial map of $S(S_M)/ff)/fd$ is proper.

The main result of section 4 establishes that if N is an object of $S(S_M/ff))/fd$ which is cofibrant and Y is a topological space, then the set of proper homotopy classes $\pi_0(Pro)(R_p N, Y)$ is isomorphic to the hom–set $Ho\big(S(S_M)\big)(N, S_p Y)$. It is interesting to remark that in section 7 we have proved that a locally finite simplicial complex X which has finite dimension and a countably infinite number of simplexes is always of the form

$X \cong R_p N$.

For the category $proSS$, we think of as a q–simplex, the pro–simplicial set $c\Delta[q]$ which is defined by $c\Delta[q](i) = \sum_{j\geq i} \Delta[q]$, $i \in \mathbf{N}$. Similar notions of q–simplex are considered for global pro–simplicial sets and for the corresponding pointed cases. Associated with these q–simplexes, there are a realization functor $R_{\chi_\infty}\colon S(S_{M_\infty}) \longrightarrow proSS$ and a singular functor $S_{\chi_\infty}\colon proSS \longrightarrow S(S_{M_\infty})$.

In 1975, E.M. Brown [Br.1] defined the proper homotopy groups ${}^B\pi_q^\infty(X)$ of a σ–compact space X with a base ray. He also considered a functor $\bar{\mathcal{P}}\colon towGps \longrightarrow Gps$ which carries the tower of homotopy groups, $\pi_q\varepsilon X$, of a tower of neighbourhoods of X at infinity to the homotopy group ${}^B\pi_q^\infty(X)$. For the case $q = 0$, $\bar{\mathcal{P}}$ is a functor from $towSet_*$ to Set_*. Here we consider new versions of the $\bar{\mathcal{P}}$ functor which are of the form, $proSet \longrightarrow Set_{\mathcal{P}c*}$, $proSet_* \longrightarrow Set_{\mathcal{P}cS^0}$, $proGps \longrightarrow Gps_{\mathcal{P}c\mathbf{N}}$, etc. The new versions are provided with the additional structure of the action of a monoid ($\mathcal{P}c*$, $\mathcal{P}cS^0$) or a near–ring $\mathcal{P}c\mathbf{N}$.

Using the shorter notation, $S = Set$, $S_* = Set_*$, the functor $\mathcal{P}\colon proS \longrightarrow S_{\mathcal{P}c*}$ induces a functor $S\mathcal{P}\colon SproS \longrightarrow S(S_{\mathcal{P}c*})$ and we have the composite $proSS \xrightarrow{F} SproS \xrightarrow{S\mathcal{P}} S(S_{\mathcal{P}c*})$, where F is naturally defined. The main result of section 5 establishes that $S\mathcal{P}\ F$ is isomorphic to the "singular" functor S_{χ_∞}. That is, S_{χ_∞} is an extension of the $\mathcal{P}$ functor and, for this reason, the functor S_{χ_∞} is also denoted by $\mathcal{P}$.

The inverse limit functor $\lim\colon towSS \longrightarrow SS$ is related with the functor $\mathcal{P}\colon towSS \longrightarrow S(S_{\mathcal{P}c*})$ in the following way: There is an element sh in the monoid $\mathcal{P}c*$ such that the simplicial subset $F_{sh}X = \{x \in \mathcal{P}X \mid x\ sh = x\}$ of elements fixed by sh is isomorphic to $\lim X$, where X is an object in $towSS$.

Section 6 contains the main technical results of the paper. It is well known that the homotopy inverse limit, holim, can be defined as a right–derived functor of the lim functor. We prove that the functor $\mathcal{P}\colon proSS \longrightarrow S(S_{\mathcal{P}c*})$ has a right–derived functor $\mathcal{P}^R\colon Ho(proSS) \longrightarrow Ho\big(S(S_{\mathcal{P}c*})\big)$. The relation above between lim and $\mathcal{P}$ induces the formula $\lim^R X = F_{sh}\mathcal{P}^R X = \{x \in \mathcal{P}^R X \mid x\ sh = x\}$ where X is an object in $towSS$.

We summarize the results of section 6 saying that there is a pair of adjoint functors

$$Ho(proSS) \underset{\mathcal{P}^R}{\overset{\mathcal{L}^L}{\rightleftarrows}} Ho\big(S(S_{\mathcal{P}c*})\big).$$

that can be composed with the pair of adjoint functors (given in section 2)

$$Ho\big(S(S_{\mathcal{P}c*})\big) \underset{U}{\overset{-\otimes\mathcal{P}c*}{\rightleftarrows}} Ho(SS)$$

to obtain the new the pair

$$Ho(proSS) \underset{\bar{\mathcal{P}}^R}{\overset{\bar{\mathcal{L}}^L}{\rightleftarrows}} Ho(SS).$$

A first consequence of the existence of these pairs of adjoint functors is that the Hurewicz homotopy groups of $\bar{\mathcal{P}}^R X$ (X is an object in $proSS$) are isomorphic to the Grossman homotopy groups of X. We also prove that π_q and $\bar{\mathcal{P}}$ "commute"; that is, $\pi_q\ \bar{\mathcal{P}}^R\ X \cong \bar{\mathcal{P}}\ \pi_q\ X$. This proves Brown's result that $\bar{\mathcal{P}}$ carries the towers of homotopy groups, $\pi_q \varepsilon X$, to the proper homotopy groups, ${}^B\pi_q^\infty X$. As a second consequence, we will translate some classical theorems of standard homotopy theory to prohomotopy theory.

Section 8 is devoted to obtaining some applications to proper homotopy theory. We use the functors R_p and S_p to transform classical theorems of standard homotopy theory into similar theorems in the proper setting. We analyse two examples, in the first the proper Whitehead Theorem is proved as a particular case of the standard Whitehead Theorem. We also remark that the cohomological version of the Whitehead Theorem with twisted coefficients implies a cohomological version in the proper setting. In the second example, we see how the standard Hurewicz Theorem implies a Hurewicz Theorem in the proper category. This method also provides a proper homology theory for the Brown proper homotopy groups. However, there are other proper homology theories that also satisfy Hurewicz Theorems for the Brown proper groups, see Remark 3 after Theorem 7.3. In this section we also analyse the relation between the proper singular functor, the right–derived functor of the $\mathcal{P}$ functor and the Edwards–Hastings functor, ε. As a consequence of this relation, we obtain a partial version of the Edwards–Hastings embedding Theorem for the proper category.

In section 9, we have developed some applications to prohomotopy theory. We introduce the notion of $\mathcal{L}$–cofibrant prospace (or pro–simplicial set). The class of $\mathcal{L}$–cofibrant prospaces contains the end prospace εX of a finite dimensional simplicial complex and Porter's Vietoris prospace VX of a compact metrisable space X which has finite covering dimension. Therefore any result about $\mathcal{L}$–cofibrant prospaces has interpretations in the proper and in the strong shape settings. In this section, we establish a Whitehead Theorem for the class of $\mathcal{L}$–cofibrant prospaces. Using the functors $\mathcal{L}^L$ and $\mathcal{P}^R$ we obtain this result as a particular case of the standard Whitehead Theorem. The algebraic condition of the theorem is given in terms of the Grossman homotopy groups. There is also an equivalent cohomological condition.

We also introduce a notion of $\bar{\mathcal{P}}$–movability that in general is weaker than the notion of movable given in [E–H]. We give an easy proof (without using $\lim^1$ functors) that for $\bar{\mathcal{P}}$–movable towers, the strong (Steenrod) homotopy groups are isomorphic to the Čech homotopy groups and the Grossman groups are also determined by the formula ${}^G\pi_q = \bar{\mathcal{P}}({}^S\pi_q)$. Therefore for $\mathcal{L}$–cofibrant $\bar{\mathcal{P}}$–movable towers the algebraic condition of the Whitehead Theorem can be expressed in terms of strong (Steenrod) homotopy groups or Čech homotopy groups.

Section 10 is devoted to obtaining some applications to strong shape theory. Recall that the Grossman homotopy groups of the prospace VX are the Quigley [Quigl] inward groups, see also [P.6]. Using the functor $\bar{\mathcal{P}}^R$ the Quigley inward groups are interpreted as the Hurewicz homotopy groups of the simplicial set $\bar{\mathcal{P}}^R VX$. Therefore defining the homology groups of X as the "singular" homology of $\bar{\mathcal{P}}^R VX$, we obtain a homology theory that satisfies the Hurewicz Theorem for the Quigley inward groups. Nevertheless, in the Remark 4) after Definition 10.1, we suggest other homology theories for the Quigley inward groups. In general $H_q\ \bar{\mathcal{P}}^R VX$ is not isomorphic to $\bar{\mathcal{P}}\ H_q\ VX$, but there are other homology theories such that H "commutes" with the $\bar{\mathcal{P}}$ functor.

It is also known that the strong (Steenrod) homotopy groups of the prospace VX are the Quigley approaching groups. Using the functor $\mathcal{P}^R$, we have that the Quigley approaching groups of a compact metrisable space X are the Hurewicz homotopy groups of $F_{sh}\mathcal{P}^R VX$. Therefore we obtain a Hurewicz Theorem if we define the homology of X as the "singular" homology of $F_{sh}\mathcal{P}^R VX$. This gives a nice homology for Quigley's approaching groups that is not isomorphic to the strong (Steenrod) homology groups used by Kodama–Koyama [K–K] to obtain a Hurewicz Theorem for these groups.

We finish the paper by giving a Whitehead Theorem for the strong shape category in terms of Quigley inward groups. For the case of $\bar{\mathcal{P}}$–movable spaces the algebraic condition can also be given in terms of Quigley approaching groups or Čech–groups.

1. Closed simplicial model categories.

The tool used in this paper is the notion of closed simplicial model category. We refer the reader to [Q.1] and [Q.2] which contain the necessary definitions, examples and the main properties of this structure.

Given a solid arrow diagram in a category C

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

it is said that i has the left lifting property (LLP) with respect to p and p is said to have the right lifting property (RLP) with respect to i if there exists a map $h\colon B \longrightarrow X$ such that $hi = f$ and $ph = g$.

A closed model category is a category C endowed with three distinguished families of maps called cofibrations, fibrations and weak equivalences satisfying certain axioms.

These axioms were considered in [Q.1] and an equivalent but different formulation was given in [Q.2].

Given a closed model category C, the homotopy category $Ho(C)$ is obtained from C by formally inverting all the weak equivalences, see [Q.1] and [G-Z].

A simplicial category is a category C endowed with a functor $Hom_C\colon C^{op} \times C \longrightarrow SS$ satisfying the axioms given in [Q.1], in particular we have that $Hom_C(X,Y)_0 \cong C(X,Y)$. Associated with a simplicial category C, we have the category $\pi_0 C$ which has the same objects that C and the hom–set is defined by $\pi_0 C(X,Y) = \pi_0 Hom_C(X,Y)$, where $\pi_0 Hom_C(X,Y)$ is the set of connected components of the simplicial set $Hom_C(X,Y)$.

A closed simplicial model category is a simplicial category which is also a closed model category and satisfies certain axioms, see [Q.1]. For a finite simplicial set K a closed simplicial category C is provided with objects $X \otimes K$, X^K for any object X in C. Associated with these objects, there are the following isomorphisms:

$Hom_C(X \otimes K, Y) \cong Hom_{SS}(K, Hom_C(X,Y))$

$Hom_C(X, Y^K) \cong Hom_{SS}(K, Hom_C(X,Y))$

Suppose that C is a closed simplicial model category and $\emptyset$ denotes the initial object and $*$ denotes the final object. An object X is said to be cofibrant if the unique map $\emptyset \longrightarrow X$ is a cofibration and an object Y is said to be fibrant if the unique map $Y \longrightarrow *$ is a fibration.

The main relation between the categories $\pi_0 C$ and HoC is given through cofibrant and fibrant objects: If X is cofibrant and Y is fibrant, then $\pi_0 C(X,Y) \cong HoC(X,Y)$.

It is said that C is a pointed category if both the initial and final objects exist and are isomorphic. In this case, for two objects X, Y in C, we always have the zero map $*: X \longrightarrow Y$ that defines a 0–simplex of $Hom_C(X, Y)$. Therefore we also have a natural functor $Hom_C: C^{op} \times C \longrightarrow SS_*$.

Examples.

1) The category SS of simplicial sets. Let $\Delta[n]$ denote the standard n–simplex, $\dot{\Delta}[n]$ the simplicial set generated by the faces of $\Delta[n]$ and $V(n,k)$ for $0 \leq k \leq n > 0$ the simplicial subset of $\Delta[n]$ generated by the $(n-1)$–faces $\partial_i: \Delta[n-1] \longrightarrow \Delta[n]$ with $0 \leq i \leq n$ and $i \neq k$. A map $f: X \longrightarrow Y$ is said to be a fibration if for all $n > 0$, it has the RLP with respect to $V(n,k) \longrightarrow \Delta[n]$, $0 \leq k \leq n$. A map $f: X \longrightarrow Y$ is said to be a trivial fibration if f has the RLP with respect to $\dot{\Delta}[n] \longrightarrow \Delta[n]$, $n \geq 0$. A map $i: A \longrightarrow B$ is said to be a cofibration (resp. trivial cofibration) if i has the RLP with respect to any trivial fibration (resp. fibration). A map f is said to be a weak equivalence if f can be factored as $f = pi$ where i is a trivial cofibration and p is a trivial fibration.

Given a simplicial set K, the object $X \otimes K$ is defined to be $X \otimes K = X \times K$.

The functor $Hom_{SS}: SS^{op} \times SS \longrightarrow SS$ is defined by $Hom_{SS}(X,Y)_n \cong \cong SS(X \otimes \Delta[n], Y)$. The object X^K is defined by $X^K = Hom_{SS}(K, X)$.

The category of pointed simplicial sets SS_* is also a closed simplicial model category. If we consider the functor $()^+: SS \longrightarrow SS_*$ which carries X to $X \sqcup *$, we have that $()^+$ is the left adjoint of the forgetful functor $U: SS_* \longrightarrow SS$. A map f is said to be a fibration (resp. weak equivalence) if Uf is a fibration in SS (resp. weak equivalence). A map is a cofibration if it has the LLP with respect to trivial fibrations.

For an object X in SS_* and K in SS, define $X \otimes K \cong X \times K^+ / ((X \times *) \cup (* \times K^+))$. $Hom_{SS_*}: SS_*^{op} \times SS_* \longrightarrow SS_*$ is defined by $Hom_{SS_*}(X,Y)_n = SS_*(X \otimes \Delta[n], Y)$, and $X^K = Hom_{SS_*}(K^+, X)$.

2) The category Top of topological spaces. Let $R: SS \longrightarrow Top$ and $S: Top \longrightarrow SS$ be the realization and singular functors, respectively. A map $f: X \longrightarrow Y$ in Top is said to be a fibration (weak equivalence) if Sf is a fibration (weak equivalence) in SS. A map $i: A \longrightarrow B$ is a cofibration if i has the LLP with respect to trivial fibrations. Given a finite simplicial set K and a topological space X, the objects $X \otimes K$ and X^K are defined by

$$X \otimes K = X \times RK,$$

$$X^K = Top(RK, X)$$

where RK is the realization of K and $Top(RK, X)$ is the mapping space of continuous maps from RK to X endowed with the compact–open topology. The functor $Hom_{Top}\colon Top^{op} \times Top \longrightarrow SS$ is defined by

$$Hom_{Top}(X, Y)_n = Top(X \times R\Delta[n], Y).$$

The category Top_* of pointed spaces also admits a closed simplicial model structure. In this case for a given simplicial set K and a pointed space X, the objects $X \otimes K$ and X^K are defined by

$$X \otimes K = X \times (RK)^+ / ((X \times *) \cup (* \times (RK)^+)$$

$$X^K = Top_*((RK)^+, X)$$

where $(RK)^+$ is the disjoint union of RK and the one–point space $*$.

3) The category $proC$ of pro–objects in C. Associated with a category C, we can consider the category $proC$ introduced by A. Grothendieck [Gro]. A study of some properties of this category can be seen in the appendix of [A–M], the monograph of [E–H] or in the books of [M–S] and [C–P].

The objects of $proC$ are functors $X\colon I \longrightarrow C$, where I is a small left filtering category and the set of morphisms from $X\colon I \longrightarrow C$ to $Y\colon J \longrightarrow C$ is given by the formula

$$proC(X, Y) = \lim_j \ \operatorname{colim}_i \ C(X_i, Y_j).$$

Edwards and Hastings [E–H] have proved that if C has the structure of a closed simplicial model category and C satisfies the condition N, see [E–H, page 45], then $proC$ inherits a natural structure of a closed simplicial model category. For a given finite simplicial set K and an object $X = \{X_i\}$ of $proC$, the objects $X \otimes K$ and X^K are defined by

$$\{X_i\} \otimes K = \{X_i \otimes K\},$$

$$\{X_i\}^K = \{X_i^K\}.$$

The functor $Hom_{proC}\colon (proC)^{op} \times proC \longrightarrow SS$ is defined by

$$Hom_{proC}(X, Y)_n = proC(X \otimes \Delta[n], Y).$$

2. The category of simplicial M–sets.

A monoid consists of a set M and an associative multiplication: $M \times M \longrightarrow M$, $(m,m') \to mm'$, with unit element 1 ($1m = m = m1$, for every $m \in M$). A 0-monoid M is a monoid with a zero element $0 \in M$ ($m0 = 0 = 0m$, for every $m \in M$). If C is a category and X is an object of C, then the hom-set $C(X,X)$ with the composition of morphisms, $(g,f) \to gf$, has a natural monoid structure. If C is a category with zero object, then $C(X,X)$ is a 0–monoid.

Examples. 1) Let Pro be the category of spaces and proper maps (a continuous map is proper if the inverse image of a closed compact subset is compact) and consider the set of natural numbers $\mathbf{N}$ provided with the discrete topology. The set of proper maps $M = Pro(\mathbf{N},\mathbf{N})$ has a natural monoid structure. Let A, B be closed subsets of a space X and assume that $\mathrm{cl}(X - A), \mathrm{cl}(X - B)$ are compact. Given two proper maps $f\colon A \to Y$, $g\colon B \to Y$, it is said that f and g have the same germ if there exists a closed subset C of X such that $\mathrm{cl}(X - C)$ is compact, $C \subset A$, $C \subset B$ and $f/C = g/C$. Let Pro_∞ denote the category of spaces and germs of proper maps, the monoid of germs of proper maps $M_\infty = Pro_\infty(\mathbf{N},\mathbf{N})$ will also be considered in this paper.
2) Let $\hat{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$ be the Alexandroff compactification of $\mathbf{N}$. Taking ∞ as a base point, $\hat{\mathbf{N}}$ becomes a pointed set. The endomorphism pointed set $Top_*(\hat{\mathbf{N}},\hat{\mathbf{N}})$ has a natural 0-monoid structure. Two pointed continuous maps $f, g : \hat{\mathbf{N}} \to \hat{\mathbf{N}}$ have the same germ at ∞ if there is $n_0 \in \mathbf{N}$ such that $f(n) = g(n)$ for every $n \geq n_0$. The set $Top_*^\infty(\hat{\mathbf{N}},\hat{\mathbf{N}})$ of germs at ∞ of continuous maps from $\hat{\mathbf{N}}$ to $\hat{\mathbf{N}}$ also becomes a 0-monoid.

Let M be a monoid and C a category. A left M–object X in C consists of an object X of C and a monoid homomorphism $M \to C(X,X) : m \to \tilde{m} : X \to X$. If M is a 0–monoid and C is a category with zero object, we suppose that an M–object X in C satisfies the additional condition $\tilde{0} = 0$. The category whose objects are the (left) M–objects in C, will be denoted by ${}_MC$. By considering monoid "antimorphisms" $M \to C(X,X)$ we have the notion of right M–object in C and the category C_M.

Let C be a category, for each object X of C we have the monoid (or 0–monoid) $M = C(X,X)$. If Y is an object of C the monoid "antimorphism"

$$C(X,X) \longrightarrow Set(C(X,Y),C(X,Y)) : \varphi \to \tilde{\varphi},\ \tilde{\varphi}(f) = f\varphi,\ f \in C(X,Y),$$

induces the structure of a right M-set on $C(X,Y)$. Therefore we have a functor $C(X,-) : C \longrightarrow Set_M$. If C is pointed (C has a zero object) we get a functor $C(X,-) : C \longrightarrow Set_{*M}$. Similarly, there are functors $C(-,Y) : C \longrightarrow {}_MSet$ or $C(-,Y) : C \longrightarrow {}_MSet_*$.

Examples. 1) For the category Pro and $M = Pro(\mathbf{N},\mathbf{N})$, we have the right M–set $Pro(\mathbf{N},\mathbf{X})$ of sequences in X converging to infinity. Similarly for $M_\infty = Pro_\infty(\mathbf{N},\mathbf{N})$ we have the right M_∞-set $Pro_\infty(\mathbf{N},\mathbf{X})$. If $|\Delta[q]|$ denotes the realization of the standard q-simplex we also get the right M–set $Pro(\mathbf{N}\times|\mathbf{\Delta}[\mathbf{q}]|,\mathbf{X})$.

2) Let $s = \prod_{n=1}^{+\infty}(\frac{-1}{n},\frac{1}{n})$ be the pseudo-interior of the Hilbert cube $Q = \prod_{n=1}^{+\infty}[\frac{-1}{n},\frac{1}{n}]$. Let X be a compact subset of s. A sequence $x\colon \mathbf{N}\to\mathbf{Q}$ converges to X if for every U, a neighbourhood of X in Q, there is n_0 such that $x_n\in U$ for every $n\geq n_0$. The sets $\{x\colon\mathbf{N}\to\mathbf{X}\mid \mathbf{x}$ converges to $X\}$ and $\{x\colon\mathbf{N}\to\mathbf{Q}-\mathbf{X}\mid\mathbf{x}$ converges to $X\}$ become M-sets for $M = Pro(\mathbf{N},\mathbf{N})$. Consider also sequences of simplexes $x\colon\mathbf{N}\times|\mathbf{\Delta}[\mathbf{q}]|\to\mathbf{Q}$ converging to X; that is, for every U, a neighbourhood of X in Q, there is n_0 such that $x(\{n\}\times|\Delta[q]|)\subset U$ for every $n\geq n_0$. The sets $ss_q(X) = \{x\colon\mathbf{N}\times|\mathbf{\Delta}[\mathbf{q}]|\to\mathbf{Q}\mid\mathbf{x}$ converges to $X\}$ and $ss_q^c(X) = \{x\colon\mathbf{N}\times|\mathbf{\Delta}[\mathbf{q}]|\to\mathbf{Q}-\mathbf{X}\mid\mathbf{x}$ converges to $X\}$ become M-sets for $M = Pro(\mathbf{N},\mathbf{N})$ and $ss(X)$, $ss^c(X)$ are simplicial M-sets associated with X.

Given a 0–monoid M, the category of right M–pointed sets, Set_{*M}, is an algebraic category, see [Pa], by considering one 0–ary operation to fix a base point $*$ and a 1–ary operation m for each $m\in M$. The relations are given by $x1 = x$, $x0 = *$, $(xm)n = x(mn)$. In the case of a monoid, we do not need the 0–ary operation and the relation $x0 = *$. By general properties of algebraic categories we have that Set_{*M} (resp., Set_M) is a complete and cocomplete category, see [Pa; page 140]. That is, the category Set_{*M} (Set_M) has limits and colimits. The categories of the form Set_{*M}, Set_M enjoy very nice properties such as the existence of exponentials and a subobject classifier. That is, these categories are elementary topoi, see [M-M].

For these categories there is a natural forgetful functor $U\colon Set_{*M}\longrightarrow Set_*$ (resp., $U\colon {}_MSet_*\longrightarrow Set_*$) and a left adjoint functor $-\odot M\colon Set_*\longrightarrow Set_{*M}$ defined by $X\odot M = X\times M/(X\times 0\cup *\times M)$. An element $(x,m)\in X\times M$ determines a unique class in $X\odot M$ that will be denoted by $x\odot m$. The forgetful functor $Set_{*M}\longrightarrow Set_*$ is faithful and preserves limits and the left adjoint functor $-\odot M$ preserves colimits. For the non-pointed case, the forgetful functor $Set_M\longrightarrow Set$ has also a left adjoint functor $-\odot M\colon Set\longrightarrow Set_M$ which is defined by $X\odot M = X\times M$. In this case we also denote an element (x,m) by $x\odot m$. We note that $X\odot M$ the free M-set generated by X is isomorphic to $\bigsqcup_{x\in X} M$ provided with the canonical right action of M.

Let $\mathcal{C}$ be a category closed under finite limits. A map $f\colon X\to Y$ is said to be an effective epimorphism if for any object T of $\mathcal{C}$, the diagram of sets

$$\mathcal{C}(Y,T) \xrightarrow{f^*} \mathcal{C}(X,T) \quad \underset{pr_2{}^*}{\overset{pr_1{}^*}{\rightrightarrows}} \quad \mathcal{C}(X \underset{Y}{\times} X, T)$$

is a difference kernel. An object P of $\mathcal{C}$ is said to be projective if $\mathcal{C}(P,X) \xrightarrow{f_*} \mathcal{C}(P,Y)$ is surjective whenever $f: X \to Y$ is an effective epimorphism. A category $\mathcal{C}$ has sufficiently many projectives, if for any object X, there is an effective epimorphism $P \longrightarrow X$ where P is a projective object. Assume that $\mathcal{C}$ is closed under colimits, an object X is said to be small if $\mathcal{C}(X,-)$ commutes with filtered colimits. A class $\mathcal{U}$ of objects of $\mathcal{C}$ is a class of generators if, for every object X, there is an effective epimorphism $Q \to X$ where Q is a sum of copies of members of $\mathcal{U}$.

For the case $\mathcal{C} = Set_{*M}$ (resp., Set_M) the class of effective epimorphisms is the class of set-theoretically surjective maps. Note that the category Set_{*M} (resp., Set_M) has a class of generators, $\mathcal{U}$, with a single object, $S^0 \odot M \cong M$ $(* \odot M \cong M)$. For later applications we also note that M is projective and small.

If $\mathcal{C}$ is a category, let $S\mathcal{C}$ denote the category of simplicial objects in $\mathcal{C}$. We also have a natural functor $in: \mathcal{C} \longrightarrow S\mathcal{C}$ which carries an object A to the simplicial object inA defined by $(inA)_q = A$ and where degeneracy and face operators are equal to the identity of A.

Quillen [Q.1] proved that if $\mathcal{C}$ is closed under finite sums, X is an object in $S\mathcal{C}$ and K is a finite simplicial set, then an object $X \otimes K$ exists, defined by

$$(X \otimes K)_n = \sum_{\sigma \in K_n} X_n$$

in which the degeneracy and face operators are defined in terms of the corresponding operators of X and K. If $\mathcal{C}$ is closed under finite limits, then dually an object X^K exists for every finite simplicial set. These have nice universal properties see Quillen [Q.1].

Therefore given a category $\mathcal{C}$ closed under finite limits and colimits, $S\mathcal{C}$ becomes a simplicial category where the natural functor

$$Hom_{S\mathcal{C}}: S\mathcal{C}^{op} \times S\mathcal{C} \longrightarrow SS$$

is defined by $Hom_{S\mathcal{C}}(A,B)_n = S\mathcal{C}(A \otimes \Delta[n], B)$. If $\mathcal{C}$ is a pointed category, we can also consider the functor $Hom_{S\mathcal{C}}: S\mathcal{C}^{op} \times S\mathcal{C} \longrightarrow SS_*$.

In order to have a shorter notation we also use $S = Set$, $S_* = Set_*$, $S_M = Set_M$, $S_{*M} = Set_{*M}$. The corresponding simplicial categories will be denoted by SS, SS_*,

$S(S_M)$, $S(S_{*M})$. We note that, for the functors $in\colon S_* \longrightarrow SS_*$, $in\colon S_{*M} \longrightarrow S(S_{*M})$, and $-\odot M\colon SS_* \longrightarrow S(S_{*M})$, there are natural isomorphisms

$$((inX)\otimes K)\odot M \cong (in(X\odot M))\otimes K,$$

where X is an object in S_* and K is a finite simplicial set.

The following result is a particular case of Theorem 4 of section 4 ch II of [Q.1].

Proposition 1. Let $\mathcal{C}$ be a category closed under finite limits and under colimits and having a set $\mathcal{U}$ of small projective generators. Let $S\mathcal{C}$ be the simplicial category of simplicial objects in $\mathcal{C}$. Define a map f in $S\mathcal{C}$ to be a fibration (weak equivalence) if $Hom(inP, f)$ is a fibration (weak equivalence) in SS for each P of $\mathcal{U}$. A map f is a cofibration if f has the left lifting property with respect to the class of trivial fibrations. Then $S\mathcal{C}$ is a closed simplicial model category.

For the case $\mathcal{C} = S_{*M}$ (or $\mathcal{C} = S_M$), we have that $\mathcal{U}$ has only a single object $S^0\otimes M \cong M$. Notice that for a map f of $S(S_{*M})$ we have that

$Hom_{S(S_{*M})}(in(S^0\otimes M), f) \cong Hom_{S(S_{*M})}(inS^0\otimes M, f)$

$\cong Hom_{S(S_{*M})}(\Delta[0]^+\odot M, f) \cong Hom_{SS_*}(\Delta[0]^+, f) = Uf$

Therefore we have the following closed simplicial model structure:

Definition 1. In the category of simplicial M-sets, $S(S_{*M})$, a map f is said to be a fibration (weak equivalence) if Uf is a fibration (weak equivalence) in SS_*. A map is said to be a cofibration if f has the LLP with respect to any trivial fibration.

Theorem 1. The category $S(S_{*M})$ together with the classes of cofibrations, fibrations and weak equivalences defined above has a natural closed simplicial model category structure.

Remark. For the non-pointed case a similar result is obtained for the category $S(S_M)$. The corresponding fibrations and weak equivalences are defined by using the forgetful functor $S(S_M) \longrightarrow SS$.

In the category SS_*, the "tensor" object $X\otimes K$ and the "function" object X^K can be defined for any simplicial set K. We apply this property to prove the following:

Lemma 1. Given f of SS_*, we have:
1) If f is a weak equivalence, then $f\odot M$ is a weak equivalence,
2) if f is a cofibration, then $f\odot M$ is a cofibration.

Proof. 1) Let f be a weak equivalence in SS_*, then we are going to prove that $f \odot inM$ is a weak equivalence in SS_*. By Proposition 3.5, ch2 of [Q.1], it suffices to prove that for any fibrant object Y of SS_* $[f \otimes inM, Y]$ is an isomorphism. This is obtained from the following isomorphisms

$$[f \otimes inM, Y] = \pi_0 Hom(f \otimes inM, Y) \cong$$
$$\cong \pi_0 Hom(f, Hom(inM, Y)) \cong [f, Hom(inM, Y)]$$

and the fact that f is a weak equivalence.

The forgetful functor $U: S(S_{*M}) \longrightarrow SS_*$ satisfies that $U(f \odot M) = f \otimes inM$. Because $U(f \odot M)$ is a weak equivalence, by Definition 1 we also have that $f \odot M$ is a weak equivalence.

2) Since $- \odot M : SS_* \longrightarrow S(S_{*M})$ is left adjoint to $U: S(S_{*M}) \longrightarrow SS_*$, and U preserves weak equivalences and fibrations, we also have that $- \odot M$ preserves cofibrations.

As a consequence of this Lemma, we obtain an induced pair of adjoint functors on the localized categories.

Theorem 2. The functors $- \odot M$ and U factorize through the homotopy categories in such a way that $- \odot M: Ho(SS_*) \longrightarrow Ho(S(S_{*M}))$ is left adjoint to $U: Ho(S(S_{*M})) \longrightarrow Ho(SS_*)$. Moreover, $- \otimes M$ preserves cofibration sequences and U preserves fibration sequences.

Remark. If M is a monoid (without zero element) the analogues of the theorems above are similarly obtained. If M is a group the closed simplicial model category $S(S_M)$ induces a nice homotopy category $Ho(S(S_M))$ to study equivariant homotopy theory.

3. Realization and singular functors.

In this section, we analyse the construction of singular and realization functors for the category of simplicial M-sets.

Recall that a monoid M can be considered as a category with one object, with morphisms the elements of M and with composition the product in the monoid M. Therefore the category of right M-sets can be considered as the functor category $Set^{M^{op}}$. Thus the category $S(S_M)$ of simplicial M-sets is the functor category $(Set^{M^{op}})^{\Delta^{op}}$, which is equivalent to the category $Set^{(M \times \Delta)^{op}}$.

Given a small category I, the functor category $Set^{I^{op}}$ is also called the category of presheaves on I. Associated with a functor $X: I^{op} \longrightarrow Set$, we recall the construction of the category of elements of X, denoted by $\int_I X$. For more details and properties of this construction, which is often called the Grothendieck construction, we refer the reader to [M-M].

The objects of $\int_I X$ are pairs (i,x) where i is an object of I and x is an element of $X(i)$. Its morphisms $(i',x') \to (i,x)$ are those morphisms $u: i' \to i$ of I for which $X(u): X(i) \to X(i')$ satisfies $X(u)x = x'$. This category has a canonical projection functor $\pi_X: \int_I X \longrightarrow I$ defined by $\pi_X(i,x) = i$.

The following result is proved in [M-M; Th2, ChI].

Theorem 1. If $\chi: I \longrightarrow \mathcal{C}$ is a functor from a small category I to a cocomplete category $\mathcal{C}$, the functor S_χ from $\mathcal{C}$ to $Set^{I^{op}}$ given by

$$S_\chi C: i \longrightarrow \mathcal{C}(\chi(i), C)$$

has a left adjoint functor $R_\chi: Set^{I^{op}} \longrightarrow \mathcal{C}$ defined for each functor X in $Set^{I^{op}}$ as the colimit

$$R_\chi X = \operatorname{colim}(\int_I X \xrightarrow{\pi_X} I \xrightarrow{\chi} \mathcal{C}).$$

For the small category $I = M \times \Delta$, the equivalence of categories $S(S_M) \simeq Set^{(M\times\Delta)^{op}}$ carries a functor $X: \Delta^{op} \longrightarrow Set_M$ to a functor $X': (M \times \Delta)^{op} \longrightarrow Set$. Similarly, for a given category $\mathcal{C}$ and a functor $\chi: \Delta \longrightarrow {}_M\mathcal{C}$ one has the corresponding functor $\chi': M \times \Delta \longrightarrow \mathcal{C}$. Since M only has one object $*$, the objects of $M \times \Delta$ are of the form $(*, [p])$. However, in the sequel, we just write $[p]$ for the object $(*, [p])$. Recall that a morphism of $M \times \Delta$ is of the form $(m, \varphi): [p] \longrightarrow [q]$, where m is an element of M and φ is a map of Δ. Observe that $(m,\varphi) = (m, id_{[q]})(1_M, \varphi) = (1_M, \varphi)(m, id_{[p]})$. Sometimes, we just write m for $(m, id_{[q]})$ and φ for $(1_M, \varphi)$ if no confusion is possible. For a functor $Y: (M\times\Delta)^{op} \longrightarrow Set$, we write $Y([p]) = Y_p$, $Y(m,\varphi) = (m,\varphi)^*$, $Y(m, id_{[p]}) = Y(m) = m^*$ and $Y(1_M, \varphi) = Y(\varphi) = \varphi^*$. Similarly for a functor $\chi': M \times \Delta \longrightarrow \mathcal{C}$, we write χ'_p instead of $\chi'([p])$.

Using this notation, we can reformulate the Theorem above as follows:

Theorem 2. If $\chi: \Delta \longrightarrow {}_M\mathcal{C}$ is a functor, where $\mathcal{C}$ is a cocomplete category, then the "singular" functor S_χ from $\mathcal{C}$ to $S(S_M)$ defined by

$$(S_\chi C)_p = \mathcal{C}(\chi'_p, C)$$

has a left adjoint (the "realization" functor) $R_\chi: S(S_M) \longrightarrow \mathcal{C}$ defined for each X an object in $S(S_M)$ as the colimit

$$R_\chi X = \operatorname{colim}(\int_{M\times\Delta} X' \xrightarrow{\pi_{X'}} M \times \Delta \xrightarrow{\chi'} \mathcal{C}).$$

In this paper, we will consider functors $\chi\colon \Delta \longrightarrow {}_M\mathcal{C}$, where $\mathcal{C}$ does not have all colimits. For these categories we will analyse those X in $S(S_M)$ for which the colimit $R_\chi X$ exists. The following properties of colimits will be useful.

Given a functor $L\colon J' \longrightarrow J$ and an object j in J, the comma category $j \downarrow L$ has as objects morphisms of the form $u\colon j \to Lj'$. A morphism from $u_0\colon j \to Lj'_0$ to $u_1\colon j \to Lj'_1$ is a morphism $v'\colon j'_0 \to j'_1$ which satisfies $L(v')u_0 = u_1$. A category J is called connected if given any two objects j_0, j_1 in J, there is a finite sequence of arrows (both directions possible) joining j_0 to j_1.

A functor $L\colon J' \longrightarrow J$ is final if for each j in J, the comma category $j \downarrow L$ is non-empty and connected. For more details concerning final functors, we refer the reader to [M] and [C-P]. In particular we will use the following:

Proposition 1. If $L : J' \longrightarrow J$ is final and $F : J \longrightarrow \mathcal{C}$ is a functor such that colimFL exists then colim$F$ exists and the canonical map colim$FL \to$ colim$F$ is an isomorphism.

Definition 1. Given an object X in $S(S_M)$, it is said that dim$X \leq n$ if for $q \geq n$ and $y \in X_q$, there are $p \leq n$, $x \in X_p$ and a surjective map $\varphi\colon [q] \to [p]$ such that $y = \varphi^* x$.

Denote by Δ/n the full subcategory of Δ determined by the objects $[0], \cdots, [n]$. Given a functor $X'\colon (M \times \Delta)^{op} \longrightarrow Set$, one defines the functor $Sk_n X'$ as the composite

$$(M \times \Delta/n)^{op} \longrightarrow (M \times \Delta)^{op} \xrightarrow{X'} Set.$$

It is easy to check the existence of a canonical induced functor $I\colon \int Sk_n X' \longrightarrow \int X'$.

Proposition 2. If X is an object in $S(S_M)$ with dim$X \leq n$, then the functor $I\colon \int Sk_n X' \longrightarrow \int X'$ is final.

Proof. Let $([q], y)$ be an object of $\int X'$. The condition dim$X \leq n$ implies that the comma category $([q], y) \downarrow I$ is non-empty. In order to prove that $([q], y) \downarrow I$ is connected it suffices to apply the Eilenberg-Zilber Lemma [G-Z, p. 26].

In this section, we work with the following notions of diagram scheme and diagram. A diagram scheme consists of a set D_0 of objects and a set D_1 of arrows together with a source map $s : D_1 \to D_0$ and a target map $t : D_1 \to D_0$. For instance a small category has the structure of a scheme diagram. A morphism $F\colon D \longrightarrow D'$ of scheme diagrams consists of a pair of maps $F_0\colon D_0 \longrightarrow D'_0$ and $F_1\colon D_1 \longrightarrow D'_1$ such that $sF_1 = F_0 s$, $tF_1 = F_0 t$. Let $\mathcal{C}$ be a category. A diagram $F\colon D \longrightarrow \mathcal{C}$ is an operation which assigns to each object of

D an object of $\mathcal{C}$ and to each arrow of D a morphism of $\mathcal{C}$. This assignment commutes with the source and target operators. The analogues of comma category and connected category are also defined for scheme diagrams and diagrams. Using these notions one has:

Lemma 2. Let J be a small category and let $I: D \longrightarrow J$ be a diagram such that I is an inclusion map. Assume that for every j in J, there is an associated morphism $u_j: j \to d_j$, where d_j is an object in D. Suppose that these morphisms satisfy:
i) If j is an object in D, then $u_j: j \to d_j$ is a morphism in D.
ii) Given a morphism $u: j_0 \to j_1$ of J, the objects $u_{j_0}: j_0 \to d_{j_0}$ and $u_{j_1}u: j_0 \to d_{j_1}$ are in the same connected component of $j_0 \downarrow I$.
If $F: J \longrightarrow \mathcal{C}$ is a functor and colimFI exists then colim$F$ exists and the canonical map colim$FI \longrightarrow$ colim$F$ is an isomorphism.

Proof. The proof is routine and is left as an exercise.

As an application of the Lemma above, for some X in $S(S_M)$ with dim$X \leq n$, we will describe a finite diagram $I: D(Sk_n X') \longrightarrow \int Sk_n X'$ which satisfies the conditions of Lemma 2. In this case, in order to prove the existence of colim$(\int X' \longrightarrow M \times \Delta \xrightarrow{\chi'} \mathcal{C})$, it suffices to prove the existence of colim$(D(Sk_n X') \longrightarrow M \times \Delta \xrightarrow{\chi'} \mathcal{C})$. First we introduce some necessary notation.

Let S_M/ff be the full subcategory of S_M determined by M–sets freely generated by finite sets. An object of S_M/ff is of the form $\{1, \cdots, n\} \otimes M \cong M \sqcup \cdots \sqcup M$. An element x of $\{1, \cdots, n\} \otimes M$ will be denoted by $x = (i, \alpha)$ where $1 \leq i \leq n$ and $\alpha \in M$. A morphism $u: \{1, \cdots, n\} \otimes M \longrightarrow \{1, \cdots, m\} \otimes M$ is determined by a map $\tau_u: \{1, \cdots, n\} \to \{1, \cdots, m\}$ and the values $u(1, 1_M) = (\tau_u(1), u_1), \ldots$, and $u(n, 1_M) = (\tau_u(n), u_n)$, where $u_1, \ldots, u_n \in M$. If X is an object in $S(S_{*M}/ff)$, we have that $X_q = \{1, \cdots, k_q\} \otimes M$ is generated by the elements $(1, 1_M), \ldots (k_q, 1_M)$.

Recall that in the category Δ we have the canonical maps $\epsilon_i: [p-1] \to [p]$, $\sigma_i: [p+1] \to [p]$, $0 \leq i \leq p$ The i^{th} face ϵ_i is defined by $\epsilon_i(j) = j$, if $j < i$, $\epsilon_i(j) = j + 1$ otherwise. The i^{th} degeneracy σ_i collapses $i + 1$ to i. For a functor $X: \Delta^{op} \longrightarrow \mathcal{C}$, one usually writes $X(\epsilon_i) = \epsilon_i^* = s_i$ and $X(\sigma_i) = \sigma_i^* = d_i$.

Given an object X in $S(S_M/ff)$ with dim$X \leq n$. The diagram $I: D(Sk_n X') \longrightarrow \int Sk_n X'$, is defined as follows:

If $n = 0$, $D(Sk_0 X')$ is given by the objects $([0], (1, 1_M)), \ldots, ([0], (k_0, 1_M))$.

If $n > 0$, for $q = 0$ and $1 \leq i \leq k_0$, we consider the following objects and arrows in $\int Sk_n X'$:

$$
\begin{array}{ccc}
([1],(\tau_{s_0}(i),s_0 1_M)) & \xrightarrow{(s_0 1_M, id_{[1]})} & ([1],(\tau_{s_0}(i),1_M)) \\
\Big\downarrow{\scriptstyle (1_M,\sigma_0)} & & \\
([0],(i,1_M)). & &
\end{array}
$$

For $0 < q < n, \quad 1 \leq i \leq k_q, \quad 0 \leq j \leq q, \quad 0 \leq l \leq q$, we take the following objects and arrows in $\int Sk_n X'$:

$$
\begin{array}{ccc}
([q+1],(\tau_{s_j}(i),s_j 1_M)) & \xrightarrow{(s_j 1_M, id_{[q+1]})} & ([q+1],(\tau_{s_j}(i),1_M)) \\
\Big\downarrow{\scriptstyle (1_M,\sigma_j)} & & \\
([q],(i,1_M)) & & \\
\Big\uparrow{\scriptstyle (1_M,\epsilon_l)} & & \\
([q-1],(\tau_{d_l}(i),d_l 1_M)) & \xrightarrow{(d_l 1_M, id_{[q-1]})} & ([q-1],(\tau_{d_l}(i),1_M))
\end{array}
$$

and for $q = n, 1 \leq i \leq k_n, 0 \leq l \leq n$, we consider the following objects and arrows:

$$
\begin{array}{ccc}
([n],(i,1_M)) & & \\
\Big\uparrow{\scriptstyle (1_M,\epsilon_l)} & & \\
([n-1],(\tau_{d_l}(i),d_l 1_M)) & \xrightarrow{(d_l 1_M, id_{[n-1]})} & ([n-1],(\tau_{d_l}(i),1_M))
\end{array}
$$

All the objects and arrows given above define a diagram $I\colon D(Sk_n X') \longrightarrow \int Sk_n X'$. If $([q],(i,m))$ is an object in $D(Sk_n X')$, then we have the map $u = (m, id_{[q]})\colon ([q],(i,m)) \longrightarrow ([q],(i,1_M))$ where $([q],(i,1_M))$ is an object in $D(Sk_n X')$. It is easy to check that the family of maps u satisfies the conditions of Lemma 2, so we obtain the following result.

Proposition 2. Let X be an object in $S(S_M/ff)$ with $\dim X \leq n$. If $\operatorname{colim}(D(Sk_n X') \longrightarrow M \times \Delta \xrightarrow{\chi'} \mathcal{C})$ exists, then $\operatorname{colim}(\int X' \longrightarrow M \times \Delta \xrightarrow{\chi'} \mathcal{C})$ exists and both colimits are isomorphic.

As a consequence of Proposition 2, for the case that $\mathcal{C}$ has finite colimits, there is a realization functor $R_\chi\colon S(S_M/ff)/fd \longrightarrow \mathcal{C}$ where $S(S_M/ff)/fd$ is the full subcategory determined by objects X in $S(S_M/ff)$ with finite dimension. Next section we will consider the case $\mathcal{C} = Pro$, where Pro is the category of spaces and proper maps. In this case we have a natural inclusion functor $Pro \longrightarrow Top$ into the category Top of spaces and continuous maps. Using the fact that Top has all colimits, we will apply Proposition 2 in order to construct a "proper" realization functor $R_\chi\colon S(S_M/ff)/fd \longrightarrow Pro$.

In this paper, we have to deal with "realization" functors which only are defined on a full subcategory of $S(S_M)$, then it will be useful to introduce the following notion of partial left adjoint functor.

Definition 1. Let $\mathcal{A}'$ be a full subcategory of a category $\mathcal{A}$. We say that $F\colon \mathcal{A}' \longrightarrow \mathcal{B}$ is a partial left adjoint to $G\colon \mathcal{B} \longrightarrow \mathcal{A}$ if for any A in $\mathcal{A}'$ and B in $\mathcal{B}$, there is a natural isomorphism

$$\mathcal{B}(FA, B) \cong \mathcal{A}(A, GB).$$

For simplicial categories we consider the following notion of simplicial adjuntion.

Definition 2. Let $\mathcal{A}$, $\mathcal{B}$ be simplicial categories and assume that $F\colon \mathcal{A} \longrightarrow \mathcal{B}$, $G\colon \mathcal{B} \longrightarrow \mathcal{A}$ are functors. We say that F is simplicial left adjoint to G, if for any A in $\mathcal{A}$ and B in $\mathcal{B}$, there is a natural simplicial isomorphism

$$Hom_{\mathcal{B}}(FA, B) \cong Hom_{\mathcal{A}}(A, GB).$$

If $\mathcal{A}'$ is a full subcategory of $\mathcal{A}$, we say that a functor $F\colon \mathcal{A}' \longrightarrow \mathcal{B}$ is partial simplicial left adjoint to $G\colon \mathcal{B} \longrightarrow \mathcal{A}$, if for any A in $\mathcal{A}'$ and B in $\mathcal{B}$, there is a natural simplicial isomorphism

$$Hom_{\mathcal{B}}(FA, B) \cong Hom_{\mathcal{A}}(A, GB).$$

4. Realization and singular functors for proper categories and procategories.

In this section, we consider the realization and singular functors associated with some covariant functors $\chi\colon \Delta \longrightarrow {}_M C$ and introduce the various notations that will be used later.

1) The standard realization and singular functor.

Let $M = \{1\}$ be the monoid having just the unit element. For this monoid it is clear that for any category C, ${}_M C = C = C_M$. If we consider the standard covariant functor $\chi = st\colon \Delta \longrightarrow Top$ defined by $st[q] = |\Delta[q]|$, we will obtain the standard realization and singular functors $R_{st}\colon SS \longrightarrow Top$, $S_{st}\colon Top \longrightarrow SS$. The functor R_{st} is simplicial left adjoint to S_{st}; that is, $Hom_{Top}(R_{st}X, Y) \cong Hom_{SS}(X, S_{st}Y)$. In this paper the standard realization functor R_{st} is denoted by R and by $|\ |$ and the standard singular functor S_{st} by S.

2) Equivariant realization and singular functors.

Given a monoid M, it can be provided with the discrete topology and a functor $Top \longrightarrow {}_MTop$ can be defined by $X \longrightarrow M \times X$. This functor is left adjoint to the forgetful functor ${}_MTop \longrightarrow Top$. If we consider the covariant functor $e = (M \times (-)) \cdot st$

$$\Delta \xrightarrow{st} Top \xrightarrow{M\times(-)} {}_MTop$$

defined by $e[q] = M \times |\Delta[q]|$, we can apply Theorem 3.2 to obtain a realization functor $R_e: S(S_M) \longrightarrow Top_M$ and a singular functor $S_e: Top_M \longrightarrow S(S_M)$.

Given a finite simplicial set K and a object X of Top_M there are objects $X \otimes K$ and X^K defined by

$$X \otimes K = X \times |K|$$

$$X^K = X^{|K|}$$

The action of M on $X \otimes K$ is defined by $(x, y)m = (xm, y)$ for $x \in X, y \in |K|$ and $m \in M$, and the action of M on X^K is given by $(\varphi m)(y) = (\varphi(y))m$ for $\varphi \in X^K, y \in |K|$ and $m \in M$. As above, R_e, S_e are a pair of simplicial adjoint functors

$$Hom_{Top_M}(R_e X, Y) \cong Hom_{S(S_M)}(X, S_e Y).$$

The equivariant homotopy category is defined to be $\pi_0(Top_M)$. Taking into account the isomorphism above, it follows that

$$\pi_0(Top_M)(R_e X, Y) \cong \pi_0(S(S_M))(X, S_e Y).$$

3) Proper realization and singular functors.

If we consider the monoid $M = Pro(\mathbf{N}, \mathbf{N})$, since the identity $id: M \longrightarrow Pro(\mathbf{N}, \mathbf{N})$ is a monoid homomorphism, it follows that $\mathbf{N}$ has the natural structure of a left M-set. The functor $- \times |\Delta[q]|: Pro \longrightarrow Pro$ induces a left M-set structure on $\mathbf{N} \times |\mathbf{\Delta[q]}| \cong \bigsqcup_{\mathbf{N}} |\mathbf{\Delta[q]}|$ by considering the composite:

$$M \longrightarrow Pro(\mathbf{N}, \mathbf{N}) \longrightarrow \mathbf{Pro}(\mathbf{N} \times |\mathbf{\Delta[q]}|, \mathbf{N} \times |\mathbf{\Delta[q]}|).$$

Therefore there is an induced functor

$$\chi = p: \Delta \longrightarrow {}_MPro, \ p[q] = \mathbf{N} \times |\mathbf{\Delta[q]}|.$$

The inclusion functor $I: Pro \longrightarrow Top$ induces a natural functor ${}_MI: {}_MPro \longrightarrow {}_MTop$ and we also have the composite:

$$\chi = c = {}_MI \cdot p: \Delta \longrightarrow {}_MPro \longrightarrow {}_MTop.$$

Since Top has colimits, applying Theorem 3.2, we obtain the continuous realization functor $R_c: S(S_M) \longrightarrow Top$ and the continuous singular functor $S_c: Top \longrightarrow S(S_M)$.

By the exponential law, there is a set isomorphism

$$Top(\mathbf{N} \times |\mathbf{\Delta[q]}|, \mathbf{X}) \cong \mathbf{Top}(|\mathbf{\Delta[q]}|, \mathbf{X_c^N})$$

where the mapping space $X_c^{\mathbf{N}}$ has the compact open topology. It is also clear that $X_c^{\mathbf{N}}$ has the structure of a right M–space, therefore we have the following diagram which is commutative up to isomorphism

$$\begin{array}{ccc} Top & \xrightarrow{()_c^{\mathbf{N}}} & Top_M \\ S_c \searrow & & \swarrow S_e \\ & S(S_M) & \end{array}$$

that is, $S_cX \cong S_e(X_c^{\mathbf{N}})$.

Recall that for X an object in $S(S_M)$, the functor $R_c: S(S_M) \longrightarrow Top$ is defined by

$$R_cX = \operatorname{colim}(\int X' \longrightarrow M \times \Delta \xrightarrow{c'} Top).$$

If X is an object in $S(S_M/ff)/fd$, by Proposition 3.2 R_cX is isomorphic to

$$\operatorname{colim}(D(Sk_n)X' \longrightarrow M \times \Delta \xrightarrow{c'} Top).$$

Since $M \times \Delta \xrightarrow{c'} Top$ factorizes as $M \times \Delta \xrightarrow{p'} Pro \xrightarrow{I} Top$ and using the fact that for any object $([q], (i, m))$ in $D(Sk_nX')$ the continuous map $c'[q] = \mathbf{N} \times |\mathbf{\Delta[q]}| \longrightarrow \mathbf{R_cX}$ is proper, it follows that $\operatorname{colim}(D(Sk_nX') \longrightarrow M \times \Delta \xrightarrow{p'} Pro)$ exists. Applying again Proposition 3.2, one has that $\operatorname{colim}(\int X' \longrightarrow M \times \Delta \xrightarrow{p'} Pro)$ exists. Therefore, for any X an object in $S(S_M/ff)/fd$ we can define $R_p: S(S_M/ff)/fd \longrightarrow Pro$ by

$$R_pX = \operatorname{colim}(\int X' \longrightarrow M \times \Delta \xrightarrow{p'} Pro).$$

On the other hand, observe that the set $X_p^{\mathbf{N}} = Pro(\mathbf{N}, \mathbf{X})$ is bijective to the subset $\{f \in \widehat{X}_c^{\widehat{\mathbf{N}}} | f^{-1}\infty = \infty\}$. We will consider on $X_p^{\mathbf{N}}$ the relative topology induced by the compact open topology of the space $\widehat{X}_c^{\widehat{\mathbf{N}}}$. It is easy to check that we have a natural set–isomorphism

$$Pro(\mathbf{N} \times |\mathbf{\Delta[q]}|, \mathbf{X}) \cong \mathbf{Top}(|\mathbf{\Delta[q]}|, \mathbf{X_p^N}).$$

These sets have also a natural structure as right M–sets, ($M = Pro(\mathbf{N},\mathbf{N})$) and the isomorphism above becomes an M–set isomorphism. Therefore we have the following diagram of functors which is commutative up to natural isomorphism

$$\begin{array}{ccc} Pro & \xrightarrow{(-)_p^{\mathbf{N}}} & Top_M \\ S_p \searrow & & \swarrow S_e \\ & S(S_M) & \end{array}$$

The pair of adjoint functors $R_c\colon S(S_M) \longrightarrow Top$, $S_c\colon Top \longrightarrow S(S_M)$ satisfies:

a) S_c preserves "function" functors: for a finite simplicial set K and an object X of Top_M, we have

$$S_c(X^K) \cong S_e((X^{|K|})_c^{\mathbf{N}}) \cong S_e((X_c^{\mathbf{N}})^{|K|}) \cong (S_e X_c^{\mathbf{N}})^K \cong (S_c X)^K$$

b) R_c preserves "tensor" functors. Let X be an object of $S(S_M)$ and let Y be a topological space

$Top(R_cX \otimes K, Y) \cong Top(R_cX, Y^K) \cong S(S_M)(X, S_c(Y^K)) \cong$

$\cong S(S_M)(X, (S_cY)^K) \cong S(S_M)(X \otimes K, S_cY) \cong Top(R_c(X \otimes K), Y)$

By the Yoneda Lemma, it follows that $R_cX \otimes K \cong R_c(X \otimes K)$. This implies that R_c is simplicial left adjoint to S_c; that is, $Hom_{Top}(R_cX, Y) \cong Hom_{S(S_M)}(X, S_cY)$.

c) $R_p\colon S(S_M/ff)/fd \longrightarrow Pro$ preserves "tensor" functors. This follows because $R_p = R_c$ on the full subcategory $S(S_M/ff)/fd$. Observe that the "tensor" functor of Top, see example 2 of section 1, induces a "tensor" functor on the subcategory Pro of spaces and proper maps. Notice that we only consider "tensor" functors associated with finite simplicial sets. Using this tensor product one can define a functor $Hom_{Pro}\colon Pro^{op} \times Pro \longrightarrow SS$ by

$$Hom_{Pro}(X, Y)_q = Pro(X \otimes \Delta[q], Y).$$

In this way Pro becomes a simplicial category and the standard proper homotopy category is defined to be $\pi_0(Pro)$.

Because $R_p\colon S(S_M/ff)/fd) \longrightarrow Pro$ preserves "tensor" functors, we have the following isomorphisms:

$Hom_{Pro}(R_pX, Y)_q \cong Pro(R_pX \otimes \Delta[q], Y) \cong Pro(R_p(X \otimes \Delta[q]), Y) \cong$

$\cong S(S_M)(X \otimes \Delta[q], S_pY) \cong Hom_{S(S_M)}(X, S_pY)_q.$

This implies that we have a simplicial isomorphism

$$Hom_{Pro}(R_pX, Y) \cong Hom_{S(S_M)}(X, S_pY)$$

and $R_p: S(S_M/ff)/fd \longrightarrow Pro$ is a partial simplicial left adjoint functor for $S_p: Pro \longrightarrow S(S_M)$. The functors R_p and S_p induce the following adjointness on the categories $\pi_0(Pro)$ and $\pi_0(S(S_M))$. If X is an object of $S(S_M/ff)/fd$ and Y an object of Pro, then

$$\pi_0(Pro)(R_pX, Y) \cong \pi_0(S(S_M))(X, S_pY).$$

The last properties give the following results:

Theorem 1. The proper realization functor $R_p: S(S_M)/ff)/fd \longrightarrow Pro$ is simplicial partial left adjoint to the proper singular functor $S_p: Pro \longrightarrow S(S_M)$.

Theorem 2. If X is a cofibrant object of $S(S_M/ff)/fd$ and Y an object of Pro, then

$$\pi_0(Pro)(R_pX, Y) \cong Ho(S(S_M))(X, S_pY)$$

Proof. Let $U: S(S_M) \longrightarrow SS$ denote the forgetful functor. Notice that

$$U(S_pY) \cong US_e(Y_p^{\mathbf{N}}) = S(Y_p^{\mathbf{N}})$$

is a fibrant object of SS. By the definition of fibration in $S(S_M)$, see Definition 2.1, it follows that S_pY is fibrant in $S(S_M)$. Therefore we have

$\pi_0(Pro)(R_pX, Y) = \pi_0\ Hom_{Pro}(R_pX, Y) \cong$

$\cong \pi_0\ Hom_{S(S_M)}(X, S_pY) \cong Ho(S(S_M))(X, S_pY)$

The last isomorphism follows from the fact that X is cofibrant and S_pY is fibrant.

4) Realization and singular functors for pro–spaces.

Let C be a category with countable sums (coproducts). Using the sum of C, we can define a functor $c: C \longrightarrow proC$ as follows, if X is an object of $C, cX: \mathbf{N} \longrightarrow \mathbf{C}$ is defined by

$$(cX)_i = \bigsqcup_{j \geq i} X$$

where $\mathbf{N}$ is considered in this case with its left filtering category structure. The standard "inclusions" of the coproduct define the natural map $(cX)_{i+1} \longrightarrow (cX)_i$. Recall that the objects of the category $(proC, C)$ are promorphisms of the form $Y \to B$, where $Y: I \to C$ is an object in $proC$ and $B: 1 = \{0\} \to C$ is a constant object. To determine a promorphism

$Y \to B$ it suffices to give a map $\varphi\colon 1 \to I$ and a morphism $f_{\varphi(0)}\colon Y_{\varphi(0)} \to B_0$. Since $cX\colon \mathbf{N} \to \mathbf{C}$ is an object in $proC$ and $(cX)_0$ is a constant object, the map $\varphi\colon 1 \to \mathbf{N}$, $\varphi(0) = 0$, and id: $(cX)_{\varphi(0)} \to (cX)_0$ determine a promorphism $c_g X\colon cX \to (cX)_o$; that is, an object in (proC,C). This defines a global (or augmented) functor $c_g\colon C \to (proC, C)$. If it is necessary to distinguish the two functors we shall use the notation $c_\infty\colon C \to proC$ and $c_g\colon C \to (proC, C)$, otherwise we just write c.

Consider the following functors

a) The functor $\chi_\infty\colon \Delta \longrightarrow {}_{M_\infty}(proSS)$.

The functor $c\colon SS \longrightarrow proSS$ gives an object $c\Delta[0]$ in $proSS$ and we can consider the monoid $M_\infty = proSS(c\Delta[0], c\Delta[0])$ which is isomorphic to $Pro_\infty(\mathbf{N}, \mathbf{N})$. As in example 3, $c\Delta[0]$ has a natural structure as a left M_∞–object. The functor $- \otimes \Delta[q]\colon proSS \longrightarrow proSS$ induces left M_∞–object structures on $c\Delta[0] \otimes \Delta[q] \cong c\Delta[q]$, so there is a functor $\chi_\infty\colon \Delta \longrightarrow {}_{M_\infty}(proSS)$, $(\chi_\infty)_q = c\Delta[q]$. Now by Theorem 3.2, we obtain a realization functor $R_{\chi_\infty}\colon S(S_{M_\infty}) \longrightarrow proSS$ which is simplicial left adjoint to the corresponding singular functor $S_{\chi_\infty}\colon proSS \longrightarrow S(S_{M_\infty})$.

b) The global functor $\chi_g\colon \Delta \longrightarrow {}_M(proSS, SS)$.

Using the global version of the c functor, $c_g\colon SS \longrightarrow (proSS, SS)$, and the monoid $M = (proSS, SS)(c_g\Delta[0], c_g\Delta[0])$ which is isomorphic to $Pro(\mathbf{N}, \mathbf{N})$, we have an induced functor $\chi_g\colon \Delta \longrightarrow {}_M(proSS, SS)$. Associated with the functor χ_g, there are a realization functor $R_{\chi_g}\colon S(S_M) \longrightarrow (proSS, SS)$ and a singular functor $S_{\chi_g}\colon (proSS, SS) \longrightarrow S(S_M)$ that induce a simplicial adjunction isomorphism

$$Hom_{(proSS,SS)}(R_{\chi_g}X, Y) \cong Hom_{S(S_M)}(X, S_{\chi_g}Y).$$

c) The functor $\chi^*_\infty\colon \Delta \longrightarrow {}_{M^*_\infty}(proSS_*)$.

Recall that for a simplicial set K, $K \sqcup \Delta[0]$ is denoted by K^+. Using the functor $c\colon SS_* \longrightarrow proSS_*$, we get the object $c\Delta[0]$ and we can consider the monoid $M^*_\infty = proSS_*(c\Delta[0]^+, c\Delta[0]^+)$ which is isomorphic to $Top^\infty_*(\widehat{\mathbf{N}}, \widehat{\mathbf{N}})$. As a consequence of Theorem 3.2, we also have a natural adjunction isomorphism

$$Hom_{proSS_*}(R_{\chi^*_\infty}X, Y) \cong Hom_{S(S_{M^*_\infty})}(X, S_{\chi^*_\infty}Y).$$

d) The functor $\chi^*_g\colon \Delta \longrightarrow {}_{M^*_g}(proSS_*, SS_*)$.

The monoid $M^*_g = (proSS_*, SS_*)(c_g\Delta[0]^+, (c_g\Delta[0]^+)$ is isomorphic to $Top_*(\widehat{\mathbf{N}}, \widehat{\mathbf{N}})$. As in the cases above, we have a functor $\chi^*_g\colon \Delta \longrightarrow {}_{M^*_g}(proSS_*, SS_*)$ and a simplicial isomorphism

$$Hom_{(proSS_*,SS_*)}(R_{\chi^*_g}X, Y) \cong Hom_{S(S_{M^*_g})}(X, S_{\chi^*_g}Y).$$

5. Brown's $\mathcal{P}$ functor and the singular functor $proSS \longrightarrow S(S_M)$

In 1975, E.M. Brown [Br.1] gave a definition of a proper fundamental group ${}^{B}\pi_1^{\infty}(X)$ of a σ–compact space X with a base ray. He also defined a functor $\bar{\mathcal{P}}: towGps \longrightarrow Gps$ that gives the relation between the tower of fundamental groups, $\pi_1\varepsilon X$, of a tower of neighbourhoods of X at infinity and the proper fundamental group. This relation in given by $\bar{\mathcal{P}}\pi_1\varepsilon X \cong {}^{B}\pi_1^{\infty}(X)$. In this section we extend this definition to other categories and study the relation with the singular functor $proSS \longrightarrow S(S_M)$.

Let C denote one of the following categories:
$Set = (sets)$,
$Set_* = (pointed\ sets)$,
$Gps = (groups)$,
$Ab = (abelian\ groups)$.
The small projective generators of these (algebraic) categories will be denoted by $*, S^0, \mathbf{N}$, $\mathbf{N}_a$, respectively.

Since C has sums, we have the functor $c: C \longrightarrow proC$ defined by $cX: \mathbf{N} \longrightarrow \mathbf{C}$, $(cX)_i = \underset{j\geq i}{\sqcup} X$. Sometimes, we will also consider the global (or augmented) version $c: C \longrightarrow$ $\longrightarrow (proC, C)$.

Let G denote the small projective generator of C and let $\mathcal{P}cG$ denote the endomorphism set

$$\mathcal{P}cG = proC(cG, cG)$$

If $C = Set$, $G = *$, $\mathcal{P}c*$ has a monoid structure. Notice that $\mathcal{P}c* \cong proSS(c\Delta[0], c\Delta[0]) \cong Pro_{\infty}(\mathbf{N}, \mathbf{N})$. In the pointed case, $C = Set_*$, $G = S^0$, the endomorphism set admits the structure of a 0–monoid, see section 2, and we have that $\mathcal{P}cS^0 \cong proSS_*(c\Delta[0]^+, c\Delta[0]^+)$ $\cong Top_*^{\infty}(\widehat{\mathbf{N}}, \widehat{\mathbf{N}})$. If $C = Gps$, $G = \mathbf{N}$, the endomorphism set has a natural near–ring structure, see [Mel, Pilz]. Finally for $C = Ab$, $G = \mathbf{N}_a$, the endormorphism set $\mathcal{P}c\mathbf{N}_a$ becomes a ring isomorphic to the ring of locally finite matrices modulo the ideal of finite matrices, see [F–W.1, F–W.2].

Let $C_{\mathcal{P}cG}$ denote one of the following categories:
If $C = Set$, $G = *$, then $Set_{\mathcal{P}c*}$ is the category of right $\mathcal{P}c*$–sets.
If $C = Set_*$, $G = S^0$, then $Set_{*\mathcal{P}cS^0}$ is the category of right $\mathcal{P}cS^0$–pointed sets.
If $C = Grp$, $G = \mathbf{N}$, then $Grp_{\mathcal{P}c\mathbf{N}}$ is the category of right $\mathcal{P}c\mathbf{N}$–groups. This category is also known as the category of right near modules over the near–ring $\mathcal{P}c\mathbf{N}$, see [Mel, Pilz].

If $C = Ab$, $G = \mathbf{N}_a$, then $Ab_{\mathcal{P}c\mathbf{N}_a}$ is the category of right $\mathcal{P}c\mathbf{N}_a$–abelian groups that is usually called the category of right $\mathcal{P}c\mathbf{N}_a$–modules.

If we consider the global (or augmented) functor $c = c_g\colon C \longrightarrow (proC, C)$, we will get the endomorphism set $\mathcal{P}_g c_g G = (proC, C)(c_g G, c_g G)$ that will also be denoted by $\mathcal{P}_g cG$ and the corresponding category $C_{\mathcal{P}_g cG}$.

Given an object X of $proC$, it is easy to check that $proC(cG, X)$ is an object of $C_{\mathcal{P}cG}$. Therefore we have a functor

$$\mathcal{P}\colon proC \longrightarrow C_{\mathcal{P}cG}$$

defined by $\mathcal{P}X = proC(cG, X)$.

The full subcategory of $proC$ determined by objects indexed by natural numbers is usually denoted by $towC$. We say that an object X of $proC$ is finitely generated if there is an effective epimorphism $Q \longrightarrow X$ where Q is a finite sum of copies of cG.

We summarize some properties of the $\mathcal{P}$ functors in the following results, see [He.1].

Theorem 1. The functor $\mathcal{P}\colon proC \longrightarrow C_{\mathcal{P}cG}$ satisfies:
i) the restriction $\mathcal{P}\colon towC \longrightarrow C_{\mathcal{P}cG}$ is faithful;
ii) the restriction $\mathcal{P}\colon towC/fg \longrightarrow C_{\mathcal{P}cG}$ is also full, where $towC/fg$ is the full subcategory of $towC$ of finitely generated towers.

Theorem 2. The functor $\mathcal{P}\colon proC \longrightarrow C_{\mathcal{P}cG}$ has a left adjoint functor $\mathcal{L}\colon C_{\mathcal{P}cG} \longrightarrow proC$.

Remark. There are similar results for the category $(proC, C)$ of global pro–objects in C and the category $C_{\mathcal{P}_g cG}$.

Since the forgetful functor $U\colon C_{\mathcal{P}cG} \longrightarrow C$ has a left adjoint functor $-\odot \mathcal{P}cG\colon C \longrightarrow$ $\longrightarrow C_{\mathcal{P}cG}$, we have the pairs of adjoint functors

$$proC \underset{\mathcal{P}}{\overset{\mathcal{L}}{\rightleftarrows}} C_{\mathcal{P}cG} \underset{U}{\overset{-\odot \mathcal{P}cG}{\rightleftarrows}} C$$

and the composites $\overline{\mathcal{L}} = \mathcal{L}(-\odot \mathcal{P}cG)$, $\overline{\mathcal{P}} = U\mathcal{P}$ give a new pair of adjoint functors

$$proC \underset{\overline{\mathcal{P}}}{\overset{\overline{\mathcal{L}}}{\rightleftarrows}} C.$$

Notice that $\overline{\mathcal{P}}\colon proGps \longrightarrow Gps$ is the functor defined by Brown and denoted in his paper [Br.1] by $\mathcal{P}$.

Remark. It is easy to check that $\overline{\mathcal{L}}G = \mathcal{L}(G \odot \mathcal{P}cG) \cong \mathcal{L}(\mathcal{P}cG) = cG$ and $\overline{\mathcal{L}}f = cf$ for any morphism $f\colon G \longrightarrow G$. Therefore $\overline{\mathcal{L}} = c$ on the full subcategory of C obtained from G by considering finite colimits of copies of G.

Given a left filtering small category I, the equivalence of categories $(C^I)^\Delta \simeq (C^\Delta)^I$ induces a natural functor $F\colon proSC \longrightarrow SproC$ defined by $(FX)_q(i) = (X(i))_q$, where X is an object in $proSC$. On the other hand the functors $\mathcal{L}\colon C_{\mathcal{P}cG} \longrightarrow proC$ and $\mathcal{P}\colon proC \longrightarrow C_{\mathcal{P}cG}$ induce functors $S\mathcal{L}\colon SC_{\mathcal{P}cG} \longrightarrow SproC$ and $S\mathcal{P}\colon SproC \longrightarrow SC_{\mathcal{P}cG}$. Next we analyse the relation between the $\mathcal{P}$ functor and the singular functor $S_{\chi_\infty}\colon proSC \longrightarrow S(C_{\mathcal{P}cG})$ defined in 4) of the previous section.

The relation between these functors is given in the following:

Theorem 3. The following diagram is commutative up to natural isomorphism,

$$\begin{array}{ccc} proSC & \xrightarrow{F} & SproC \\ S_{\chi_\infty} \searrow & & \swarrow S\mathcal{P} \\ & S(C_{\mathcal{P}cG}). & \end{array}$$

Proof. We are going to use the fact that the functor $c\colon C \longrightarrow proC$ agrees with $\overline{\mathcal{L}}$ in some cases (see Remark above). We also consider several functors of the form $in\colon \mathcal{C} \longrightarrow S\mathcal{C}$, where $(inX)_q = X$ and the face and degeneracy operators are equal to the identity of X. We have the following isomorphisms

$$
\begin{aligned}
(S_{\chi_\infty} X)_q &= proSC(\chi_\infty[q], X) \\
&\cong proSC(c(inG \otimes \Delta[q]), X) \\
&\cong proSC(c\ inG \otimes \Delta[q], X) \\
(1) &\cong SproC(F(c\ inG \otimes \Delta[q]), FX) \\
&\cong SproC(F\ c\ inG \otimes \Delta[q], FX) \\
&\cong SproC(in\ cG \otimes \Delta[q], FX) \\
&\cong SproC(in\overline{\mathcal{L}}\ G \otimes \Delta[q], FX) \\
&\cong SproC(in\mathcal{L}(G \odot \mathcal{P}cG) \otimes \Delta[q], FX) \\
&\cong SproC(S\mathcal{L}\ in(G \odot \mathcal{P}cG) \otimes \Delta[q], FX) \\
&\cong SproC(S\mathcal{L}(in(G \odot \mathcal{P}cG) \otimes \Delta[q]), FX) \\
&\cong S(S_{\mathcal{P}cG})(in(G \odot \mathcal{P}cG) \otimes \Delta[q], S\mathcal{P}\ FX) \\
&\cong (S\mathcal{P}\ FX)_q
\end{aligned}
$$

The isomorphism (1) follows from the fact that $c\ inG \otimes \Delta[q]$ is a finite dimensional pro–object. If $S_{\leq q}\mathcal{C}$ denotes the category of q–truncated simplicial objects in $\mathcal{C}$; that is, functors $(\Delta/q)^{op} \longrightarrow \mathcal{C}$ where Δ/q is the full subcategory of Δ determined by the objects $[0], [1], \ldots, [q]$. It is not hard to check that

$$
\begin{aligned}
proSC(c\ inG \otimes \Delta[q], X) &\cong proS_{\leq q}C(c\ inG \otimes \Delta[q], X) \\
(2) &\cong S_{\leq q}\ proC(F(c\ inG \otimes \Delta[q]), FX) \\
&\cong SproC(F(c\ inG \otimes \Delta[q]), FX).
\end{aligned}
$$

The isomorphism (2) is a consequence of the theorem of C.V. Meyer, see [Mey], that says that $pro(C^D)$ is equivalent to $(proC)^D$ if D is a finite category and C has finite limits.

Remark. As a consequence of Theorem 3, we observe that the singular functor S_{χ_∞} is calculated dimensionwise by the $\mathcal{P}$ functor. In this way, the functor S_{χ_∞} can be considered as an extension of the functor $\mathcal{P}\colon proC \longrightarrow C_{\mathcal{P}cG}$. For this reason, in the sequel, the functors S_{χ_∞}, S_{χ_g}, $S_{\chi^*_\infty}$, $S_{\chi^*_g}$ will be denoted by $\mathcal{P}$ and the corresponding realization functors R_{χ_∞}, R_{χ_g}, $R_{\chi^*_\infty}$, $R_{\chi^*_g}$ by $\mathcal{L}$.

The forgetful functors $Set_* \longrightarrow Set$, $Gps \longrightarrow Set_*$ and $Ab \longrightarrow Gps$ have left adjoint functors denoted by

$$
\begin{aligned}
(\ )^+\colon Set &\longrightarrow Set_*, \\
f\colon Set_* &\longrightarrow Gps, \\
ab\colon Gps &\longrightarrow Ab.
\end{aligned}
$$

We consider the induced functors

$$pro(\ )^+ \colon proSet \longrightarrow proSet_*,$$
$$prof \colon proSet_* \longrightarrow proGps$$
$$pro(ab) \colon proGps \longrightarrow proAb$$

and the induced monoid homomorphisms

$$proSet(c*, c*) \longrightarrow proSet_*(cS^0, cS^0) \longrightarrow proGrp(c\mathbf{N}, c\mathbf{N}) \longrightarrow proAb(c\mathbf{N}_a, c\mathbf{N}_a).$$

Using the isomorphism $Pro_\infty(\mathbf{N}, \mathbf{N}) \cong proS(c*, c*)$, the proper map $sh \colon \mathbf{N} \longrightarrow \mathbf{N}$, $sh(i) = i+1, i \in \mathbf{N}$, define an element of $proS(c*, c*)$. The monoid homomorphisms above determine new canonical elements in the other monoids. Any one of these elements will be denoted by sh and it will be called the shift operator.

For C any of the categories with which we are working, we define a functor $F_{sh} \colon C_{\mathcal{P}cG} \longrightarrow C$ by

$$F_{sh}\ X = \{x \in X \mid x\ sh = x\}.$$

It is easy to check the functor diagram

$$\begin{array}{ccc} towC & \xrightarrow{\mathcal{P}} & C_{\mathcal{P}cG} \\ \lim \searrow & & \swarrow F_{sh} \\ & C & \end{array}$$

is commutative up to natural isomorphism, where lim is the standard inverse limit. We can also prove the following result.

Theorem 4. The following diagram

$$\begin{array}{ccc} towSC & \xrightarrow{\mathcal{P}} & S(C_{\mathcal{P}cG}) \\ \lim \searrow & & \swarrow F_{sh} \\ & SC & \end{array}$$

is commutative up to natural isomorphism.

Proof. By Theorem 3, $\mathcal{P}$ is isomorphic to $S\mathcal{P}\ F$. It is clear that $S\ \lim \cong SF_{sh}\ S\mathcal{P}$. Since the diagram

$$\begin{array}{ccc} tow\ SC & \xrightarrow{F} & Stow\ C \\ \lim \searrow & & \swarrow S\lim \\ & SC & \end{array}$$

is commutative, we have that

$$F_{sh}\ \mathcal{P} \cong F_{sh}\ S\mathcal{P}\ F \cong SF_{sh}\ S\mathcal{P}\ F \cong S\lim\ F \cong \lim.$$

Remark. The functors $\lim: proSC \longrightarrow SC$, $\mathcal{P}: proSC \longrightarrow S(C_{\mathcal{P}cG})$, $F_{sh}: S(C_{\mathcal{P}cG}) \longrightarrow$ $\longrightarrow SC$ have left adjoint functors.

6. Derived functors of $\mathcal{L}$ and $\mathcal{P}$

In this section we analyse the properties of the pair of adjoint functors $\mathcal{L} = R_{\chi^*_\infty}: S(S_{*_{\mathcal{P}cS^0}}) \longrightarrow proSS_*$, $\mathcal{P} = S_{\chi^*_\infty}: proSS_* \longrightarrow S(S_{*_{\mathcal{P}cS^0}})$ with respect to the closed model structures of these categories. In the category $proSS_*$ we consider the structure given by Edwards–Hastings [E–H] and the category $S(S_{*_{\mathcal{P}cS^0}})$ is provided with the structure given in section 2.

Recall that if $\mathcal{A}'$ is a full subcategory of $\mathcal{A}$, we say that a functor $F: \mathcal{A}' \longrightarrow \mathcal{B}$ is a partial left adjoint to the functor $G: \mathcal{B} \longrightarrow \mathcal{A}$ if for any A of $\mathcal{A}'$ and B of $\mathcal{B}$, there is a natural isomorphism $\mathcal{B}(FA, B) \cong \mathcal{A}(A, GB)$. If $\mathcal{A}, \mathcal{B}$ are simplicial categories and this isomorphism extends to a simplicial isomorphism $Hom_{\mathcal{B}}(FA, B) \cong Hom_{\mathcal{A}}(A, GB)$, it is said that F is a partial simplicial left adjoint functor to G.

Lemma 1. The restriction of the functor $c: SS_* \longrightarrow proSS_*$ to the full subcategory of finite simplicial sets is a partial simplicial left adjoint functor to $\bar{\mathcal{P}} = U\mathcal{P}: proSS_* \longrightarrow SS_*$.

Proof. For a finite simplicial set X, we have the isomorphisms

$Hom_{proSS_*}(cX, Y) \cong Hom_{proSS_*}(c\ \Delta[0]^+ \otimes X, Y) \cong$

$\cong Hom_{SS_*}(X, Hom_{proSS_*}(c\ \Delta[0]^+, Y)) \cong Hom_{SS_*}(X, \bar{\mathcal{P}}Y)$.

Lemma 2. The functor $\mathcal{P}: proSS_* \longrightarrow S(S_{*_{\mathcal{P}cS^0}})$ satisfies

1) If $p: E \longrightarrow B$ is a fibration in $proSS_*$ in the sense of Edwards–Hastings, then $\mathcal{P}p$ is a fibration in $S(S_{*_{\mathcal{P}cS^0}})$.

2) Let $p\colon E \longrightarrow B$ be a level morphism in $towSS_*$ ($p = \{p_i\colon E_i \longrightarrow B_i \mid i \in \mathbf{N}\}$) such that each $p_i\colon E_i \longrightarrow B_i$ is a fibration in SS_*. Then $\mathcal{P}p$ is a fibration in $S(S_{*_{\mathcal{P}cS^0}})$.
Proof. By Definition 2.1, $\mathcal{P}p$ is a fibration in $S(S_{*_{\mathcal{P}cS^0}})$ if and only if $U\mathcal{P}p = \bar{\mathcal{P}}p$ is a fibration in SS_*. By Lemma 1, c is partial left adjoint to $\bar{\mathcal{P}}\colon proSS_* \longrightarrow SS_*$. Therefore the existence of a lift in the commutative diagram

$$\begin{array}{ccc} V(n,k) & \longrightarrow & \bar{\mathcal{P}}E \\ \big\downarrow & & \big\downarrow \\ \Delta[n] & \longrightarrow & \bar{\mathcal{P}}B \end{array}$$

is equivalent to the existence of a lift in the corresponding commutative diagram

$$\begin{array}{ccc} cV(n,k) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ c\Delta[n] & \longrightarrow & B \end{array}$$

In case 1), the lift exists because $proSS_*$ is a closed model category and $cV(n,k) \longrightarrow c\Delta[n]$ is a trivial cofibration. For case 2), taking into account that the bonding morphisms of $cV(n,k)$, $c\Delta[n]$ are injections, $\lim\, cV(n,k) = \emptyset = \lim\, c\Delta[n]$, $cV(n,k) \longrightarrow c\Delta[n]$ is a levelwise morphism and that for each $i \geq 0$, $p_i\colon E_i \longrightarrow B_i$ is a fibration, it is easy to find a lift in the diagram above.

In the following Lemma for a given closed model category $\mathcal{C}$, we use Quillen′s notation $\mathcal{C}_f$ to denote the full subcategory of fibrant objects. We are also going to use the following notation and results: Let $SS_*^{\mathbf{N}}$ denote the category of functors $\mathbf{N} \longrightarrow SS_*$ and natural transformations. Given an object $Y\colon \mathbf{N} \longrightarrow SS_*$, consider $\mathbf{N}^+ = \{-1\} \cup \mathbf{N}$ and define $Y^+\colon \mathbf{N}^+ \longrightarrow SS_*$ by $Y^+_{-1} = *$ and $Y^+_i = Y_i$ if $i \geq 0$. For an injective increasing map $\varphi\colon \mathbf{N} \longrightarrow \mathbf{N}$, define $\bar{\varphi}\colon \mathbf{N} \longrightarrow \mathbf{N}^+$ by $\bar{\varphi}(j) = -1$ if $j \leq \varphi(0)$ and $\bar{\varphi}(j) = i$ if $\varphi(i) <$

$< j \leq \varphi(i+1)$. Now we define an object $Y^*\varphi\colon \mathbf{N} \longrightarrow SS_*$ by $(Y^*\varphi)_j = Y^+_{\bar\varphi(j)}$. There is a natural morphism $Y \longrightarrow Y^*\varphi$ and $towSS_*$ is equivalent to the category of left fractions $\Sigma^{-1}SS_*^{\mathbf{N}}$ associated with the family of morphisms of the form $Y \longrightarrow Y^*\varphi$ (see [G-Z]). As a consequence of this fact we have that

$$Hom_{towSS^*}(X,Y) \cong \operatorname{colim}_\varphi\ Hom_{SS_*^{\mathbf{N}}}(X, Y^*\varphi)$$

A more detailed description of these results is contained in [He.1].

We also have the functor $c\colon SS_* \longrightarrow SS_*^{\mathbf{N}}$ defined as usual by $(cX)_i = \bigsqcup_{j\geq i} X$ and the functor $p\colon SS_*^{\mathbf{N}} \longrightarrow SS_*$ defined by $pY = \prod_{i=0}^{+\infty} Y_i$. It is easy to check that c is left adjoint to p.

These results are applied to prove the second part of the following Lemma that will be useful to find the relation between the proper singular functor and the right–derived functor of the $\mathcal{P}$ functor.

Lemma 3. The functor $\mathcal{P}\colon proSS_* \longrightarrow S(S_{*_{\mathcal{P}cS^0}})$ satisfies

1) If f is a weak equivalence in $(proSS_*)_f$, then $\mathcal{P}f$ is a weak equivalence in $S(S_{*_{\mathcal{P}cS^0}})$. Moreover, $\mathcal{P}f$ is a homotopy equivalence in $S(S_{*_{\mathcal{P}cS^0}})$.

2) If $f = \{f_i\colon X_i \longrightarrow Y_i\}$ is a level map such that for each $i \geq 0$, $f_i\colon X_i \longrightarrow Y_i$ is a weak equivalence in SS_* and X_i, Y_i are fibrant in SS_*, then $\mathcal{P}f$ is a weak equivalence in $S(S_{*_{\mathcal{P}cS^0}})$.

Proof. 1) Since $(proSS_*)_f = (proSS_*)_{cf}$ and f is a weak equivalence, it follows that f is a homotopy equivalence. Because $\mathcal{P}\colon proSS_* \longrightarrow S(S_{*_{\mathcal{P}cS^0}})$ induces a functor $\pi_0(proSS_*) \longrightarrow \pi_0(S(S_{*_{\mathcal{P}cS^0}}))$, we get that $\mathcal{P}f$ is also a homotopy equivalence. Therefore $\mathcal{P}f$ is a weak equivalence in $S(S_{*_{\mathcal{P}cS^0}})$.

2) In order to prove that $\mathcal{P}f\colon \mathcal{P}X \longrightarrow \mathcal{P}Y$ is a weak equivalence it suffices to show that $U\mathcal{P}f\colon U\mathcal{P}X \longrightarrow U\mathcal{P}Y$ is a weak equivalence. Since for each $i \geq 0$, X_i, Y_i are fibrant, applying 2) of Lemma 2, one has that $\mathcal{P}X, \mathcal{P}Y$ are fibrant in $S(S_{*_{\mathcal{P}cS^0}})$. Therefore $U\mathcal{P}X$, $U\mathcal{P}Y$ are fibrant in SS_* and we obtain the following isomorphisms:

$$\begin{aligned}
&[\Delta[q]/\dot\Delta[q], U\mathcal{P}f] = \pi_0\ Hom_{SS_*}\ (\Delta[q]/\dot\Delta[q], U\mathcal{P}f) = \\
&= \pi_0\ Hom_{SS_*}(\Delta[q]/\dot\Delta[q], \operatorname{colim}_\varphi\ p(f^*\varphi)) \cong \\
&\cong \pi_0\ \operatorname{colim}_\varphi\ Hom_{SS_*}(\Delta[q]/\dot\Delta[q], p(f^*\varphi)) \cong \\
&\cong \pi_0\ \operatorname{colim}_\varphi\ p(Hom_{SS_*}(\Delta[q]/\dot\Delta[q], f^*\varphi(j))) \cong \\
&\cong \operatorname{colim}_\varphi\ p\ \pi_0\ Hom_{SS_*}(\Delta[q]/\dot\Delta[q], f^+_{\bar\varphi(j)}) \cong \\
&\cong \operatorname{colim}_\varphi\ p[\Delta[q]/\dot\Delta[q], f^+_{\bar\varphi(j)}]
\end{aligned}$$

where we have taken into account that

$$\begin{aligned}
&U\mathcal{P}X = Hom_{proSS_*}(c\Delta[0]^+, X) \cong \\
&\cong Hom_{towSS_*}(c\Delta[0]^+, X) \cong
\end{aligned}$$

$\cong \operatorname{colim}_\varphi \ Hom_{SS_*^{\mathbf{N}}}(c\Delta[0]^+, X^*\varphi) \cong$
$\cong \operatorname{colim}_\varphi \ Hom_{SS_*}(\Delta[0]^+, p(X^*\varphi)) \cong$
$\cong \operatorname{colim}_\varphi \ p(X^*\varphi)$
and that for maps a similar expression is obtained.
Finally, since each $[\Delta[q]/\dot{\Delta}[q], f^+_{\bar{\varphi}(j)}]$ is an isomorphism, we obtain that $[\Delta[q]/\dot{\Delta}[q], U\mathcal{P}f]$ is an isomorphism. Therefore $U\mathcal{P}f$ is a weak equivalence and by the definition of weak equivalence in $S(S_{*_{\mathcal{P}cS^0}})$ it follows that $\mathcal{P}f$ is also a weak equivalence in $S(S_{*_{\mathcal{P}cS^0}})$.

Remark. Since $\mathcal{L}: S(S_{*_{\mathcal{P}cS^0}}) \longrightarrow proSS_*$ is left adjoint to $\mathcal{P}: proSS_* \longrightarrow S(S_{*_{\mathcal{P}cS^0}})$ and $proSS_*$, $S(S_{*_{\mathcal{P}cS^0}})$ are closed model categories, it is easy to check that $\mathcal{L}$ preserves cofibrations. Using that $\mathcal{L}$ is simplicial left adjoint to $\mathcal{P}$, we also get that $\mathcal{L}$ carries a weak equivalence between cofibrant objects into a weak equivalence.

Notice that Lemma 2, Lemma 3 and the Remark after Lemma 3 prove that the functor $\mathcal{L}: S(S_{*_{\mathcal{P}cS^0}}) \longrightarrow proSS_*$ and $\mathcal{P}: proSS_* \longrightarrow S(S_{*_{\mathcal{P}cS^0}})$ satisfy the conditions of Theorem 4.3 of ch.I of [Q.1]. Therefore we have the following:

Theorem 1. The functor $\mathcal{L}: S(S_{*_{\mathcal{P}cS^0}}) \longrightarrow proSS_*$ induces a left–derived functor $\mathcal{L}^L: Ho(S(S_{*_{\mathcal{P}cS^0}})) \longrightarrow Ho(proSS_*)$ and $\mathcal{P}: proSS_* \longrightarrow S(S_{*_{\mathcal{P}cS^0}})$ induces a right–derived functor $\mathcal{P}^R: Ho(proSS_*) \longrightarrow Ho(S(S_{*_{\mathcal{P}cS^0}}))$ such that $\mathcal{L}^L$ is left adjoint to $\mathcal{P}^R$. Moreover, $\mathcal{L}^L$ preserves cofibration sequences and $\mathcal{P}^R$ preserves fibration sequences.

Recall that by Theorem 2.2 we also have the following pair of adjoint functors

$$Ho(S(S_{*_{\mathcal{P}cS^0}})) \underset{U}{\overset{-\,\odot\, M}{\rightleftarrows}} Ho(SS_*)$$

The composition of the two pairs of functor gives a new pair of adjoint functors $\bar{\mathcal{L}}^L = \mathcal{L}^L(-\odot M): Ho(SS_*) \longrightarrow Ho(proSS_*)$ and $\bar{\mathcal{P}}^R = U\mathcal{P}^R: Ho(proSS_*) \longrightarrow Ho(SS_*)$. Therefore we have:

Corollary 1. The functor $\bar{\mathcal{L}} = \mathcal{L}(-\odot M): SS_* \longrightarrow proSS_*$ has a left–derived functor $\bar{\mathcal{L}}^L: Ho(SS_*) \longrightarrow Ho(proSS_*)$ and $\bar{\mathcal{P}} = U\mathcal{P}: proSS_* \longrightarrow SS_*$ has a right–derived functor $\bar{\mathcal{P}}^R: Ho(proSS_*) \longrightarrow Ho(SS_*)$ such that $\bar{\mathcal{L}}^L$ is left adjoint to $\bar{\mathcal{P}}^R$. Moreover $\bar{\mathcal{L}}^L$ preserves cofibration sequences and $\bar{\mathcal{P}}^R$ preserves fibration sequences.

Notice that for a finite simplicial set X, we have that

$$Ho(proSS_*)(\bar{\mathcal{L}}^L X, Y) \cong Ho(SS_*)(X, \bar{\mathcal{P}}^R Y).$$

It is easy to check that $c\colon SS_* \longrightarrow proSS_*$ preserves cofibrations, then

$$Ho(proSS_*)(cX, Y) \cong Ho(proSS_*)(cX, Y')$$

where $Y \longrightarrow Y'$ is a weak equivalence and Y' is a fibrant object in $proSS_*$. Since cX is cofibrant and Y' is fibrant, we have

$$Ho(proSS_*)(cX, Y') \cong \pi_0(proSS_*)(cX, Y').$$

Applying Lemma 1, it follows that

$\pi_0(proSS_*)(cX, Y') \cong \pi_0(SS_*)(X, \bar{\mathcal{P}}Y') \cong$
$\cong \pi_0(SS_*)(X, \bar{\mathcal{P}}^R Y) \cong Ho(SS_*)(X, \bar{\mathcal{P}}^R Y).$

Therefore as a consequence of these isomorphisms, we have the following

Theorem 2. Let $Ho(SS_*)/f$ be the full subcategory of $Ho(SS_*)$ determined by finite simplicial sets. Then $c\colon Ho(SS_*)/f \longrightarrow Ho(proSS_*)$ is a partial left adjoint to $\bar{\mathcal{P}}^R\colon Ho(proSS_*) \longrightarrow Ho(SS_*)$. The functors $\bar{\mathcal{L}}^L$ and c agree up to natural isomorphism on the subcategory $Ho(SS_*)/f$, moreover, c preserves cofibration sequences associated with a map between finite simplicial sets.

Corollary 2. The following diagram

$$\begin{array}{ccc} Ho(proSS_*) & \xrightarrow{\bar{\mathcal{P}}^R} & Ho(SS_*) \\ {}^{G}\pi_q^{\infty} \searrow & & \swarrow \pi_q \\ & Gps & \end{array}$$

is commutative up to natural isomorphism, where π_q denotes the standard qth homotopy group and ${}^{G}\pi_q^{\infty}$ denotes the qth Grossman homotopy group, defined by ${}^{G}\pi_q^{\infty}(X) =$ $= Ho(proSS_*)(cS^q, X),\ S^q = \Delta[q]/\dot{\Delta}[q]$.

Proof. By Theorem 2 above, c is partial left adjoint to $\bar{\mathcal{P}}^R$, so

$${}^{G}\pi_q^{\infty}(X) = Ho(proSS_*)(cS^q, X) \cong Ho(SS_*)(S^q, \bar{\mathcal{P}}^R X) \cong \pi_q(\bar{\mathcal{P}}^R X).$$

Theorem 3 (Brown). The following diagram

$$\begin{array}{ccc} Ho(towSS_*) & \xrightarrow{tow\pi_q} & towGps \\ {}^G\pi_q^\infty \searrow & & \swarrow \bar{\mathcal{P}} \\ & Gps & \end{array}$$

is commutative up to natural isomorphism, where $tow\pi_q$ is the natural prolongation of the functor π_q to the category of towers.

Proof. We use again the fact that $towSS_*$ can be obtained as a category of left fractions of the category $SS_*^{\mathbf{N}}$, see the notation given before Lemma 3 and [He.1].

$$\begin{aligned} {}^G\pi_q^\infty(X) &= Ho(towSS_*)(cS^q, X) \\ &\cong \pi_0 Hom_{towSS_*}(cS^q, RX) \\ &\cong \pi_0 \ \mathrm{colim}_\varphi \ Hom_{SS_*^{\mathbf{N}}}(cS^q, (RX)^*\varphi) \\ &\cong \pi_0 \ \mathrm{colim}_\varphi \ Hom_{SS_*^{\mathbf{N}}}(S^q, p((RX)^*\varphi)) \\ &\cong \mathrm{colim}_\varphi \ \pi_0 \ Hom_{SS_*^{\mathbf{N}}}(S^q, p((RX)^*\varphi)) \\ &\cong \mathrm{colim}_\varphi \ \pi_q \ (p((RX)^*\varphi)) \\ &\cong \mathrm{colim}_\varphi \ p((\pi_q(RX))^*\varphi) \cong \mathrm{colim}_\varphi \ p((\pi_q X)^*\varphi) \\ &\cong U \ \mathcal{P} \ tow\pi_q \ X \cong \bar{\mathcal{P}} \ tow\pi_q \ X \end{aligned}$$

where $\pi_q(RX) = \{\pi_q(RX(i)) \mid i \geq 0\} \cong \{\pi_q(X(i)) \mid i \geq 0\} = \pi_q X$.

In Theorem 5.4, we have seen that the functors $\mathcal{P}\colon towSS_* \longrightarrow S(S_{*_{\mathcal{P}cS^0}})$ and $\lim\colon towSS_* \longrightarrow SS_*$ are related by the functor $F_{sh}\colon S(S_{*_{\mathcal{P}cS^0}}) \longrightarrow SS_*$ in such a way that $\lim \cong F_{sh} \ \mathcal{P}$. The following result gives an induced relation between the right–derived functor $\lim^R = \mathrm{holim}$ of the lim functor and the right–derived functor $\mathcal{P}^R$ of the $\mathcal{P}$ functor. We refer the reader to [E–H] for the definition and properties of the functor $\mathrm{holim}\colon Ho(towSS_*) \longrightarrow SS_*$.

Theorem 4. The functor $\text{holim} = \lim^R\colon Ho(towSS_*) \longrightarrow Ho(SS_*)$ can be factorized as

$$\begin{array}{ccc} Ho(towSS_*) & \xrightarrow{\lim^R} & Ho(SS_*) \\ \mathcal{P}^R \searrow & & \nearrow F_{sh} \\ & \pi_0(S(S_{*_{\mathcal{P}cS^0}})) & \end{array}$$

Proof. We have proved that $\mathcal{P}\colon (towSS_*)_f \longrightarrow S(S_{*_{\mathcal{P}cS^0}})$ sends weak equivalences into simplicial homotopy equivalences. Since F_{sh} preserves finite limits, it follows that F_{sh} preserves homotopy relations defined by cocylinders ($Y^{\Delta[1]}$). Therefore F_{sh} induces a functor $F_{sh}\colon \pi_0(S(S_{*_{\mathcal{P}cS^0}})) \longrightarrow Ho(SS_*)$.

Given an object X in $towSS_*$, we have that

$$F_{sh}\mathcal{P}^R X = F_{sh}\mathcal{P}RX \overset{(1)}{\cong} \lim RX = \text{holim} X$$

where (1) is a consequence of Theorem 5.4 and we have used the definition of holim given by Edwards–Hastings [E–H, page 133].

7. Simplicial complexes and simplicial M–sets.

In this section we consider noncompact simplicial complexes X satisfying

1) X is locally finite. Each point of $x \in X$ has a neighbourhood U which has points in common with only a finite number of simplexes.

2) X has finite dimension.

3) X has a countably infinite number of simplexes.

A simplicial complex of this type is homeomorphic to a subspace of some euclidean space $\mathbf{R}^m$ which is the union of countably many simplexes of dimensions 0 through n. Two simplexes have empty intersection or they meet in a common face and the countable family of simplexes is locally finite. A simplicial complex is said to be n–dimensional if it contains at least one n–simplex but none of higher dimension. A simplicial complex X is said to be n–dimensional at infinity if for every finite subcomplex K of X, there is at least an n–simplex of $X - K$ and none of higher dimension. In this section, simplicial complex means a simplicial complex satisfying conditions 1), 2) and 3).

Recall the functor $R_p\colon S(S_M/ff)/fd \longrightarrow Pro$ defined in 3) of section 4, where $M = Pro(\mathbf{N}, \mathbf{N})$. In this section for each simplicial complex X, we construct a simplicial M–set, N, in $S(S_M/ff)/fd$ such that $R_pN \cong X$. The simplicial M–set N satisfies that for each dimension $q \geq 0$, N_q is (ff), a free M–set over a finite set. The condition fd means that N has finite dimension, that is, there is n such that for $q \geq n$ every simplex of N_q is degenerate. The simplicial M–set N will be proved to be cofibrant in the closed model structure of $S(S_M)$.

Let X be a simplicial complex (satisfying 1), 2) and 3)) such that both the dimension of X and the dimension of X at infinity are equal to n. We can define a simplicial M–set, N, associated with X as follows: Define $N_0 = M$, $N_1 = s_0M \sqcup M$, where s_0M is a copy of M and $\sqcup$ denotes the sum of M–sets. For a k with $0 \leq k \leq n$, define

$$\begin{aligned} N_k = & s_{k-1}s_{k-2}\dots s_0M \ \sqcup \\ & (\bigsqcup_{k>i_{k-2}>\cdots>i_0\geq 0} s_{i_{k-2}}\cdots s_{i_0}M) \ \sqcup \\ & \cdots \\ & (\bigsqcup_{k>i_1>i_0\geq 0} s_{i_1}s_{i_0}M) \ \sqcup \\ & (\bigsqcup_{k>i_0\geq 0} s_{i_0}M) \ \sqcup \\ & M \end{aligned}$$

where any $s_{i_r}\dots s_{i_0}M$ is a copy of M. For $k > n$, N_k is similarly defined except that the last M above is removed.

The degeneracy operators of N are defined using the identity of M. Given M or a copy of M of the form $s_{i_{r-1}}\cdots s_{i_0}M$ with $k > i_{r-1} > \cdots > i_0 \geq 0$ and $k \geq i \geq 0$ we use the relations $s_is_j = s_{j+1}s_i$ if $i \leq j$ to find a copy $s_{i_{r-1}+1}\cdots s_i \cdots s_{i_0}M$ such that $i_{r-1}+1 > \cdots > i > \cdots i_0$. Then s_i is defined from $s_{i_{r-1}}\cdots s_{i_0}M$ to $s_{i_{r-1}+1}\cdots s_i\cdots s_{i_0}M$ by the "identity" map.

To define the face operator we consider two cases: If we have a copy of M of the from $s_{i_r}\cdots s_{i_0}M$ or if we have M. In the first case we use the relations $d_is_j = s_{j-1}d_i$ if $i < j$, $d_is_j = id$ if $i = j$ or $i = j+1$ and $d_is_j = s_jd_{i-1}$ if $i > j+1$ to transform an expression of the form $d_is_{i_r}\cdots s_{i_0}M$ into an expression of the form $s_{j_{r-1}}\cdots s_{j_0}M$. Then the restriction of the face operator d_i to $s_{i_r}\cdots s_{i_0}M$ is defined by the "identity" map from $s_{i_r}\cdots s_{i_0}M$ to $s_{j_{r-1}}\dots s_{j_0}M$.

Now we have to define the face operators for the term of N_k $(1 \leq k \leq n)$ equal to M. It is in this step where we use the combinatorial structure of the simplicial complex X.

Given a simplicial complex X (satisfying 1), 2) and 3)) such that the dimension of X and the dimension of X at infinity are equal to n, firstly, a enumeration can be chosen for the countable set of 0–simplexes of X, E_0^0, E_1^0, E_2^0, etc. This enumeration induces a unique order to the finite set of vertexes of each k–simplex E^k of X. Therefore for each k–simplex E^k of X the different faces $d_0E^k, d_1E^k, \dots, d_kE^k$ are well determined. We also choose and enumeration for the countable set of 1–simplexes of X, the countable set of 2–simplexes, etc, and finally for the countable set of n–simplexes.

If $0 < k \leq n$, $0 \leq i \leq k$, for each $l \in \mathbf{N}$ the face $d_i E_l^k$ is equal to some $E_{\varphi_i l}^{k-1}$. This defines a proper map $\varphi_i\colon \mathbf{N} \longrightarrow \mathbf{N}$; that is, an element $\varphi_i \in M$. The restriction of the face operator d_i applies the term M of N_k into the term M of N_{k-1}. Since M is a right M–set freely generated by $1 \in M$, it suffices to define $d_0 1 = \varphi_0, \ldots, d_k 1 = \varphi_k$.

The simplicial M–set, N, satisfies that $R_p N \cong X$, where $R_p\colon S(S_M/ff)/fd \longrightarrow Pro$ is the realization functor defined in 3) of section 4. The reason of this fact is that space X admits the following inductive construction: We start with a "proper" 0-simplex $\mathbf{N} \times |\mathbf{\Delta[0]}|$. We attach a "proper" 1-simplex to obtain the 1-skeleton, and continue in this way to obtain the n-skeleton of X . On the other hand, if we look at the definiton of $R_p N$ and take into account Proposition 3.2, we have to consider the diagram $D(Sk_n N)$, see Section 3. In this case, because $s_j 1_M = 1_M$ we can reduce again $D(Sk_n N)$ to a diagram that contains exactly the necessary instructions to attach each face $\mathbf{N} \times |\partial_\mathbf{i} \mathbf{\Delta[q]}|$ of the "proper q-simplex $\mathbf{N} \times |\mathbf{\Delta[q]}|$.

Notice that the realization functor satisfies

$$R_p(\Delta[q] \odot M) \cong \mathbf{N} \times |\Delta[q]| \cong \bigsqcup_{\mathbf{N}} |\Delta[q]|,$$

$$R_p(\dot{\Delta}[q] \odot M) \cong \mathbf{N} \times |\dot{\Delta}[q]| \cong \bigsqcup_{\mathbf{N}} |\dot{\Delta}[q]|.$$

For the case that both the dimension of X and the dimension of X at infinity are equal to n, we have constructed a simplicial M–set, N, such that $R_p N \cong X$. For the general case we have that $\dim X = m \geq n$, where n is the dimension of X at infinity. We note that there are finitely many simplexes of dimension greater than n. Using the construction above we can find a simplicial M-set, N', such that $R_p N' \cong sk_n X$. In order to attach the simplexes of dimension greater than n, for each $q \geq n$, we are going to construct a simplicial M-set, $\Delta_1[q]$ such that $R_p\ \Delta_1[q] \cong |\Delta[q]| \sqcup (\bigsqcup_1^{+\infty} *)$. Now instead of attaching $|\Delta[q]|$ by using a map $|\dot{\Delta}[q]| \to sk_{q-1}X$, we attach $|\Delta[q]| \sqcup (\bigsqcup_1^{+\infty} *)$ by using a proper map $|\dot{\Delta}[q]| \sqcup (\bigsqcup_1^{+\infty} *) \longrightarrow sk_{q-1}X$.

We note that if Y is a simplicial M-set, we have the following isomorphisms:

$$Hom_{S(S_M))}(\Delta[q] \odot M, Y) \cong Hom_{SS}(\Delta[q], UY) \cong UY_q$$

where U is right adjoint to $- \odot M$. Therefore each element $y \in Y_q$ determines a map $f_y\colon \Delta[q] \odot M \to Y$. Recall that for each simplicial set Z, we have that $(Z \odot M)_q \cong Z_q \odot M \cong Z_q \times M$, and an element (z, m) of $Z_q \odot M$ is also denoted by $z \odot m$. If i_q denotes the identity of $[q]$ and $sh\colon \mathbf{N} \to \mathbf{N}$ is an element of M defined by $sh(i) = i+1$, we have that the element $i_q \odot sh$ of $(\Delta[q] \odot M)_q$ determines a map $f_{i_q \odot sh}\colon \Delta[q] \odot M \to \Delta[q] \odot M$. We also consider the restriction of $f_{i_q \odot sh}$ to the corresponding $(q-1)$-skeletons that will be denoted by $sk_{q-1}(f_{i_q \odot sh})$. On the other hand the final map $*\colon \Delta[q] \to \Delta[0]$ induces a map $* \odot M\colon \Delta[q] \odot M \longrightarrow \Delta[0] \odot M$. Using this notation the simplicial M-set, $\Delta_1[q]$ is determined by the pushout

$$\begin{array}{ccc}
\Delta[q] \odot M & \xrightarrow{* \odot M} & \Delta[0] \odot M \\
{\scriptstyle f_{i_q \odot sh}} \downarrow & & \downarrow \\
\Delta[q] \odot M & \longrightarrow & \Delta_1[q]
\end{array}$$

and similarly one also has the pushout

$$\begin{array}{ccc}
\dot{\Delta}[q] \odot M & \xrightarrow{* \odot M} & \Delta[0] \odot M \\
{\scriptstyle sk_{q-1}(f_{i_q \odot sh})} \downarrow & & \downarrow \\
\dot{\Delta}[q] \odot M & \longrightarrow & \dot{\Delta}_1[q]
\end{array}$$

It is easy to check that

$$R_p\ \Delta_1[q] \cong |\Delta[q]| \sqcup (\coprod_1^{+\infty} *),$$

$$R_p\ \dot{\Delta}_1[q] \cong |\dot{\Delta}[q]| \sqcup (\coprod_1^{+\infty} *).$$

If we suppose that we have a simplicial M-set, N', such that $R_p N' \cong sk_n X$. Because there are finitely many simplexes with dimension greater than n, we can consider pushouts of the form $N'' = N' \sqcup_{\dot{\Delta}_1[p]} \Delta_1[p]$ to obtain finally the desired N.

Notice that the simplicial M–set N has the following skeletal structure

$$sk_0 N \subset sk_1 N \subset \cdots \subset sk_n N \subset \cdots \subset sk_m N$$

where if $l \leq n$, $sk_l N$ is obtained from $sk_{l-1}N$ by a pushout of the form

$$\begin{array}{ccc} \dot{\Delta}[l] \odot M & \longrightarrow & sk_{l-1}N \\ \downarrow & & \downarrow \\ \Delta[l] \odot M & \longrightarrow & sk_l N \end{array}$$

where the map $\Delta[l] \odot M \to sk_l N$ is determined by the adjoint isomorphisms by the identity 1 of M considered as an element of the term M of $(sk_l N)_l \cong N_l$. If $l > n$, N_l is obtained from N_{l-1} by a pushout of the form

$$\begin{array}{ccc} \bigsqcup_{finite} \dot{\Delta}_1[l] & \longrightarrow & sk_{l-1}N \\ \downarrow & & \downarrow \\ \bigsqcup_{finite} \Delta_1[l] & \longrightarrow & sk_l N \end{array}$$

Since $\dot{\Delta}_1[l] \longrightarrow \Delta_1[l]$ is a retract of $\dot{\Delta}[l] \odot M \longrightarrow \Delta[l] \odot M$, which is a cofibration, it follows that $\dot{\Delta}_1[l] \longrightarrow \Delta_1[l]$ and $\dot{\Delta}[l] \odot M \longrightarrow \Delta[l] \odot M$ are cofibrations in $S(S_M)$. Therefore N is a cofibrant object in $S(S_M)$. It is also clear that N is an object of $S(S_M/ff)/fd$.

Then we have proved the following:

Theorem 1. For any simplicial complex X, there is an object N in $S(S_M/ff)/fd$ which is cofibrant in $S(S_M)$ and such that $R_p N \cong X$.

We are going to analyse the relationship between the proper realization functor $R_p\colon S(S_M/ff)/fd \longrightarrow Pro$ and the realization functor $\mathcal{L} = R_{\chi^g_\infty}\colon S(S_M) \longrightarrow (proSS, SS)$. Consider the Edwards–Hastings embedding $\varepsilon\colon Pro \longrightarrow (proTop, Top)$ and the restrictions $\varepsilon\colon Pro_\sigma \longrightarrow (proTop, Top)$ and $\varepsilon\colon PC \longrightarrow (proTop, Top)$, where Pro_σ is the full subcategory of Pro determined by locally compact, σ–compact Hausdorff spaces and PC is the full subcategory of Pro_σ determined by spaces that admits a triangulation as a simplicial complex satisfying the conditions 1), 2) and 3) of the beginning of the section.

Edwards and Hastings [E–H; Proposition 6.2.7] proved that the induced functors

$\varepsilon\colon \pi_0(Pro_\sigma) \longrightarrow Ho_{St}(proTop, Top)$

$\varepsilon\colon \pi_0((Pro_\sigma)_\infty) \longrightarrow Ho_{St}(proTop)$

are full embeddings, where $\pi_0(Pro_\sigma)$ and $\pi_0((Pro_\sigma)_\infty)$ are defined dividing by proper homotopies and germs of proper homotopies and $Ho_{St}(proTop, Top)$, $Ho_{St}(proTop)$ are obtained by the inversion of the weak equivalences of $(proTop, Top)$ (*resp.*, $proTop$) of the closed model structure defined by Edwards–Hasting [E–H] on these procategories and induced by the Strøm closed model structure of Top.

If one considers the closed simplicial model structure of Top defined by Quillen [Q.1], using the Edwards–Hastings method there are induced closed simplicial model structures on the categories $(proTop, Top)$ and $proTop$. Let $Ho_Q(proTop, Top)$, $Ho_Q(proTop)$ denote the corresponding localized categories. Using these new closed model structures, there are also full embeddings

$\varepsilon\colon \pi_0(PC) \longrightarrow Ho_Q(proTop, Top)$

$\varepsilon\colon \pi_0((PC)_\infty) \longrightarrow Ho_Q(proTop)$

if we consider the restriction of ε to spaces that admit a triangulation as a simplicial complex.

The standard realization and singular functor $Top \underset{S}{\overset{R}{\rightleftarrows}} SS$ induce equivalences of categories

$$Ho_Q(proTop, Top) \underset{S}{\overset{R}{\rightleftarrows}} Ho(proSS, SS)$$

$$Ho_Q(proTop) \underset{S}{\overset{R}{\rightleftarrows}} Ho(proSS)\,.$$

Therefore we also have the full embeddings

$S\varepsilon\colon \pi_0(PC) \longrightarrow Ho(proSS, SS)$

$S\varepsilon\colon \pi_0((PC)_\infty) \longrightarrow Ho(proSS)$

The following proposition relates the proper realization functor $R_p\colon S(S_M/ff)/fd \longrightarrow$ $\longrightarrow PC$ and the realization functor $\mathcal{L}\colon S(S_M) \longrightarrow (proSS, SS)$.

Proposition 1. Let X be an object of PC and let N be a simplicial M–set associated with X by the construction given in this section ($R_pN \cong X$). Then $\mathcal{L}N$ is isomorphic to $S\varepsilon X$ in the category $Ho(proSS, SS)$.

Proof. Let X be a simplicial complex and assume that the set of vertexes of each simplex of X is provided with a fixed order. We can define a simplicial set sX by

$$(sX)_q = \{f\colon |\Delta[q]| \longrightarrow X \mid f \text{ is a simplicial, order–preserving map}\}$$

It is well known that $sX \longrightarrow SX$ is a weak equivalence in SS. Therefore if X is an object of PC provided with an enumeration for the countable set of its vertexes and $X = X(0) \supset X(1) \supset \cdots$ is a decreasing sequence of subcomplexes such that $X(i+1) \subset \mathrm{Int}X(i)$, $i \geq 0$, and $\cap X(i) = \emptyset$, we have that $s\varepsilon' X \longrightarrow S\varepsilon X$ is a weak equivalence in $(proSS, SS)$, where $s\varepsilon' X = \{sX(i)\}$.

Assume that X is an object in PC with $\dim X = m$ and the dimension of X at infinity is equal to n ($m \geq n$). Suppose that X is provided with the corresponding enumerations for the countable sets of 0–simplexes, 1–simplexes, … , and n–simplexes. Then we have the following pushouts

$$\begin{array}{ccc}
R_p(\dot{\Delta}[l] \odot M) \cong \bigsqcup_{\mathbf{N}} |\dot{\Delta}[l]| & \longrightarrow & sk_{l-1}X \\
\downarrow & & \downarrow \\
R_p(\Delta[l] \odot M) \cong \bigsqcup_{\mathbf{N}} |\Delta[l]| & \longrightarrow & sk_l X
\end{array} \qquad 1 \leq l \leq n$$

and for $n < l \leq m$

$$\begin{array}{ccc}
\bigsqcup_{finite} R_p\dot{\Delta}_1[l] & \longrightarrow & sk_{l-1}X \\
\downarrow & & \downarrow \\
\bigsqcup_{finite} R_p\ \Delta_1[l] & \longrightarrow & sk_l X
\end{array}$$

The "functor" $s\varepsilon'$ preserves these colimits and we have, in $(proSS, SS)$, the pushouts

$$\begin{array}{ccc} s\varepsilon'(R_p(\dot{\Delta}[l]\odot M)) & \longrightarrow & s\varepsilon'(sk_{l-1}X) \\ \downarrow & & \downarrow \\ s\varepsilon'(R_p(\Delta[l]\odot M)) & \longrightarrow & s\varepsilon'(sk_l X) \end{array} \qquad 1\leq l\leq n$$

$$\begin{array}{ccc} s\varepsilon'(\bigsqcup_{finite} R_p(\dot{\Delta}_1[l])) & \longrightarrow & s\varepsilon'(sk_{l-1}X) \\ \downarrow & & \downarrow \\ s\varepsilon'(\bigsqcup_{finite} R_p(\Delta_1[l])) & \longrightarrow & s\varepsilon'(sk_l X) \end{array} \qquad n< l\leq m$$

The left adjoint $\mathcal{L}\colon S(S_M)\longrightarrow (proSS, SS)$ preserves colimits, so for the simplicial M–set, N, we have the sequence

$\mathcal{L}sk_0N\subset \mathcal{L}sk_1N\subset\cdots\subset\mathcal{L}sk_nN\subset\cdots\subset\mathcal{L}sk_mN\cong\mathcal{L}N$

and the pushouts

$$\begin{array}{ccc} \mathcal{L}(\dot{\Delta}[l]\odot M) & \longrightarrow & \mathcal{L}(sk_{l-1}N) \\ \downarrow & & \downarrow \\ \mathcal{L}(\Delta[l]\odot M) & \longrightarrow & \mathcal{L}(sk_l N) \end{array} \qquad 1\leq l\leq n$$

$$\begin{array}{ccc}\mathcal{L}(\underset{finite}{\sqcup}\dot{\Delta}_1[l]) & \longrightarrow & \mathcal{L}(sk_{l-1}N) \\ \downarrow & & \downarrow \\ \mathcal{L}(\underset{finite}{\sqcup}\Delta_1[l]) & \longrightarrow & \mathcal{L}(sk_l N) \qquad n < l \leq m\end{array}$$

But we have that

$\mathcal{L}(\dot{\Delta}[l] \odot M) \cong s\varepsilon'(R_p(\dot{\Delta}[l] \odot M))$

$\mathcal{L}(\Delta[l] \odot M) \cong s\varepsilon'(R_p(\Delta[l] \odot M))$

$\mathcal{L}(\dot{\Delta}_1[l]) \cong s\varepsilon'(R_p\dot{\Delta}_1[l])$

$\mathcal{L}(\Delta_1[l]) \cong s\varepsilon'(R_p\Delta_1[l])$

Then by induction it follows that $\mathcal{L}(sk_0N) \cong s\varepsilon' sk_0X$, $\mathcal{L}(sk_1N) \cong s\varepsilon' sk_1X, \ldots,$ and finally $\mathcal{L}N \cong s\varepsilon' X$. Therefore $S\varepsilon X$ is isomorphic in $Ho(proSS, SS)$ to $\mathcal{L}(N)$ where N is an object of $S(S_M/ff)/fd$ which is cofibrant in $S(S_M)$.

8. Applications to proper homotopy theory.

Associated with the monoid $M = Pro(\mathbf{N}, \mathbf{N})$, we have introduced the proper realization functor $R_p: S(S_M/ff)/fd \longrightarrow Pro$ and the proper singular functor $S_p: Pro \longrightarrow S(S_M)$. Given an object N of $S(S_M/ff)/fd$ and a space Y, by Theorem 4.1, we have that $\pi_0(Pro)(R_pN, Y) \cong \pi_0(S(S_M))(N, S_pY)$. If N is also a cofibrant object in $S(S_M)$, then Theorem 4.2 implies that $\pi_0(Pro)(R_pN, Y) \cong Ho(S(S_M))(N, S_pY)$. Consequently, in some cases, the problem of computing sets of proper homotopy classes is translated from the proper homotopy category $\pi_0(Pro)$ to the category of fractions $Ho(S(S_M))$.

We note that the definition of the functor S_p is given by sequences of singular simplexes converging to infinity. Therefore the use of the functors R_p and S_p will be more convenient for spaces which are first countable at infinity. For more general spaces we have to use nets instead of sequences and the category $S(S_M)$ would have to be modified to one of the form $S(S_{\mathcal{M}})$ where $\mathcal{M}$ is a category of "proper maps" between directed sets. In any case, many of the more important applications of the proper homotopy theory are concerning with noncompact spaces which are first countable at infinity.

An important class of these latter spaces are the simplicial complexes considered in section 7. Recall that PC denotes the category of proper maps between spaces that admit a simplicial decomposition with a countably infinite number of simplexes, we also assume that this triangulation is locally finite and has finite dimension. By Theorem 7.1, a simplicial complex X of PC is of form $X \cong R_p N$ where N is an object of $S(S_M/ff)/fd$ which is cofibrant in $S(S_M)$. Then it follows that

$$\pi_0(Pro)(X,Y) \cong \pi_0(Pro)(R_p N, Y) \cong Ho(S(S_M))(N, S_p Y).$$

In order to define the proper homotopy groups of a space X, we choose a base sequence $\sigma\colon \mathbf{N} \longrightarrow X$ converging to infinity. Associated with X, one has the simplicial M–set, $S_p X$, and the forgetful functor $U\colon S(S_M) \longrightarrow SS$ gives the simplicial set $\bar{S}_p X = U S_p X$. Notice that σ is a 0–simplex of $\bar{S}_p X$. We consider the following definition of proper homotopy groups ${}^p\pi_q(X,\sigma)$

Definition 1. Let X be a space and $\sigma\colon \mathbf{N} \longrightarrow X$ a proper map, then the qth proper homotopy group is defined by

$${}^p\pi_q(X,\sigma) := \pi_q(\bar{S}_p X, \sigma).$$

Remarks. 1) For the category Pro_∞ of germs of proper maps and the monoid $M_\infty = Pro_\infty(\mathbf{N},\mathbf{N})$, we have similar notions and results. For instance, we can consider the proper homotopy groups at infinity ${}^p\pi_q^\infty(X,\sigma)$ of a space X and base sequence σ.
2) E.M. Brown [Br.1] defined the proper homotopy groups ${}^B\pi_q^\infty(X,\alpha)$ of a space X with a proper base ray $\alpha\colon [0,+\infty) \longrightarrow X$. If S^q denotes the q-sphere and $*$ is a base point of S^q, we can consider the Brown q-sphere ${}^BS^q = ([0,\infty)\times\{*\})\cup(\mathbf{N}\times\mathbf{S^q})$. It is easy to check that the inclusion $\mathbf{N}\times\mathbf{S^q} \to ([\mathbf{0},\infty)\times\{*\})\cup(\mathbf{N}\times\mathbf{S^q})$ induces a group isomorphism $\eta_\alpha\colon {}^B\pi_q^\infty(X,\alpha) \to {}^p\pi_q^\infty(X,\alpha/\mathbf{N})$. We note that if $\alpha,\alpha'\colon [0,\infty) \to X$ are two proper rays such that $\alpha/\mathbf{N} = \alpha'/\mathbf{N}$, we have the group isomorphism $\theta = \eta_{\alpha'}^{-1}\eta_\alpha\colon {}^B\pi_q^\infty(X,\alpha) \to {}^B\pi_q^\infty(X,\alpha')$. However, two different choices of base ray can lead to non-isomorphic progroups. We refer the reader to Siebenmann's thesis [Sie.1]. He considers a space X (an infinity cylinder with an infinity string of circles) and two proper maps $\alpha,\alpha'\colon [0,\infty) \to X$ that lead to non-isomorphic progroups

$$G = tow\pi_1(\varepsilon(X,\alpha)) \ncong tow\pi_1(\varepsilon(X,\alpha')) = G'$$

Siebenmann shows that for α, $\lim G$ is a cyclic infinite group, and for α', $\lim G'$ is a trivial group. Recall that if one consider the functor $\mathcal{P}\colon towGrp \longrightarrow Grp_{\mathcal{P}c\mathbf{N}}$, for the progroups G, G' we have the group isomorphisms:

$$\lim G \cong \{x \in \mathcal{P}G \mid x\ sh = x\}.$$

$$\lim G' \cong \{x' \in \mathcal{P}G' \mid x'sh = x'\}.$$

Therefore, as a consequence of Siebenmman's example we obtain that $\mathcal{P}G$ is not isomorphic to $\mathcal{P}G'$ in the category $Grp_{\mathcal{P}c\mathbf{N}}$ (notice that any morphism in $Grp_{\mathcal{P}c\mathbf{N}}$ has to 'commute' with the shift operator sh).

On the other hand, by Theorem 6.3, one has canonical isomorphisms

$$\bar{\mathcal{P}}G \cong {}^{B}\pi_1^{\infty}(X, \alpha),$$

$$\bar{\mathcal{P}}G' \cong {}^{B}\pi_1^{\infty}(X, \alpha').$$

In the Siebenmann example we have that the group isomorphism θ does not preserve the action of sh, hence θ is not a morphism of the category $Grp_{\mathcal{P}c\mathbf{N}}$.

Now we obtain the following version of the Whitehead Theorem in the proper setting.

Theorem 1. Let $f: X \longrightarrow Y$ be a proper map between simplicial complexes (that is, f is a morphism of PC). Then f is a proper homotopy equivalence if and only if ${}^{p}\pi_q(f): {}^{p}\pi_q(X, \sigma) \longrightarrow {}^{p}\pi_q(Y, f\sigma)$ is an isomorphism for all $q \geq 0$ and for every base sequence σ.

Proof. Let Z be a object in PC. By Theorem 7.1, there is an object N in $S(S_M/ff)/fd$ which is cofibrant in $S(S_M)$ and such that $Z \cong R_pN$. Using Theorem 7.2 we obtain the following commutative diagram

$$\begin{array}{ccc} \pi_0(Pro)(R_pN, X) & \cong & Ho(S(S_M))(N, S_pX) \\ {\scriptstyle f_*}\Big\downarrow & & \Big\downarrow{\scriptstyle (S_pf)_*} \\ \pi_0(Pro(R_pN, Y)) & \cong & H_0(S(S_M))(N, S_pY) \end{array}$$

By the definition of ${}^{p}\pi_q$, one has that $\bar{S}_pf = US_pf$ is a weak equivalence in SS. Taking into account the definition of weak equivalence in $S(S_M)$, we have that S_pf is a weak equivalence. Therefore $(S_pf)_*$ is an isomorphism in the diagram above and this implies that f_* is also isomorphism. This follows for any $Z \cong R_pN$ and by the Yoneda Lemma, one obtains, that f is an isomorphism in $\pi_0(Pro)$; that is, f is a proper homotopy equivalence.

Remarks. 1) A similar version of this proper Whitehead Theorem can be proved for germs of proper maps and the proper homotopy groups at infinity ${}^p\pi_q^\infty$.
2) Siebenmann [Sie.2], Farrel–Taylor–Wagoner [F–T–W], Edwards–Hastings [E–H] and Bassendoski [Bas] have proved different versions of the proper Whitehead Theorem. There are also other versions of the Whitehead Theorem for prospaces and pro–simplicial sets that can be applied to proper homotopy. Extremiana–Hernández–Rivas [E–H–R.1] gave a version that only uses strong (Steenrod) proper homotopy groups. Baues [Ba.2] and Ayala–Domínguez–Quintero have given a Whitehead Theorem for spaces with a base tree.
3) Let $\pi = {}^p\pi_1(X)$ and assume that there is an action of π on an abelian group A. One can define proper cohomology of X with twisted coefficients by ${}^pH^q(X;A) := H^q(\bar{S}_pX;A)$. It is clear that the cohomological version of the standard Whitehead Theorem implies a similar version for the proper category.

The following result gives the relation between the proper singular functor, the right–derived functor of the $\mathcal{P}$ functor and the Edwards–Hasting functor.

Theorem 2. Let Pro_σ be the full subcategory of Pro determined by locally compact, σ–compact Hausdorff spaces. Then the following diagram

$$\begin{array}{ccc} \pi_0(Pro_\sigma) & \xrightarrow{S\varepsilon} & Ho(proSS, SS) \\ S_p \searrow & & \swarrow \mathcal{P}^R \\ & Ho(S(S_M)) & \end{array}$$

is commutative up to natural isomorphism.

Proof. Let X be an object in Pro_σ. From the topological properties of X, we infer that $\varepsilon X \cong \{X_i \mid i\in\mathbf{N}\}$. Therefore $S\varepsilon X \cong \{SX_i\}$. Using the properties of the model structure of $(towSS, SS)$, one has a levelwise map $\{f_i\colon SX_i \longrightarrow (RS\varepsilon X)_i\}$ such that $RS\varepsilon X$ is a fibrant object, and for each $i \geq 0$, SX_i and $(RS\varepsilon X)_i$ are fibrant objects, and f_i is a weak equivalence. We can now apply Lemma 6.3, to obtain that

$$\mathcal{P}S\varepsilon X \cong \mathcal{P}\{S\varepsilon X_i\} \longrightarrow \mathcal{P}(RS\varepsilon X) = \mathcal{P}^R S\varepsilon X$$

is a weak equivalence; that is, an isomorphism in $Ho(S(S_M))$.

On the other hand, one has the isomorphisms

$$
\begin{aligned}
(\mathcal{P}S\varepsilon X)_q &\cong (proSS, SS)(\mathcal{L}(\Delta[q] \odot M), S\varepsilon X) \\
&\cong (proTop, Top)(|c\Delta[q]|, \varepsilon X) \\
&\cong (proTop, Top)(\varepsilon(\mathbf{N} \times |\Delta[q]|), \varepsilon X) \\
&\cong Pro(\mathbf{N} \times |\Delta[q]|, X) \\
&\cong (S_p X)_q
\end{aligned}
$$

Therefore $\mathcal{P}S\varepsilon X$ is isomorphic to S_pX and $S_pX \longrightarrow \mathcal{P}^RS\varepsilon X$ is an isomorphism in $Ho(S(S_M))$.

A partial version of the Edwards–Hastings embedding Theorem can be obtained as a Corollary.

Corollary 1. Let X be an object in PC, then $\pi_0(Pro)(X,Y) \cong Ho(proSS, SS)(S\varepsilon X, S\varepsilon Y)$ for any space Y in Pro_σ.
Proof. By Theorem 7.1 there is an object N in $S(S_M)$ such that $R_pN \cong X$ and

$$\pi_0(Pro)(X,Y) \cong Ho(S(S_M))(N, S_pY).$$

By the above Theorem, S_pY is isomorphic to $\mathcal{P}^RS\varepsilon Y$ in $Ho(S(S_M))$, so one has

$$Ho(S(S_M))(N, S_pY) \cong Ho(S(S_M))(N, \mathcal{P}^RS\varepsilon Y \cong Ho(proSS, SS)(\mathcal{L}N, S\varepsilon Y)$$

In the last isomorphism, we have taking into account that N is cofibrant in $S(S_M)$. Applying Proposition 7.1, one has that $\mathcal{L}N$ is isomorphic to $S\varepsilon X$. Therefore

$$\pi_0(Pro)(X,Y) \cong Ho(proSS, SS)(S\varepsilon X, S\varepsilon Y).$$

Different homology theories can be defined in order to have Hurewicz Theorems. If one considers the following definition, we have that the standard Hurewicz Theorem implies a proper Hurewicz Theorem. Recall that $\bar{S}_pX$ denotes the simplicial set US_pX where $U\colon S(S_M) \longrightarrow SS$ is the forgetful functor and S_p is the proper singular functor.

Definition 2. Let X be a space, the *qth* proper homology group of X is defined by

$${}^pH_q(X) := H_q(\bar{S}_pX).$$

Theorem 3 (Proper Hurewicz Theorem). Let X be a noncompact space and suppose that X is properly 0–connected (${}^p\pi_0(X,\sigma) = *$ for some base sequence σ). Then there is a homomorphism ${}^p\pi_q(X) \longrightarrow {}^pH_q(X)$ for each $q \geq 0$ such that

1) For $q = 1$, ${}^p\pi_1(X) \longrightarrow {}^pH_1(X)$ is up to isomorphism the natural epimorphism from a group to its abelianization. The first proper homology group is isomorphic to the abelianization of the proper fundamental group.
2) If X is properly $(n-1)$–connected, $n \geq 2$, (${}^p\pi_q(X) \cong 0$ for $q \leq n-1$), then the Hurewicz homomorphism ${}^p\pi_n(X) \longrightarrow {}^pH_n(X)$ is an isomorphism and ${}^p\pi_{n+1}(X) \longrightarrow {}^pH_{n+1}(X)$ is an epimorphism.

Remarks. 1) If X is a space and $\alpha\colon [0, +\infty) \longrightarrow X$ a base ray, such that α is a "proper cofibration". Then the pushout

$$\begin{array}{ccc} \varepsilon[0,+\infty) & \longrightarrow & * \\ {\scriptstyle \varepsilon\alpha}\downarrow & & \downarrow \\ \varepsilon X & \longrightarrow & \varepsilon' X \end{array}$$

defines an object $\varepsilon' X$ of $(proTop_*, Top_*)$ and $\varepsilon X \longrightarrow \varepsilon' X$ is a weak equivalence in $(proTop, Top)$. Then $\bar{\mathcal{P}}^R S\varepsilon X \longrightarrow \bar{\mathcal{P}} S\varepsilon' X$ is a weak equivalence in SS and we have that $\pi_q(\bar{\mathcal{P}}^R S\varepsilon X) \cong \pi_q(\bar{\mathcal{P}}^R S\varepsilon' X)$. The proper homotopy groups satisfy

$${}^p\pi_q(X) = \pi_q(S_pX) \cong \pi_q(\bar{\mathcal{P}}^R S\varepsilon X) \cong$$
$$\cong \pi_q(\bar{\mathcal{P}}^R S\varepsilon' X) \cong \bar{\mathcal{P}}(pro\pi_q, \pi_q)(\varepsilon' X).$$

That is the functor π_q commutes with the $\bar{\mathcal{P}}$ functor.
2) The functor H_q does not commute with the $\bar{\mathcal{P}}$ functor. Take X obtained from the semiopen interval $[0, +\infty)$ by attaching one 1–sphere at each nonnegative integer. In this case, the natural map ${}^pH_1(X) \longrightarrow \bar{\mathcal{P}}((proH_1, H_1)\varepsilon' X)$ is not an isomorphism.
3) We can also consider the following functor

$$Pro \xrightarrow{S\varepsilon} (proSS, SS) \xrightarrow{f} (proSA, SA) \xrightarrow{\mathcal{P}^R} S(A_{\mathcal{P}c\mathbf{N}}) \xrightarrow{U} SA$$

that induces another proper homology theory that also satisfies a Hurewicz Theorem. In this case, the functor H_q "commutes" with $\bar{\mathcal{P}}$.
4) Other useful proper invariants are the Strong (Steenrod) homotopy groups of a rayed space that can be defined by $\pi_q(F_{sh}\mathcal{P}^R\varepsilon' X)$ or by $\pi_q\lim^R \varepsilon' X$, see [H-P.1, H-P.2]. Other alternative definition can be seen in [Če]. A proper homology theory for these groups, that satisfy the Hurewicz Theorem, can be defined by $H_q(F_{sh}\mathcal{P}^R\varepsilon' X)$. Other Hurewicz Theorems for the strong homotopy groups are proved in [E-H-R.2].

9. Applications to prohomotopy theory.

In this section, in order to prove new versions of standard theorems for the homotopy category $Ho(proSS_*)$, we will use the pair of adjoint functors

$$Ho(S(S_{*_{\mathcal{P}cS^0}})) \underset{\mathcal{P}^R}{\overset{\mathcal{L}^L}{\rightleftarrows}} Ho(proSS_*).$$

Definition 1. An object X of $proSS_*$ is said to be $\mathcal{L}$–cofibrant if X is isomorphic in $Ho(proSS_*)$ to some $\mathcal{L}G$, where G is a cofibrant object in $S(S_{*_{\mathcal{P}cS^0}})$. If G is cofibrant and $dim\ G \leq k$, then X is said to be $\mathcal{L}$–k–cofibrant.

There are many versions of the Whitehead theorem in prohomotopy theory. On one side, there are theorems that give algebraic conditions to ensure that a morphism of $proHo(Top)$ is an isomorphism, see for instance [Rau] and [M-S]. On the other side, there are theorems of the same type for a morphism of $Ho(proTop)$. The monograph of Edwards–Hastings [E–H] and the papers of Grossman [Gr.1, Gr.2, Gr.3] include some versions of the last type for maps between towers that satisfy additional conditions on (co) dimension or movability. Here we prove a slightly different version of the Whitehead theorem for $\mathcal{L}$–cofibrant objects. In general, an $\mathcal{L}$–cofibrant object is not necessarily isomorphic to a tower. The algebraic condition of our result is given in terms of Grossman homotopy groups or equivalent cohomological conditions.

Theorem 1. Let X, Y be $\mathcal{L}$–cofibrant objects in $proSS_*$ and let $u\colon X \longrightarrow Y$ be a map in $Ho(proSS_*)$. If $\mathcal{P}^R u$ is an isomorphism in $Ho(S(S_{*_{\mathcal{P}cS^0}}))$, then u is an isomorphism in $Ho(proSS_*)$.

Proof. It suffices to prove that for any cofibrant object G of $S(S_{*_{\mathcal{P}cS^0}})$ the induced map

$$u_*\colon Ho(proSS_*)(\mathcal{L}G, X) \longrightarrow Ho(proSS_*)(\mathcal{L}G, Y)$$

is an isomorphism. Because $\mathcal{L}^L$ is left adjoint to $\mathcal{P}^R$, this condition is equivalent to showing that

$$(\mathcal{P}^R u)_*\colon Ho((S_{*_{\mathcal{P}cS^0}}))(G, \mathcal{P}^R X) \longrightarrow Ho((S_{*_{\mathcal{P}cS^0}}))(G, \mathcal{P}^R Y)$$

is an isomorphism. This follows because $\mathcal{P}^R u$ is an isomorphism by hypothesis.

Consider a simplicial q–sphere, for instance $S^q = \dot{\Delta}[q+1]$, and recall that $\mathcal{L}(S^q \odot \mathcal{P}cS^0) \cong cS^q$. Given an object X in $proSS_*$, the qth Grossman homotopy group of X (see [Gr.3]) can be defined by ${}^G\pi_q^\infty(X) = Ho(proSS_*)(cS^q, X)$. It is clear that ${}^G\pi_q^\infty(X) \cong Ho((S_{*_{\mathcal{P}cS^0}}))(S^q \odot \mathcal{P}cS^0, \mathcal{P}^R X) \cong Ho(SS_*)(S^q, \bar{\mathcal{P}}^R X) \cong \pi_q(\bar{\mathcal{P}}^R X)$. A pro–pointed simplicial set is said to be (Grossman) 0–connected if ${}^G\pi_0^\infty(X)$ is trivial.

Corollary 1. Let X, Y be $\mathcal{L}$–cofibrant objects in $proSS_*$ and assume that X and Y are 0–connected (${}^G\pi_0^\infty = 0$). If $u\colon X \longrightarrow Y$ is a morphism in $Ho(proSS_*)$, then u is an isomorphism if and only if ${}^G\pi_q^\infty X \longrightarrow^G \pi_q^\infty Y$ is an isomorphism for all $q \geq 1$.

Proof. By Theorem 1, u is an isomorphism if and only if $\mathcal{P}^R u$ is an isomorphism. It is easy to check that $\mathcal{P}^R u$ is an isomorphism in $Ho(S(S_{*_{\mathcal{P}cS^0}}))$ if and only if $\bar{\mathcal{P}}^R u = U\mathcal{P}^R u$ is an isomorphism in $Ho(SS_*)$. We note that the simplicial sets $\bar{\mathcal{P}}^R X$ and $\bar{\mathcal{P}}^R Y$ are 0–connected. Therefore this is equivalent to saying that $\pi_q \bar{\mathcal{P}}^R u$ is an isomorphism for $q \geq 1$. Since $\pi_q \bar{\mathcal{P}}^R u = {}^G\pi_q^\infty u$, we get the algebraic condition of the Corollary.

Remark. We can define the cohomology of a pro–simplicial set X with twisted coefficients in A by $H^q(X; A) := H^q(\bar{\mathcal{P}}^R X; A)$ where A is a π–module and $\pi = {}^G\pi_1^\infty X$. It is clear that we can give a cohomological version of the Whitehead Theorem for $\mathcal{L}$–cofibrant objects.

Recall that the natural "inclusion" $Ho(SS) \longrightarrow Ho(proSS)$ is left adjoint to the homotopy limit $\lim^R\colon Ho(proSS) \longrightarrow Ho(SS)$ and for the case of towers $\lim^R$ factorizes as $\lim^R = F_{sh}\mathcal{P}^R$. We also have similar functors and properties for the pointed case.

Given an object X in $proSS_*$ the qth strong homotopy group of X is defined by ${}^S\pi_q(X) = \pi_q(\lim^R X)$. If X is a tower we have that $\pi_q(\lim^R X) = \pi_q(F_{sh}\mathcal{P}^R X)$. The Čech homotopy groups of X are defined by $\check{\pi}_q(X) = \lim\, \pi_q X$, where $\pi_q X$ denotes the progroup $pro\pi_q X$. If X is a tower then $\check{\pi}_q(X) = \lim\, \pi_q X \cong F_{sh}\ \mathcal{P}\ \pi_q X \cong F_{sh}\ \pi_q\ \mathcal{P}^R X$. Then the homotopy groups $\pi_q \mathcal{P}^R X$ determines the Čech homotopy groups $\check{\pi}_q(X)$ (the homotopy group $\pi_q \mathcal{P}^R X$ is provided in a natural way with a shift operator sh).

For an object X in $proSS_*$, we have the natural map $\lim^R X \longrightarrow X$ in $proSS_*$. Consider the following notion.

Definition 2. An object X in $proSS_*$ is said to be $\bar{\mathcal{P}}$–movable if the induced map $\bar{\mathcal{P}}^R \lim^R X \longrightarrow \bar{\mathcal{P}}^R X$ is a weak equivalence in SS_*.

Proposition 1. Let X be an object of $towSS_*$ and assume that X is $\bar{\mathcal{P}}$–movable, then
i) The Čech homotopy groups $\check{\pi}_q(X)$ are isomorphic to the strong homotopy groups ${}^S\pi_q(X)$.

ii) The strong homotopy groups determine the Grossman homotopy groups by the formula ${}^G\pi_q^\infty(X) \cong \bar{\mathcal{P}}({}^S\pi_q(X))$.

Proof. To prove i), consider the following isomorphisms

$$\begin{aligned}\check{\pi}_q(X) &= \lim\, \pi_q\ X \cong F_{sh}\ \mathcal{P}\ \pi_q\ X \cong \\ &\cong F_{sh}\ \pi_q\ \mathcal{P}^R\ X \cong F_{sh}\ \pi_q\ \mathcal{P}^R\ \lim^R\ X \cong \\ &\cong F_{sh}\ \mathcal{P}(\pi_q\ \lim^R\ X) \cong \pi_q\ \lim^R\ X \cong \\ &\cong {}^S\pi_q(X)\end{aligned}$$

Part ii) follows from the isomorphisms

$${}^G\pi_q^\infty(X) = \pi_q(\bar{\mathcal{P}}^R\ X) \cong \pi_q(\bar{\mathcal{P}}^R\ \lim^R\ X) \cong \bar{\mathcal{P}}\ \pi_q\ \lim^R\ X \cong \bar{\mathcal{P}}({}^S\pi_q(X)).$$

Now we obtain the following Whitehead Theorem for $\bar{\mathcal{P}}$–movable prosimplicial sets.

Corollary 2. Let X, Y be objects in $towSS_*$ and assume that X, Y are $\mathcal{L}$–cofibrant and $\bar{\mathcal{P}}$–movable. Suppose also that X and Y are $\check{\pi}$–0–connected. If $u\colon X \longrightarrow Y$ is a morphism in $Ho(proSS_*)$, then the following conditions are equivalent
i) u is an isomorphism in $Ho(proSS_*)$
ii) $\check{\pi}_q(X) \longrightarrow \check{\pi}_q(Y)$ is an isomorphism for all $q \geq 1$
iii) ${}^S\pi_q(X) \longrightarrow {}^S\pi_q(Y)$ is an isomorphism for all $q \geq 1$.

Proof. Since X, Y are $\bar{\mathcal{P}}$–movable, we infer by the Proposition above that $\check{\pi}_q(X) \cong {}^S\pi_q(X)$ and similarly for Y. Because ${}^S\pi_q(X)$ is isomorphic to ${}^S\pi_q(Y)$, by the Proposition above it follows that ${}^G\pi_q^\infty(X) \cong \bar{\mathcal{P}}({}^S\pi_q\ X) \cong \bar{\mathcal{P}}({}^S\pi_q\ Y) \cong {}^G\pi_q^\infty(Y),\ \ q \geq 1$. Applying Corollary 1 we have that condition iii) implies that u is an isomorphism.

Remarks. 1) Let X be a topological space and $\alpha\colon [0, +\infty) \to X$ be a "proper" cofibration. In Remark 1) after Theorem 8.3 we have considered the pointed pro-simplicial set $S\varepsilon'(X, \alpha)$. We say that X is $\bar{\mathcal{P}}$-movable at infinity if $S\varepsilon'(X, \alpha)$ is $\bar{\mathcal{P}}$-movable. As a consequence of Proposition 1, we have that for a space X, which is $\bar{\mathcal{P}}$-movable at infinity, the strong (Steenrod) homotopy group

$${}^S\pi_q(X, \alpha) \cong \pi_0(Pro_\infty)((S^q \times [0, +\infty), * \times [0, +\infty)), (X, \alpha))$$

is isomorphic to the proper Čech group

$$\check{\pi}_q(X, \alpha) = \lim \pi_q \varepsilon(X, \alpha).$$

Therefore we also have the following proper Whitehead Theorem: Let (X,α) (Y,β) be two "well rayed" simplicial complexes (objects in PC) and let $f\colon(X,\alpha)\to(Y,\beta)$ be the germ of a proper map. Assume that i) X, Y have finite dimension, ii) X, Y are $\bar{\mathcal{P}}$-movable at infinity, iii) X, Y have one Freudenthal end. Then f is a proper homotopy equivalence at infinity if and only if $\check{\pi}_q(f)\colon\check{\pi}_q(X,\alpha)\longrightarrow\check{\pi}_q(Y,\beta)$ is an isomorphism for $q\geq 1$.
A Whitehead Theorem involving only strong (Steenrod) proper homotopy groups was proved in [E-H-R.1].
2) If we define the $H\bar{\mathcal{P}}$-homology groups of a pro-simplicial set X by $H_q^{\bar{\mathcal{P}}}(X)=H_q(\bar{\mathcal{P}}^R X)$, we also have a homology theory that satisfies the Hurewicz Theorem for the Grossman homotopy groups. We will analyse this case in the following section for the pro-simplicial set VX associated with a compact metrisable space X.
3) Many of the notions and theorems of this section can also be obtained for the non pointed case $proSS$, and the corresponding global (augmented) categories $(proSS,SS)$, $(proSS_*,SS_*)$.

10. Applications to strong shape theory.

First we recall the definitions of the Čech nerve CX of a space X and the Vietoris nerve VX that was introduced by Porter [P.1].

Given a space X, consider the directed set $covX$. An element of $covX$ is an open covering $\mathcal{U}$ of X. If $\mathcal{U},\vartheta\in covX$, it is said that ϑ refines $\mathcal{U}$ $(\vartheta\geq\mathcal{U})$ if for any $V\in\vartheta$, there is some $U\in\mathcal{U}$ such that $V\subset U$. Given a space X and an open covering $\mathcal{U}$, $(CX)_{\mathcal{U}}$, denotes a simplicial set such that a typical n–simplex is given by $(U_0,\dots,U_n)$ where $U_0,\dots,U_n\in\mathcal{U}$ and $U_0\cap\cdots\cap U_n\neq\emptyset$. The correspondence $X\longrightarrow\{(CX)_U \mid U\in covX\}$ defines a functor $C\colon Top\longrightarrow proHo(SS)$.

If $\mathcal{U}$ is an open covering of the space X, the Vietoris nerve of $\mathcal{U},(VX)_{\mathcal{U}}$, is the simplicial set in which an n–simplex is an ordered $(n+1)$–tuple $(x_0,\dots,x_{n+1})$ of points contained in an open set $U\in\mathcal{U}$. One important difference with the Čech nerve is that if ϑ refines $\mathcal{U}$ there is a canonical map $(VX)_\vartheta\longrightarrow(VX)_{\mathcal{U}}$ in SS, in the case of the Čech nerve the corresponding map $(CX)_\vartheta\longrightarrow(CX)_{\mathcal{U}}$ has to be considered only in $Ho(SS)$.

Using the Vietoris functor $V\colon Top\longrightarrow proSS$, one can define the category, $StSh(Top)$, of strong shape of topological spaces by taking as objects the topological spaces and for two spaces X,Y the hom–set $StSh(X,Y)$ is defined by

$$StSh(X,Y)=Ho(proSS)(VX,VY),$$

where $proSS$ is provided with the closed model structure given by Edwards–Hastings [E–H]. We shall also use the Dowker Theorem [E–H; page 125], which asserts that for an open covering $\mathcal{U}$ of a topological space the Vietoris nerve $(VX)_{\mathcal{U}}$ is isomorphic to the Čech nerve $(CX)_{\mathcal{U}}$ in the category $Ho(SS)$.

It is not difficult to check that if X is a compact metrisable space, then there is a cofinal sequence $\dots, \mathcal{U}_2, \mathcal{U}_1, \mathcal{U}_0$ of open coverings in $covX$. Therefore $CX = \{(CX)_{\mathcal{U}} \mid \mathcal{U} \in covX\}$ is isomorphic to $C'X = \{(CX)_{\mathcal{U}_i} \mid i \in \mathbf{N}\}$ in $proHo(SS)$ and $VX = \{(VX)_{\mathcal{U}} \mid \mathcal{U} \in covX\}$ is isomorphic to $V'X = \{(VX)_{\mathcal{U}_i} \mid i \in \mathbf{N}\}$ in $proSS$ and in $Ho(proSS)$.

Recall that if X is a compact metrisable space we can assume (up to homeomorphism) that X is a subspace of $s = \prod_{n=1}^{+\infty}(\frac{-1}{n}, \frac{1}{n})$, the pseudo-interior of the Hilbert cube $Q = \prod_{n=1}^{+\infty}[\frac{-1}{n}, \frac{1}{n}]$. Consider the open neighbourhoods of X in Q

$$NX = \{U \mid X \subset U, U \text{ is an open subset of } Q\}$$

as an object of $proTop$. Since X is a compact space, there is a cofinal sequence of neighbourhoods $N'X = \{U_i \mid X \subset U_i,\ i \in \mathbf{N}\}$ such that NX is isomorphic to $N'X$ in $proTop$. Applying the singular functor we get $SNX = \{SU \mid U \in NX\}$ which is isomorphic to VX in $Ho(proSS)$. It is also interesting to remark that the natural inclusion

$$N'^cX = \{U_i - X \mid X \subset U_i,\ i \in \mathbf{N}\} \subset \{U_i \mid X \subset U_i,\ i \in \mathbf{N}\} = N'X$$

is an isomorphism in $Ho(proTop)$ and therefore $SN'X$ and SN'^cX are isomorphic in $Ho(proSS)$.

Notice that by considering the functor $Ho(towSS) \longrightarrow towHo(SS)$ and the Dowker Theorem we have that for a compact metrisable space X, $C'X$ and $V'X$ are isomorphic in $towHo(SS)$. If we choose representative maps of the bounding maps of $C'X$, we obtain an object $C''X$ in the category $towSS$ and by Theorem 5.2.9 of [E–H] we also have that $C''X$ and $V'X$ are isomorphic in $Ho(towSS)$. Therefore for a compact metrisable space the objects $VX, SNX, V'X, SN'X, SN'^cX, C''X$ are isomorphic in $Ho(proSS)$.

Recall that in the Example 2 of section 2, we introduced the simplicial M–sets $ss(X)$ and $ss^c(X)$ for a compact subset X of the pseudo–interior of the Hilbert cube. The following result gives a geometric interpretation of the simplicial M–set $\mathcal{P}^R VX$.

Proposition 1. Let X be a compact subset of the pseudo–interior of the Hilbert cube, then $\mathcal{P}^R VX$, $ss(X)$, and $ss^c(X)$ are isomorphic in the category $Ho\big(S(S_M)\big)$, where $M = Pro(\mathbf{N}, \mathbf{N})$.

Proof. Since VX is isomorphic to SNX, then $\mathcal{P}^R VX \cong \mathcal{P}^R SNX \cong \mathcal{P}^R SN'X \cong$ $\cong \mathcal{P}^R SN'^cX$. The objects $SN'X = \{SU_i \mid i \in \mathbf{N}\}$ and $SN'^cX = \{S(U_i - X) \mid i \in \mathbf{N}\}$ satisfy that for each $i \in \mathbf{N}$, SU_i and $S(U_i - X)$ are fibrant in SS. By Lemma 6.3, we infer that $\mathcal{P}^R SN'X$ is isomorphic to $\mathcal{P}SN'X$ and $\mathcal{P}^R SN'^cX$ is isomorphic to $\mathcal{P}SN'^cX$ in the category $Ho\big(S(S_M)\big)$. Now it is easy to check that $\mathcal{P}SN'X$ is isomorphic to $ss(X)$ and $\mathcal{P}SN'^cX$ is isomorphic to $ss^c(X)$ in the category $S(S_M)$.

Remark. From the definition of $ss^c(X)$ and $S_p(Q{-}X)$, it is clear that $ss^c(X) = S_p(Q{-}X)$.

To define invariants for the strong shape category, consider the following functors

$$StSh(CM) \xrightarrow{V} Ho(proSS) \xrightarrow{\mathcal{P}^R} Ho\big(S(S_M)\big) \xrightarrow{U} Ho(SS)$$

where $StSh(CM)$ is the strong shape category for compact metrisable spaces and M is the monoid $\mathcal{P}c*$. Recall that we also use the notation $U\mathcal{P}^R = \bar{\mathcal{P}}^R$. We also note that a base point $*$ of a compact metrisable space determines a base point of $\bar{\mathcal{P}}^R VX$.

Definition 1. The $\pi\bar{\mathcal{P}}$–homotopy groups of a pointed compact metrisable space are defined by

$$\pi_q^{\bar{\mathcal{P}}}(X) = \pi_q(\bar{\mathcal{P}}^R VX)$$

and the $H\bar{\mathcal{P}}$–homology groups of X (non pointed) by

$$H_q^{\bar{\mathcal{P}}}(X) = H_q(\bar{\mathcal{P}}^R VX).$$

Remarks. 1) The $\pi\bar{\mathcal{P}}$–homotopy groups $\pi_q^{\bar{\mathcal{P}}}(X)$ are isomorphic to the "inward" groups ${}^Q\pi_q^I(X)$ of Quigley [Quig, P.6].
2) Notice that π and $\bar{\mathcal{P}}$ commute; that is $\pi_q\bar{\mathcal{P}}^R VX \cong \bar{\mathcal{P}}\pi_q VX$, where $\pi_q VX$ denotes the homotopy progroup $pro\pi_q VX$.
3) In general H and $\bar{\mathcal{P}}$ do not commute; that is, $H_q\bar{\mathcal{P}}^R VX \ncong \bar{\mathcal{P}}H_q VX$, where $H_q VX$ denotes the pro–abelian group $proH_q VX$.
4) To define homology theories, we can consider functors into a category of simplicial objects in an abelian category, for instance the free abelian functor $f\colon Set \longrightarrow Ab$ induces natural functors $f\colon SS \longrightarrow SA$, $f\colon proSS \longrightarrow proSA$ where SA is the category of simplicial abelian groups. We also have the free functor $f\colon S(S_{\mathcal{P}c*}) \longrightarrow S(S_{\mathcal{P}c\mathbf{N}})$ where $S(A_{\mathcal{P}c\mathbf{N}})$ denotes the category of simplicial objects in $A_{\mathcal{P}c\mathbf{N}}$ ($Ab = A$). Therefore we have the following simplicial objects to define homology of a pro–simplicial set X

$a)$	$f\ U\ \mathcal{P}^R\ X$	in	SA
$b)$	$U\ f\ \mathcal{P}^R\ X$	in	SA
$c)$	$U\ \mathcal{P}^R\ f\ X$	in	SA
$d)$	$f\ \mathcal{P}^R\ X$	in	$SA_{\mathcal{P}c\mathbf{N}}$
$e)$	$\mathcal{P}^R\ f\ X$	in	$SA_{\mathcal{P}c\mathbf{N}}$
$f)$	$f\ X$	in	$proSA$

For the cases c) and e) we have that H and $\bar{\mathcal{P}}$ commute. The homology in cases d) and e) has a natural structure as a $\mathcal{P}c\mathbf{N}$–module. Recall that $\mathcal{P}c\mathbf{N}$ is ring of locally finite matrices modulo the ideal of finite matrices, see [F–W].

As an immediate consequence of the definition one has that the $H\bar{\mathcal{P}}$–homology satisfies the Hurewicz Theorem for the inward groups of Quigley.

Theorem 1. Let X be a compact metrisable space and assume that X is ${}^Q\pi^I$–0–connected $\left({}^Q\pi_0^I(X) = 0\right)$, then there is a canonical homomorphism ${}^Q\pi_q^I(X) \longrightarrow H_q^{\bar{\mathcal{P}}}(X)$ such that
1) For $q = 1, {}^Q\pi_1^I(X) \longrightarrow H_1^{\bar{\mathcal{P}}}(X)$ is the abelianization of ${}^Q\pi_1^I(X)$.
2) If X is ${}^Q\pi^I$–$(n-1)$–connected, $n \geq 2$, (that is, ${}^Q\pi_q^I(X) = 0$, $q \leq n-1$), then ${}^Q\pi_n^I(X) \longrightarrow H_n^{\bar{\mathcal{P}}}(X)$ is an isomorphism and ${}^Q\pi_{n+1}^I(X) \longrightarrow H_{n+1}^{\bar{\mathcal{P}}}(X)$ is an epimorphism.

Proof. It suffices to apply the standard Hurewicz Theorem to the simplicial set $\bar{\mathcal{P}}^R VX$.

Remark. There are other homologies that satisfy Hurewicz theorems for the inward groups of Quigley. For instance consider the $H\bar{\mathcal{P}}f$–homology groups, $H_q^{\bar{\mathcal{P}}f}(X) = H_q(\bar{\mathcal{P}}^R fX)$.

It is also interesting to analyse the family of invariants obtained when one consider the commutative diagram

$$\begin{array}{ccccc} & & Ho(towSS) & & \\ & \swarrow & & \searrow \lim^R & \\ Ho(proSS) & \xrightarrow[\mathcal{P}^R]{} & \pi_0\big(S(S_M)\big) & \xrightarrow[F_{sh}]{} & Ho(SS) \end{array}$$

For a compact metrisable space the prosimplicial set VX is isomorphic to a tower of simplicial sets, then $\text{holim}VX = \lim^R VX \cong F_{sh}\mathcal{P}^R VX$. That is $\lim^R VX$ is a sub–simplicial set of $\bar{\mathcal{P}}^R VX$. The inclusion $\lim^R VX \subset \bar{\mathcal{P}}^R VX$ induces many relations between the homotopy invariants of $\lim^R VX$ and the invariants of $\bar{\mathcal{P}}^R VX$.

Definition 2. The $\pi F\mathcal{P}$–homotopy groups of a pointed metrisable space are defined by

$$\pi_q^{F\mathcal{P}}(X) = \pi_q(F_{sh}\ \mathcal{P}^R VX)$$

and the $HF\mathcal{P}$–homology groups of X (non pointed) by

$$H_q^{F\mathcal{P}}(X) = H_q(F_{sh}\mathcal{P}^R VX)$$

Remarks. 1) The $\pi F\mathcal{P}$–homotopy groups $\pi_q^{F\mathcal{P}}(X)$ are isomorphic to the approaching groups ${}^Q\pi_q^A(X)$ defined by Quigley [Quig, P.6].
2) The functors π_q and F_{sh} do not commute. There are spaces X such that $\pi_q F_{sh}\mathcal{P}^R VX \ncong F_{sh}\pi_q\mathcal{P}^R VX$. Notice that $F_{sh}\pi_q\mathcal{P}^R VX \cong F_{sh}\mathcal{P}\pi_q VX \cong$ $\cong \lim \pi_q VX$ is isomorphic to the Čech homotopy group $\check{\pi}_q(X)$. Therefore the $F\pi\mathcal{P}$–homotopy groups of X, ${}^F\pi_q^{\mathcal{P}}(X) = F_{sh}\pi_q\mathcal{P}VX$ are up to isomorphism the Čech homotopy groups.
3) We can consider the following simplicial objects, in different abelian categories, associated with a pro–simplicial set X.

a) $fF_{sh}\mathcal{P}^R X = f\lim^R X$ in SA,
b) $F_{sh}f\mathcal{P}^R X$ in SA,
c) $F_{sh}\mathcal{P}^R fX = \lim^R fX$ in SA.

The $HF\mathcal{P}f$–homology groups (or $H\lim f$) $H_q^{F\mathcal{P}f}(X) = H_q(F_{sh}\mathcal{P}^R fX)$ are the strong (or Steenrod) homology groups ${}^SH_q(X)$, see [E–H; pag 208], [Co] and [P.4]. We can also consider $F\mathcal{P}H$–homology groups ${}^{F\mathcal{P}}H_q(X) = F_{sh}\ \mathcal{P}\ H_q X \cong \lim\ H_q X \cong \lim\ proH_q X \cong \check{H}_q(X)$ which are isomorphic to the Čech homology groups.

There are Theorems of Hurewicz type for the approaching groups ${}^Q\pi_q^A(X)$ of Quigley and the strong (Steenrod) homology groups ${}^SH_q(X)$, see the paper of Kodama–Koyama [K–K]. We can also prove that the $HF\mathcal{P}$–homology groups satisfy a Hurewicz Theorem for the approaching groups of Quigley.

Theorem 2. Let X be a compact metrisable space and assume that X is ${}^Q\pi^A$–0–connected (that is, ${}^Q\pi_0^A(X) = 0$), then there is a canonical homomorphism ${}^Q\pi_q^A(X) \longrightarrow H_q^{F\mathcal{P}}(X)$ such that
1) For $q = 1$, ${}^Q\pi_1^A(X) \longrightarrow H_q^{F\mathcal{P}}(X)$ is the abelianization of ${}^Q\pi_1^A(X)$
2) If X is ${}^Q\pi^A$–$(n-1)$–connected, $n \geq 2$, (that is, ${}^Q\pi_q^A(X) = 0$, $q \leq n-1$), then ${}^Q\pi_n^A(X) \longrightarrow H_n^{F\mathcal{P}}(X)$ is an isomorphism and ${}^Q\pi_{n+1}^A(X) \longrightarrow H_{n+1}^{F\mathcal{P}}(X)$ an epimorphism.
Proof. This is a particular case of the standard Hurewicz Theorem.

If X is a compact metrisable space VX is isomorphic to $C''X$ in $Ho(proSS)$. If the covering dimension of X is finite, then $C''X$ is isomorphic to a tower of finite simplicial sets of dimension less than or equal to the covering dimension of X. It is not hard to check that a tower of finite simplicial sets of dimension $\leq n$ (for some n) is a $\mathcal{L}$–cofibrant object in the sense of Definition 9.1. Therefore if X is a compact metrisable space and X has finite covering dimension we have that VX is an $\mathcal{L}$–cofibrant object. As a consequence of the Whitehead Theorem proved in section 9, we obtain the following version of the Whitehead Theorem for the strong shape category.

Theorem 3. Let X, Y be compact metrisable spaces with finite covering dimension. Assume also that X and Y are ${}^Q\pi^I$–0–connected $\left({}^Q\pi_0^I = 0\right)$. A strong shape morphism $f\colon X \longrightarrow Y$ (that is a map $f\colon VX \longrightarrow VY$ in $Ho(proSS)$) is a strong shape isomorphism if and only if $f_*\colon {}^Q\pi_q^I(X) \longrightarrow {}^Q\pi_q^I(Y)$ is an isomorphism for $q \geq 1$.

Remarks. 1) For a compact metrisable space X, let $\pi = {}^Q\pi_1^I(X)$ be the fundamental inward group and let A be a π–module. Define the cohomology of X with twisted coefficient in A by $H^q(X; A) = H^q(\bar{\mathcal{P}}^R VX; A)$. Then in the Theorem above we can give the following equivalent condition

i) $f_*\colon {}^Q\pi_1^I(X) \longrightarrow {}^Q\pi_1^I(Y)$ is an isomorphism

ii) $f^*\colon H^q(Y; A) \longrightarrow H^q(X; A)$ is an isomorphism for $q \geq 0$ and any twisted coefficients A.

2) The functors $F_{sh}\mathcal{P}^R$ and $\bar{\mathcal{P}}^R$ can be used to transform many notions and results of standard homotopy theory into strong shape notions and results. We have just included some canonical examples about Hurewicz and Whitehead Theorems.

Definition 3. A compact metrisable space X is said to be $\bar{\mathcal{P}}$–movable if VX is $\bar{\mathcal{P}}$–movable (see Definition 9.2).

An immediate consequence of Corollary 9.2 is the following Whitehead Theorem

Theorem 4. Let X, Y be compact metrisable spaces and assume that X and Y have finite covering dimension and that X and Y are $\bar{\mathcal{P}}$–movable. Suppose also that X, Y are $\check{\pi}$–0–connected ($\check{\pi}_0 = 0$). If $f\colon X \longrightarrow Y$ is a strong shape morphism the following conditions are equivalent

i) f is a strong shape isomorphism,

ii) $\check{\pi}_q(X) \longrightarrow \check{\pi}_q(Y)$ is an isomorphism for $q \geq 1$,

iii) ${}^Q\pi^A(X) \longrightarrow^Q \pi_q^A(Y)$ is an isomorphism for $q \geq 1$.

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