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A Whittaker-Shannon-Kotel'nikov sampling theorem related to the Dunkl transform on the real line*,†

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Notation

J_α denotes the **Bessel function** of the first kind and order $\alpha > -1$.
A small variation of the so-called modified Bessel function of the first kind and order α (usually denoted by I_α):

$$\mathcal{I}_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^\infty \frac{(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}, \qquad z \in \mathbb{C}.$$

Also, let us take

$$E_\alpha(z) = \mathcal{I}_\alpha(z) + \frac{z}{2(\alpha+1)}\mathcal{I}_{\alpha+1}(z), \qquad z \in \mathbb{C}.$$

For $\alpha = -1/2$, we have $E_{-1/2}(z) = e^z$.

The Dunkl transform on the real line

For $\alpha \ge -1/2$, the **Dunkl operator** on the real line is defined by

$$\Lambda_\alpha f(x) = \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right).$$

In the particular case $\alpha = -1/2$, we have $\Lambda_{-1/2} = d/dx$.
For every $\lambda \in \mathbb{C}$, the function $E_\alpha(\lambda x)$ is the unique solution of the initial value problem

$$\begin{cases} \Lambda_\alpha f(x) = \lambda f(x), & x \in \mathbb{R}, \\ f(0) = 1 \end{cases}$$

(see [3, 4, 5]). The $E_\alpha(\lambda x)$ is called the **Dunkl kernel**.
Given the measure

$$d\mu_\alpha(x) = (2^{\alpha+1}\Gamma(\alpha+1))^{-1}|x|^{2\alpha+1}\,dx,$$

the **Dunkl transform** on the real line is given by

$$\mathcal{F}_\alpha f(y) = \int_{\mathbb{R}} f(x)E_\alpha(-iyx)\,d\mu_\alpha(x), \qquad y \in \mathbb{R}.$$

The classical Fourier transform corresponds to the case $\alpha = -1/2$.
There exists an extension for $\alpha > -1$ (see [6]).

The orthogonal system

Let $\{s_j\}_{j\ge1}$ be the increasing sequence of positive zeros of $J_{\alpha+1}(x)$, and take $s_{-j} = -s_j$ and $s_0 = 0$.
With them, let us define the functions

$$e_{\alpha,j}(r) = \frac{2^{\alpha/2}(\Gamma(\alpha+1))^{1/2}}{|\mathcal{I}_\alpha(is_j)|}E_\alpha(is_j r), \qquad j \in \mathbb{Z}\setminus\{0\}, \qquad r \in (-1,1),$$

and $e_{\alpha,0}(r) = 2^{(\alpha+1)/2}(\Gamma(\alpha+2))^{1/2}$.
Using this notation, we have

> **Theorem 1.** *Let $\alpha > -1$. Then, the sequence of functions $\{e_{\alpha,j}\}_{j\in\mathbb{Z}}$ is a complete orthonormal system in $L^2((-1,1),d\mu_\alpha)$.*

When $\alpha = -1/2$, this is the classical exponential system defining Fourier series, i.e., $e_{-1/2,j}(r) = e^{i\pi jr}$.

From the theorem, for $f \in L^2((-1,1),d\mu_\alpha)$, we have

$$f(r) = \sum_{j=-\infty}^{\infty} a_j(f)e_{j,\alpha}(r), \qquad a_j(f) = \int_{-1}^1 f(t)\overline{e_{j,\alpha}(t)}\,d\mu_\alpha(t).$$

The main tool to prove the orthogonality ([1, Lemma 1]):

Lemma. *Let $\alpha > -1$ and $x, y \in \mathbb{C}$. Then, for $x \ne y$,*

$$\int_{-1}^1 E_\alpha(ixr)\overline{E_\alpha(iyr)}\,d\mu_\alpha(r) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\,\frac{x\mathcal{I}_{\alpha+1}(ix)\mathcal{I}_\alpha(iy) - y\mathcal{I}_{\alpha+1}(iy)\mathcal{I}_\alpha(ix)}{x-y},$$

and, for $x = y$,

$$\int_{-1}^1 |E_\alpha(ixr)|^2\,d\mu_\alpha(r) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left(\frac{x^2}{2(\alpha+1)}\mathcal{I}_{\alpha+1}^2(ix) - (2\alpha+1)\mathcal{I}_{\alpha+1}(ix)\mathcal{I}_\alpha(ix) + 2(\alpha+1)\mathcal{I}_\alpha^2(ix)\right).$$

The sampling theorem

Now, as usually in sampling theory, we take the **space of Paley-Wiener type** that, under our setting, is defined as

$$PW_\alpha = \left\{ f \in L^2(\mathbb{R},d\mu_\alpha) : f(x) = \int_{-1}^1 u(y)E_\alpha(ixy)\,d\mu_\alpha(y),\ u \in L^2((-1,1),d\mu_\alpha)\right\}$$

endowed with the norm of $L^2(\mathbb{R},d\mu_\alpha)$.

In this way, our sampling theorem is

> **Theorem 2.** *If $f \in PW_\alpha$, $\alpha > -1$, then f has the representation*
>
> $$f(x) = f(s_0)\mathcal{I}_{\alpha+1}(ix) + \sum_{j\in\mathbb{Z}\setminus\{0\}} f(s_j)\frac{x\mathcal{I}_{\alpha+1}(ix)}{2(\alpha+1)\mathcal{I}_\alpha(is_j)(x-s_j)},$$
>
> *that converges in the norm of $L^2(\mathbb{R},d\mu_\alpha)$. Moreover, the series converges uniformly in compact subsets of $\mathbb{R}$.*

In the case $\alpha = -1/2$, we get the classical Shannon sampling theorem: If f is band-limited to the interval $(-1,1)$, i.e.,

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-1}^1 u(y)e^{iyt}\,dy,$$

then f can be recovered by mean of the values $f(j\pi)$, $j \in \mathbb{Z}$, by mean of

$$f(t) = \sum_{j=-\infty}^{\infty} f(j\pi)\frac{\sin(t-j\pi)}{t-j\pi}.$$

An example

For $\alpha, \beta, \alpha+\beta > -1$, we have

$$\int_0^\infty \frac{J_{\alpha+\beta+2n+1}(t)}{t^{\alpha+\beta+1}}\frac{J_\alpha(xt)}{(xt)^\alpha}t^{2\alpha+1}\,dt = \frac{\Gamma(n+1)}{2^\beta\Gamma(\beta+n+1)}(1-x^2)^\beta P_n^{(\alpha,\beta)}(1-2x^2)\chi_{[0,1]}(x), \qquad n = 0,1,2,\dots,$$

where $P_n^{(\alpha,\beta)}$ denotes the n-th Jacobi polynomial of order (α,β), and $\chi_{[0,1]}$ is the characteristic function of the interval $[0,1]$. From this formula, it follows that

$$x^{2n}E_{\alpha+\beta+2n+1}(ix) \in PW_\alpha.$$

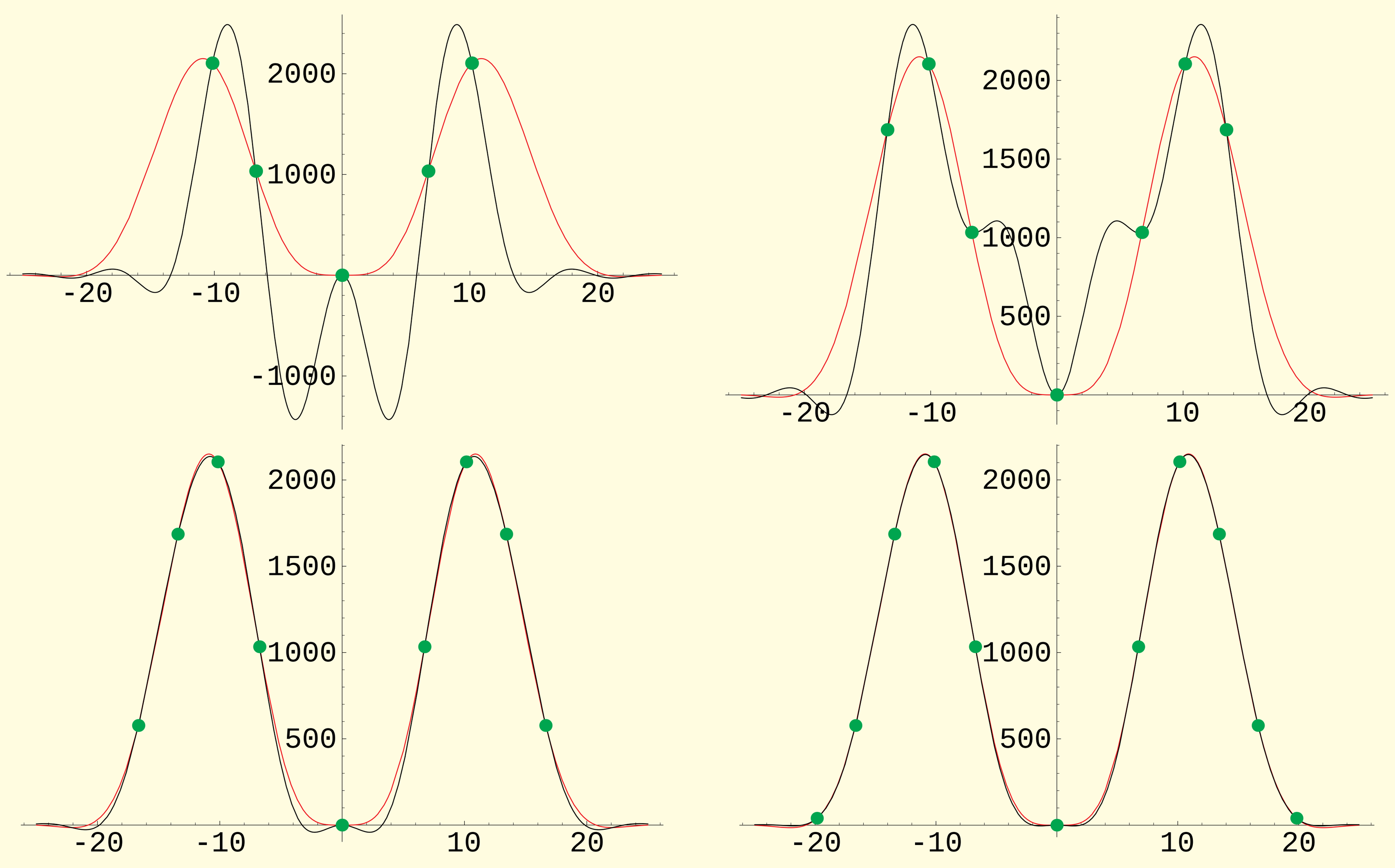
Then, from the sampling theorem,

$$x^{2n}E_{\alpha+\beta+2n+1}(ix) = \sum_{j\in\mathbb{Z}\setminus\{0\}} s_j^{2n}E_{\alpha+\beta+2n+1}(is_j)\frac{x\mathcal{I}_{\alpha+1}(ix)}{2(\alpha+1)\mathcal{I}_\alpha(is_j)(x-s_j)},$$

valid for $\alpha, \beta, \alpha+\beta > -1$, and $n = 1, 2, \dots$; and, for $n = 0$,

$$E_{\alpha+\beta+1}(ix) = \mathcal{I}_{\alpha+1}(ix) + \sum_{j\in\mathbb{Z}\setminus\{0\}} E_{\alpha+\beta+1}(is_j)\frac{x\mathcal{I}_{\alpha+1}(ix)}{2(\alpha+1)\mathcal{I}_\alpha(is_j)(x-s_j)}.$$

Pictures. Taking $f(x) = x^{2n}E_{\alpha+\beta+2n+1}(ix)$ with $\alpha = 2.3$, $\beta = 8.4$ and $n = 2$, we present some pictures showing f (red), the points of sampling (green), and partial sums $\sum_{j=-k}^{k}$ of the recovering formula with $k = 2, 3, 4$ and 5, respectively:



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