WEIGHTED WEAK BEHAVIOUR OF FOURIER-JACOBI SERIES

José J. Guadalupe [1], Mario Pérez [1] and Juan L. Varona [2]

Abstract. *Let $w(x) = (1-x)^\alpha(1+x)^\beta$ be a Jacobi weight on the interval $[-1,1]$ and $1 < p < \infty$. If either $\alpha > -1/2$ or $\beta > -1/2$ and p is an endpoint of the interval of mean convergence of the associated Fourier-Jacobi series, we show that the partial sum operators S_n are uniformly bounded from $L^{p,1}$ to $L^{p,\infty}$, thus extending a previous result for the case that both $\alpha, \beta > -1/2$. For $\alpha, \beta > -1/2$, we study the weak and restricted weak (p,p)-type of the weighted operators $f \longrightarrow uS_n(u^{-1}f)$, where u is also a Jacobi weight.*

§1. Introduction and main results.

Let w be a Jacobi weight on the interval $[-1,1]$, that is,

$$w(x) = (1-x)^\alpha(1+x)^\beta, \qquad \alpha, \beta > -1$$

and let $1 < p < \infty$; $S_n f$ stands for the n-th partial sum of the Fourier series associated to the Jacobi polynomials, orthonormal on $[-1,1]$ with respect to w. It is well known that $S_n f$ converges to f for every $f \in L^p(w)$ if and only if the partial sum operators S_n are uniformly bounded in $L^p(w)$, i.e., there exists a constant $C > 0$ such that

$$\|S_n f\|_{L^p(w)} \le C\|f\|_{L^p(w)} \quad \forall n \ge 0,\ \forall f \in L^p(w) \tag{1}$$

(throughout this paper, we will denote by C a constant independent of f, n, etc., but not necessarily the same at each occurrence). Furthermore, there exists an open interval (p_0, p_1) such that this boundedness holds if and only if p belongs to (p_0, p_1) (see [6]). The assumption that either $\alpha > -1/2$ or $\beta > -1/2$ is equivalent to $1 < p_0 < p_1 < \infty$. More precisely, in this case (1) holds if and only if

$$\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}$$

when $\alpha \ge \beta$ (and the analogous inequality with α replaced by β if $\beta \ge \alpha$).

In this paper we examine the behaviour of S_n at the endpoints of the interval of mean convergence. In order to do this we need some classical definitions and notations. Given

---

[1] Dpto. de Matemáticas. Universidad de Zaragoza. 50009 Zaragoza. Spain.

[2] Dpto. de Matemática Aplicada. Colegio Universitario de La Rioja. 26001 Logroño. Spain.
E-mail: JVARONA@CC.UNIZAR.ES

1991 Mathematics Subject Classification: 42C10.

Research supported by DGICYT under grant PB89-0181-C02-02

THIS PAPER WERE PUBLISHED IN: Math. Nachrichten **158** (1992), 161–174.

a measure μ and $1 \le p < \infty$, the space $L^p_*(\mu) = L^{p,\infty}(\mu)$ is defined to be the space of measurable functions such that

$$\|f\|_{L^p_*(\mu)} = \sup_{y>0} y\,[\mu(\{x : |f(x)| > y\})]^{1/p} < \infty.$$

An operator T is of weak (p,p)-type if $T : L^p(\mu) \longrightarrow L^p_*(\mu)$ is bounded. Now, let f^* be the nonincreasing rearrangement of f, given by $f^*(t) = \inf\{s : \lambda(s) \le t\}$, where λ denotes the distribution function of f. Then, the Lorentz space $L^{p,r}(\mu)$ is the class of all measurable functions f satisfying

$$\|f\|^*_{p,r} = \left(\frac{r}{p} \int_0^\infty \left[t^{1/p} f^*(t)\right]^r \frac{dt}{t} \right)^{1/r} < \infty,$$

where $1 \le p < \infty$, $1 \le r < \infty$. An operator T is of restricted weak (p,p)-type if $T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\mu)$ is bounded, which is equivalent to $\|T\chi_E\|_{L^p_*(\mu)} \le C\|\chi_E\|_{L^p(\mu)}$ for all characteristic functions χ_E, with $C > 0$ independent of E. We refer the reader to [11] for further information on these topics.

If both $\alpha, \beta > -1/2$, the authors proved (see [2]) that the n-th partial sum operators are uniformly of restricted weak (p,p)-type but not of of weak (p,p)-type when p is an endpoint of the interval of mean convergence. In theorems 2 and 3 we extend this result to weighted case $f \longrightarrow uS_n(u^{-1}f)$, where u is also a Jacobi weight, that is,

$$u(x) = (1-x)^a(1+x)^b, \qquad a, b \in \mathbb{R}.$$

Now, the weighted uniform boundedness

$$\|uS_n f\|_{L^p(w)} \le C\|uf\|_{L^p(w)} \quad \forall f \in L^p(u^p w),\ \forall n \ge 0$$

holds (see [6]) if and only if

$$\begin{aligned} |a + (\alpha+1)(\frac{1}{p} - \frac{1}{2})| &< \min\{\frac{1}{4}, \frac{\alpha+1}{2}\}, \\ |b + (\beta+1)(\frac{1}{p} - \frac{1}{2})| &< \min\{\frac{1}{4}, \frac{\beta+1}{2}\}. \end{aligned} \tag{2}$$

Via Pollard's formula, these operators can be related to the Hilbert transform. Then, the theory of A_p weights is used, as well as some classical dyadic-type decomposition of the interval $[-1, 1]$.

In the general case $\alpha > -1/2$, $\alpha \ge \beta$ (the case $\beta \ge \alpha$ follows by symmetry), we prove in theorem 1 that the n-th partial sum operators are uniformly of restricted weak (p,p)-type when p is an endpoint of the interval of mean convergence, thus extending the above cited result (the question of the weak boundedness had already been answered in the negative in [2]). Now, however, uniform bounds are not available for Jacobi polynomials; therefore, a uniform weighted norm inequality is needed for operators of the form $f \longrightarrow u_n H(v_n f)$, where H is the Hilbert transform and (u_n), (v_n) are two sequences of weights involving

Jacobi polynomials or their bounds. This is achieved by studying the A_p constants of the pairs of weights (u_n, v_n), as well as some $L^{p,\infty}$ norms.

Concerning mixed weak norm inequalities for the Hilbert transform, we can state the following property, which can be proved in the same way as theorem 3 of [7] (throughout this paper, the Hilbert transform, as well as A_p classes of weights, are taken on $[-1,1]$): assume that $u_1(x), u_2(x), v(x) \geq 0$, $1 < p < \infty$ and there is a constant $C > 0$ such that

$$\|u_2 Hg\|_{L^p_*(u_1)} \leq C\|g\|_{L^p(v)} \quad \forall g \in L^p(v);$$

then, there exists another constant $B > 0$ which depends only on C, such that for every interval I

$$\|u_2\chi_I\|_{L^p_*(u_1)} \left(\int_{-1}^{1} \frac{v(x)^{-1/(p-1)}}{(|I| + |x - x_I|)^q} dx\right)^{1/q} \leq B, \tag{3}$$

x_I being the centre of I.

Let us state the main results of this paper. By symmetry, there is no loss of generality in assuming $\alpha \geq \beta$. Regarding the restricted weak type, by standard arguments it is enough to consider just one of the endpoints of the interval of mean convergence, as we remark below.

Theorem 1. *Let $\alpha > -1/2$, $\beta > -1$, $\alpha \geq \beta$. If $p = \frac{4(\alpha+1)}{2\alpha+1}$, there exists a constant $C > 0$ such that for every measurable set E and for every $n \geq 0$*

$$\|S_n\chi_E\|_{L^p_*(w)} \leq C\|\chi_E\|_{L^p(w)}.$$

Theorem 2. *Let $\alpha, \beta \geq -1/2$, $u(x) = (1-x)^a(1+x)^b$, $1 < p < \infty$. If the inequalities*

$$-\frac{1}{4} \leq a + (\alpha+1)(\frac{1}{p} - \frac{1}{2}) < \frac{1}{4}, \qquad -\frac{1}{4} \leq b + (\beta+1)(\frac{1}{p} - \frac{1}{2}) < \frac{1}{4}$$

hold, then there exists a constant $C > 0$ such that

$$\|uS_n(u^{-1}\chi_E)\|_{L^p_*(w)} \leq C\|\chi_E\|_{L^p(w)}$$

for every $n \geq 0$ and every measurable set $E \subseteq [-1,1]$.

Remark. For $1 < p < \infty$ and $1/p + 1/q = 1$, it is easy to see that

$$\|uS_n(u^{-1}\chi_E)\|_{L^p_*(w)} \leq C\|\chi_E\|_{L^p(w)} \quad \forall n \geq 0,\ \forall E \subseteq [-1,1]$$

if and only if

$$\|u^{-1}S_n(u\chi_E)\|_{L^q_*(w)} \leq C\|\chi_E\|_{L^q(w)} \quad \forall n \geq 0,\ \forall E \subseteq [-1,1].$$

This allows us to derive, from theorem 2, the same result for the case

$$-\frac{1}{4} < a + (\alpha+1)(\frac{1}{p} - \frac{1}{2}) \leq \frac{1}{4}, \qquad -\frac{1}{4} < b + (\beta+1)(\frac{1}{p} - \frac{1}{2}) \leq \frac{1}{4}$$

as well as the analog of theorem 1 for $p = \frac{4(\alpha+1)}{2\alpha+3}$.

Theorem 3. *Let $\alpha, \beta \geq -1/2$, $u(x) = (1-x)^a(1+x)^b$, $1 < p < \infty$. If there exists a constant $C > 0$ such that for every $f \in L^p(u^p w)$ and for every $n \geq 0$*

$$\|uS_n f\|_{L^p_*(w)} \leq C\|uf\|_{L^p(w)},$$

then the inequalities

$$|a + (\alpha+1)(\frac{1}{p} - \frac{1}{2})| < \frac{1}{4}, \qquad |b + (\beta+1)(\frac{1}{p} - \frac{1}{2})| < \frac{1}{4}$$

are verified.

§2. Preliminary lemmas.

A basic tool in the study of Fourier series on the interval $[-1, 1]$ is Pollard's decomposition of the kernels $K_n(x,t)$ (see [9], [6]): if $\{P_n\}_{n\geq 0}$ is the sequence of polynomials orthonormal with respect to $w(x)dx$ and $\{Q_n\}_{n\geq 0}$ is the sequence of polynomials associated to $(1-x^2)w(x)dx$, then

$$K_n(x,t) = r_n T_{1,n}(x,t) + s_n T_{2,n}(x,t) + s_n T_{3,n}(x,t),$$

where

$$T_{1,n}(x,t) = P_{n+1}(x)P_{n+1}(t),$$

$$T_{2,n}(x,t) = (1-t^2)\frac{P_{n+1}(x)Q_n(t)}{x-t},$$

$$T_{3,n}(x,t) = (1-x^2)\frac{P_{n+1}(t)Q_n(x)}{t-x}$$

and $\{r_n\}$, $\{s_n\}$ are bounded sequences. In fact, for any measure μ on $[-1,1]$ with $\mu' > 0$ *a.e.* (in particular, for $w(x)dx$),

$$\lim_{n\to\infty} r_n = -1/2, \qquad \lim_{n\to\infty} s_n = 1/2$$

(this can be deduced from [9] and [10] or [4]). Therefore, we can write

$$S_n f = r_n W_{1,n} f + s_n W_{2,n} f - s_n W_{3,n} f,$$

where

$$W_{1,n} f(x) = P_{n+1}(x)\int_{-1}^{1} P_{n+1}(t)f(t)w(t)dt,$$

$$W_{2,n} f(x) = P_{n+1}(x)H((1-t^2)Q_n(t)f(t)w(t), x)$$

and

$$W_{3,n} f(x) = (1-x^2)Q_n(x)H(P_{n+1}(t)f(t)w(t), x),$$

H being the Hilbert transform on the interval $[-1,1]$. Thus, the study of S_n can be reduced to that of $W_{i,n}$ $(i=1,2,3)$. Due to the definition of $W_{2,n}$ and $W_{3,n}$, we will need to show the uniform boundedness of the Hilbert transform with pairs of weights (u_n, v_n).

The boundedness of the Hilbert transform can be stated in terms of Muckenhoupt's classes of weights A_p (see [3] and [8]; throughout this paper, these will be A_p classes on the interval $[-1,1]$): if u, v are two weights, $1<p<\infty$ and $(u^\delta, v^\delta) \in A_p$ for some $\delta > 1$, then H is a bounded operator from $L^p(v)$ into $L^p(u)$, with a constant which depends only on the A_p constant of (u^δ, v^δ).

Therefore, we will say that a sequence $\{(u_n, v_n)\}_n$ belongs *uniformly* to an A_p class if $(u_n, v_n) \in A_p$ $\forall n$ with a constant that does not depend on n.

The polynomials P_n satisfy the estimate

$$|P_n(x)| \le C(1-x+n^{-2})^{-(2\alpha+1)/4}(1+x+n^{-2})^{-(2\beta+1)/4} \quad \forall n, \forall x \in [-1,1] \tag{4}$$

with a constant $C>0$ independent of x and n. An analogous estimate is verified by Q_n, with $\alpha+1$ and $\beta+1$ instead of α and β. In this case, as $2\alpha+3>0$ and $2\beta+3>0$, we can remove the n's and get

$$|Q_n(x)| \le C(1-x)^{-(2\alpha+3)/4}(1+x)^{-(2\beta+3)/4} \quad \forall n, \forall x \in [-1,1]. \tag{5}$$

In this context, the following result will be useful.

Lemma 4. *Let $\{x_n\}$ be a sequence of positive numbers with $\lim_{n\to\infty} x_n = 0$. Let $1<p<\infty$, $r, s, R, S \in \mathbb{R}$. Then,*

$$(|x|^r(|x|+x_n)^s, |x|^R(|x|+x_n)^S) \in A_p([-1,1]) \; uniformly$$

if and only if

$$\begin{array}{lll} -1<r; & R<p-1; & R\le r; \\ -1<r+s; & R+S<p-1; & R+S\le r+s. \end{array}$$

Proof. According to its definition,

$$(|x|^r(|x|+x_n)^s, |x|^R(|x|+x_n)^S) \in A_p([-1,1]) \; uniformly$$

if and only if there exists a constant $C>0$ such that

$$\int_a^b |x|^r(|x|+x_n)^s dx \, (\int_a^b [|x|^R(|x|+x_n)^S]^{-1/(p-1)} dx)^{p-1} \le C(b-a)^p \tag{6}$$

for all $-1 \le a < b \le 1$, $\forall n \ge 1$. Integrability conditions imply the above inequalities. In turn, if those inequalities hold we can easily deduce (6) from the estimate

$$\int_a^b x^\gamma (x+c)^\mu dx \le \begin{cases} Kb^{\gamma+\mu}(b-a) & \text{if } c \le b \\ Kb^\gamma c^\mu (b-a) & \text{if } b \le c \end{cases},$$

valid for $0 \le a < b \le 1$, $0 \le c \le 1$, $\gamma > -1$, $\gamma + \mu > -1$, with a constant K which depends only on γ, μ. ■

The same property holds if we replace x by $x - a$, with $a \in [-1, 1]$. Even more, it is not difficult to show that in order to see whether a finite product of this type of expressions is uniformly in A_p, we only need to check the above inequalities for each factor of the form

$$(|x-a|^r(|x-a|+x_n)^s, |x-a|^R(|x-a|+x_n)^S)$$

separately.

We will eventually need to show that some of the operators are not of strong or weak type. In this sense, the following lemma (see [5]) will be used:

Lemma 5. *Let* supp $d\alpha = [-1, 1]$, $\alpha' > 0$ *a.e. in* $[-1, 1]$, *and* $0 < p \le \infty$. *There exists a constant* $C > 0$ *such that if* g *is a Lebesgue-measurable function on* $[-1, 1]$, *then*

$$\|\alpha'(x)^{-1/2}(1-x^2)^{-1/4}\|_{L^p(|g|^p dx)} \le C \liminf_{n\to\infty} \|P_n\|_{L^p(|g|^p dx)}.$$

There is a weak version of this property: it is a consequence of Kolmogorov's condition (see [1], lemma V.2.8, p. 485) and the previous lemma.

Lemma 6. *Let* supp $d\alpha = [-1, 1]$, $\alpha' > 0$ *a.e. in* $[-1, 1]$, *and* $0 < p < \infty$. *There exists a constant* $C > 0$ *such that if* g, h *are Lebesgue-measurable functions on* $[-1, 1]$, *then*

$$\|\alpha'(x)^{-1/2}(1-x^2)^{-1/4}g(x)\|_{L^p_*(|h|^p dx)} \le C \liminf_{n\to\infty} \|P_n g\|_{L^p_*(|h|^p dx)}.$$

The following lemma will be useful to estimate some weighted L^p_* norms:

Lemma 7. *Let* $1 \le p < \infty$, $r, s \in \mathbb{R}$, $a > 0$. *Then,*

$$\chi_{(0,a)}(x)x^r \in L^p_*(x^s dx) \iff pr + s + 1 \ge 0, \quad (r, s) \ne (0, -1).$$

Moreover, in this case there is a constant K *depending on* r, s, p *such that*

$$\|\chi_{(0,a)}(x)x^r\|_{L^p_*(x^s dx)} = K a^{r+(s+1)/p}.$$

Proof. Since

$$\|\chi_{(0,a)}(x)x^r\|^p_{L^p_*(x^s dx)} = \sup_{y>0} y^p \int_A x^s dx$$

with $A = \{x;\ 0 < x < a,\ x^r > y\}$, the proof is reduced to a simple calculation of that integral, depending on the sign of $pr + s + 1$ and r. ■

Finally, this lemma will be used in the study of the operator $W_{2,n}$:

Lemma 8. *Let $\alpha > -1$, $1 < p < \infty$, $1/p + 1/q = 1$, $0 < r < 1$, $n \in \mathbb{N}$. If $(\alpha+1)(1/p - 1/2) < 1/4$, then there exists a constant C, independent of r and n, such that*

$$\|(1-t)(1-t+n^{-2})^{-(2\alpha+3)/4}\chi_{(r,1)}(t)\|_{L^q_*((1-t)^\alpha)} \le C(1-r)^{1-(\alpha+1)/p}(1-r+n^{-2})^{(2\alpha+1)/4}.$$

Proof. a) Case $\alpha \ge -1/2$. Since $-(2\alpha+3)/4 < 0$,

$$\|(1-t)(1-t+n^{-2})^{-(2\alpha+3)/4}\chi_{(r,1)}(t)\|_{L^q_*((1-t)^\alpha)} \le$$

$$\le \|(1-t)^{(1-2\alpha)/4}\chi_{(r,1)}(t)\|_{L^q_*((1-t)^\alpha)}.$$

By lemma 7 and taking into account that $(2\alpha+1)/4 \ge 0$, we have

$$\|(1-t)^{(1-2\alpha)/4}\chi_{(r,1)}(t)\|_{L^q_*((1-t)^\alpha)} \le C(1-r)^{(1-2\alpha)/4+(\alpha+1)/q} \le$$

$$\le C(1-r)^{1-(\alpha+1)/p}(1-r+n^{-2})^{(2\alpha+1)/4}.$$

b) Case $1-r \le n^{-2}$. Then, by lemma 7 and the inequality $1-r+n^{-2} \le 2n^{-2}$, we obtain

$$\|(1-t)(1-t+n^{-2})^{-(2\alpha+3)/4}\chi_{(r,1)}(t)\|_{L^q_*((1-t)^\alpha)} \le$$

$$\le (n^{-2})^{-(2\alpha+3)/4}\|(1-t)\chi_{(r,1)}(t)\|_{L^q_*((1-t)^\alpha)} \le$$

$$\le C(n^{-2})^{-(2\alpha+3)/4}(1-r)^{1+(\alpha+1)/q} \le$$

$$\le C(1-r+n^{-2})^{-(2\alpha+3)/4}(1-r)^{1-(\alpha+1)/p}(1-r)^{\alpha+1} \le$$

$$\le C(1-r)^{1-(\alpha+1)/p}(1-r+n^{-2})^{(2\alpha+1)/4}.$$

c) Case $\alpha < -1/2$ and $n^{-2} \le 1-r$. Then $1-2\alpha \ge 0$ and $1-r+n^{-2} \le 2(1-r)$. Thus,

$$\|(1-t)(1-t+n^{-2})^{-(2\alpha+3)/4}\chi_{(r,1)}(t)\|_{L^q_*((1-t)^\alpha)} \le$$

$$\le \|(1-t+n^{-2})^{(1-2\alpha)/4}\chi_{(r,1)}(t)\|_{L^q_*((1-t)^\alpha)} \le$$

$$\le (1-r+n^{-2})^{(1-2\alpha)/4}\|\chi_{(r,1)}(t)\|_{L^q_*((1-t)^\alpha)} \le$$

$$\le C(1-r+n^{-2})^{(1-2\alpha)/4}(1-r)^{(\alpha+1)/q} =$$

$$= C(1-r)^{1-(\alpha+1)/p}(1-r+n^{-2})^{(2\alpha+1)/4}(1-r+n^{-2})^{-\alpha}(1-r)^{\alpha} \le$$

$$\le C(1-r)^{1-(\alpha+1)/p}(1-r+n^{-2})^{(2\alpha+1)/4}. \blacksquare$$

§3. Proof of theorems 1 and 2.

The proof of theorem 1 consists of lemmas 9, 10 and 11 below. In order to prove theorem 2, analogous weighted lemmas can be shown using that, in the case $\alpha, \beta \ge -1/2$, not only the polynomials Q_n but also the P_n satisfy an uniform estimate similar to (5).

Lemma 9. *Under the hypothesis of theorem 1, there exists a constant C such that*

$$\|W_{1,n}f\|_{L^p_*(w)} \leq C\|f\|_{L^p(w)} \qquad \forall f \in L^p(w),\ \forall n \in \mathbb{N}.$$

Lemma 10. *Under the hypothesis of theorem 1, there exists a constant C such that*

$$\|W_{3,n}f\|_{L^p(w)} \leq C\|f\|_{L^p(w)} \qquad \forall f \in L^p(w),\ \forall n \in \mathbb{N}.$$

Lemma 11. *Under the hypothesis of theorem 1, there exists a constant C such that for every measurable set $E \subseteq [-1,1]$ and for every $n \geq 0$*

$$\|W_{2,n}\chi_E\|_{L^p_*(w)} \leq C\|\chi_E\|_{L^p(w)} \tag{7}$$

Proof of lemma 9. From its definition, we have

$$\|W_{1,n}f\|_{L^p_*(w)} \leq \|P_{n+1}\|_{L^p_*(w)}\|P_{n+1}\|_{L^q(w)}\|f\|_{L^p(w)},$$

where $1/p + 1/q = 1$. So, we only need to prove

$$\|P_n\|_{L^p_*(w)} \leq C \quad \forall n \in \mathbb{N}$$

and

$$\|P_n\|_{L^q(w)} \leq C \quad \forall n \in \mathbb{N},$$

which follows from lemma 7, (4) and the dominate convergence theorem. ■

Proof of lemma 10. It is clear that

$$\|W_{3,n}f\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$$

for every $f \in L^p(w)$ if and only if

$$\|Hg\|_{L^p((1-x^2)^p|Q_n|^p w)} \leq C\|g\|_{L^p(|P_{n+1}|^{-p}w^{1-p})}$$

for every $g \in L^p(|P_{n+1}|^{-p}w^{1-p})$. Using again (4) and its analogous for Q_n, it is enough to obtain

$$\|Hg\|_{L^p(u_n)} \leq C\|g\|_{L^p(v_n)} \quad \forall n,\ \forall g \in L^p(v_n),$$

with

$$u_n(x) = (1-x)^{p+\alpha}(1-x+n^{-2})^{-p(2\alpha+3)/4}(1+x)^{p+\beta}(1+x+n^{-2})^{-p(2\beta+3)/4}$$

and

$$v_n(x) = (1-x)^{\alpha(1-p)}(1-x+n^{-2})^{p(2\alpha+1)/4}(1+x)^{\beta(1-p)}(1+x+n^{-2})^{p(2\beta+1)/4}.$$

Now, we only need to prove that

$$((1-x)^{(p+\alpha)\delta}(1-x+n^{-2})^{-p(2\alpha+3)\delta/4}, (1-x)^{\alpha(1-p)\delta}(1-x+n^{-2})^{p(2\alpha+1)\delta/4}) \in A_p$$

and

$$((1+x)^{(p+\beta)\delta}(1+x+n^{-2})^{-p(2\beta+3)\delta/4}, (1+x)^{\beta(1-p)\delta}(1+x+n^{-2})^{p(2\beta+1)\delta/4}) \in A_p$$

uniformly in n, for some $\delta > 1$. This can be deduced from lemma 4. ■

Proof of lemma 11. From $p = \frac{4(\alpha+1)}{2\alpha+1}$ and $\alpha \geq \beta$, it follows

$$-\frac{1}{4} \leq (\alpha+1)(\frac{1}{p}-\frac{1}{2}) < \frac{1}{4} \tag{8}$$

and

$$-\frac{1}{4} \leq (\beta+1)(\frac{1}{p}-\frac{1}{2}) < \frac{1}{4}. \tag{9}$$

We will prove that (8) and (9) imply (7). By the symmetry of these inequalities, we can consider only the case $E \subseteq [0,1]$.

a) We will show first that there exists a constant $C > 0$ such that

$$\|\chi_{[-3/4,3/4]}W_{2,n}(\chi_E)\|_{L^p_*(w)} \leq C\|\chi_E\|_{L^p(w)}$$

for all $n \in \mathbb{N}$ and every measurable set $E \subseteq [0,1]$.

Since $1-x$ and $1+x$ are bounded away from 0 and ∞ on $[-3/4,3/4]$, from (4) and the definition of $W_{2,n}$ we get

$$\|\chi_{[-3/4,3/4]}W_{2,n}(\chi_E)\|_{L^p_*(w)} \leq C\|\chi_{[-3/4,3/4]}H((1-t^2)\chi_E Q_n w)\|_{L^p((1-x)^r(1+x)^s)}$$

for any previously fixed r, s. If we find r, s such that

$$H : L^p((1-x)^{\alpha-p(2\alpha+1)/4}(1+x)^{\beta-p(2\beta+1)/4}) \longrightarrow L^p((1-x)^r(1+x)^s) \tag{10}$$

is bounded, then from (5) it would follow

$$\|\chi_{[-3/4,3/4]}H((1-t^2)\chi_E Q_n w)\|_{L^p((1-x)^r(1+x)^s)} \leq \|H((1-t^2)\chi_E Q_n w)\|_{L^p((1-x)^r(1+x)^s)} \leq$$

$$\leq C\|(1-x^2)\chi_E Q_n w\|_{L^p((1-x)^{\alpha-p(2\alpha+1)/4}(1+x)^{\beta-p(2\beta+1)/4})} \leq C\|\chi_E\|_{L^p(w)},$$

as we want to show. In order to get (10), it is enough to have

$$((1-x)^{r\delta}(1+x)^{s\delta}, (1-x)^{\delta[\alpha-p(2\alpha+1)/4]}(1+x)^{\delta[\beta-p(2\beta+1)/4]}) \in A_p$$

for some $\delta > 1$. This is equivalent, by lemma 4, to the following conditions:

$$\begin{array}{ll} -1 < r; & -1 < s; \\ \alpha - p(2\alpha+1)/4 < p-1; & \beta - p(2\beta+1)/4 < p-1; \\ \alpha - p(2\alpha+1)/4 \leq r; & \beta - p(2\beta+1)/4 \leq s. \end{array}$$

It is easy to see that the second row inequalities hold, while for the others we only need to take r and s large enough.

b) Now, we are going to prove that there exists a constant $C > 0$ such that

$$\|\chi_{[-1,-3/4]} W_{2,n}(\chi_E)\|_{L^p_*(w)} \le C\|\chi_E\|_{L^p(w)}$$

for all $n \in \mathbb{N}$ and every measurable set $E \subseteq [0,1]$.

As $E \subseteq [0,1]$, we can drop the denominator $x-t$ in $W_{2,n}(\chi_E)$ and, using the inequality (5), we get

$$|\chi_{[-1,-3/4]}(x) W_{2,n}(\chi_E, x)| \le$$

$$\le C\chi_{[-1,-3/4]}(x)|P_{n+1}(x)| \int_{-1}^{1} (1-t)^{1-(2\alpha+3)/4} \chi_E(t) w(t) dt \le$$

$$\le C\chi_{[-1,-3/4]}(x)|P_{n+1}(x)| \ \|(1-t)^{(1-2\alpha)/4}\|_{L^q(w)} \|\chi_E\|_{L^p(w)} \le$$

$$\le C\|\chi_E\|_{L^p(w)} \chi_{[-1,-3/4]}(x)|P_{n+1}(x)|.$$

Therefore

$$\|\chi_{[-1,-3/4]} W_{2,n}(\chi_E)\|_{L^p_*(w)} \le$$

$$\le C\|\chi_E\|_{L^p(w)} \ \|\chi_{[-1,-3/4]}(1+x+n^{-2})^{-(2\beta+1)/4}\|_{L^p_*(w)} \le C\|\chi_E\|_{L^p(w)},$$

by the dominate convergence and lemma 7.

c) We must show now that there exists a constant $C > 0$ such that

$$\|\chi_{[3/4,1]} W_{2,n}(\chi_E)\|_{L^p_*(w)} \le C\|\chi_E\|_{L^p(w)}$$

for all $n \in \mathbb{N}$ and every measurable set $E \subseteq [0,1]$. Let us define, for $k = 2, 3, \ldots$ the sets

$$I_k = [1-2^{-k}, 1-2^{-k-1}),$$

$$J_{k1} = [0, 1-2^{-k+1}), \qquad J_{k2} = [1-2^{-k+1}, 1-2^{-k-2}), \qquad J_{k3} = [1-2^{-k-2}, 1].$$

For each $k \ge 2$, J_{ki} $(i = 1,2,3)$ are non-overlapping sets such that $[0,1] = J_{k1} \cup J_{k2} \cup J_{k3}$. The sets I_k are also disjoint and $\bigcup_{k\ge 2} I_k = [3/4, 1)$. The following properties are easy to check:

$$\forall k \ge 2,\ \forall x \in I_k, \quad 2^{-k-1} \le 1-x \le 2^{-k}; \tag{11}$$

$$\forall k \ge 2,\ \forall t \in J_{k2}, \quad 2^{-k-2} \le 1-t \le 2^{-k+1}; \tag{12}$$

$$\forall k \ge 2,\ \forall x \in I_k,\ \forall t \in J_{k1}, \quad 2^{-k+1} \le 1-t \le 2(x-t) \le 2(1-t); \tag{13}$$

$$\forall k \ge 2,\ \forall x \in I_k,\ \forall t \in J_{k3}, \quad 1-t \le 2^{-k-2} \le t-x \le 2^{-k}. \tag{14}$$

We can write

$$\chi_{[3/4,1]} W_{2,n}(\chi_E) = \sum_{k\ge 2} \chi_{I_k} W_{2,n}(\chi_E \chi_{J_{k1}}) + \sum_{k\ge 2} \chi_{I_k} W_{2,n}(\chi_E \chi_{J_{k2}}) + \sum_{k\ge 2} \chi_{I_k} W_{2,n}(\chi_E \chi_{J_{k3}}).$$

We prove that each term is bounded.

c1) If $x \in I_k$, from (13) and Hölder's inequality for Lorentz spaces it follows

$$|H((1-t^2)\chi_E\chi_{J_{k1}}Q_n w, x)| \le C\int_{-1}^{1}\chi_E\chi_{J_{k1}}|Q_n|w \le$$

$$\le C\|\chi_E\|_{L^p(w)}\|\chi_{J_{k1}}Q_n\|_{L^q_*(w)}.$$

From the estimates (4) for Q_n, property (8), lemma 7 and using that $1 \le 1+t \le 2$ for $t \in J_{k1}$, we obtain

$$\|\chi_{J_{k1}}Q_n\|_{L^q_*(w)} \le C\|\chi_{J_{k1}}(1-t+n^{-2})^{(\alpha+1)/q-(2\alpha+3)/4}(1-t+n^{-2})^{-(\alpha+1)/q}\|_{L^q_*(w)} \le$$

$$\le C(1-x+n^{-2})^{(\alpha+1)/q-(2\alpha+3)/4}\|\chi_{J_{k1}}(1-t+n^{-2})^{-(\alpha+1)/q}\|_{L^q_*(w)} \le$$

$$\le C(1-x+n^{-2})^{(\alpha+1)/q-(2\alpha+3)/4}.$$

Therefore, if $x \in I_k$ then

$$|H((1-t^2)\chi_E\chi_{J_{k1}}Q_n w, x)| \le C\|\chi_E\|_{L^p(w)}(1-x+n^{-2})^{(\alpha+1)/q-(2\alpha+3)/4},$$

where the constant C does not depend on k. Since the I_k are disjoint, this implies

$$|\sum_{k\ge 2}\chi_{I_k}(x)W_{2,n}(\chi_E\chi_{J_{k1}}, x)| \le$$

$$\le C(1-x+n^{-2})^{-(2\alpha+1)/4}\|\chi_E\|_{L^p(w)}(1-x+n^{-2})^{(\alpha+1)/q-(2\alpha+3)/4} =$$

$$= C(1-x+n^{-2})^{-(\alpha+1)/p}\|\chi_E\|_{L^p(w)} \le C(1-x)^{-(\alpha+1)/p}\|\chi_E\|_{L^p(w)}.$$

Then, by lemma 7,

$$\|\sum_{k\ge 2}\chi_{I_k}W_{2,n}(\chi_E\chi_{J_{k1}})\|_{L^p_*(w)} \le C\|\chi_E\|_{L^p(w)}.$$

c2) Let $k \ge 2$. By (4) and (11),

$$\|\chi_{I_k}W_{2,n}(\chi_E\chi_{J_{k2}})\|_{L^p_*(w)} = C\|\chi_{I_k}P_{n+1}H((1-t^2)\chi_E\chi_{J_{k2}}Q_n w)\|_{L^p_*(w)} \le$$

$$\le C(2^{-k}+n^{-2})^{-(2\alpha+1)/4}(2^{-k})^{\alpha/p}\|\chi_{I_k}H((1-t^2)\chi_E\chi_{J_{k2}}Q_n w)\|_{L^p_*(dx)} \le$$

$$\le C(2^{-k}+n^{-2})^{-(2\alpha+1)/4}(2^{-k})^{\alpha/p}\|H((1-t^2)\chi_E\chi_{J_{k2}}Q_n w)\|_{L^p(dx)}.$$

Since the Hilbert transform is bounded in $L^p(dx)$, this expression can be bounded, using (4) and (12), by

$$C(2^{-k}+n^{-2})^{-(2\alpha+1)/4}(2^{-k})^{\alpha/p}\|(1-x^2)\chi_E\chi_{J_{k2}}Q_n w\|_{L^p(dx)} \le$$

$$\leq C(2^{-k}+n^{-2})^{-(2\alpha+1)/4}(2^{-k})^{\alpha+1}(2^{-k}+n^{-2})^{-(2\alpha+3)/4}\|(1-x)^{\alpha/p}\chi_E\chi_{J_{k2}}\|_{L^p(dx)} \leq$$
$$\leq C\|\chi_E\chi_{J_{k2}}\|_{L^p(w)}.$$

Now, as the functions $\chi_{I_k}W_{2,n}(\chi_E\chi_{J_{k2}})$ have non-overlapping support and $\sum_{k\geq 2}\chi_{J_{k2}} \leq 3$, we get

$$\|\sum_{k\geq 2}\chi_{I_k}W_{2,n}(\chi_E\chi_{J_{k2}})\|^p_{L^p_*(w)} \leq \sum_{k\geq 2}\|\chi_{I_k}W_{2,n}(\chi_E\chi_{J_{k2}})\|^p_{L^p_*(w)} \leq$$
$$\leq C\sum_{k\geq 2}\|\chi_E\chi_{J_{k2}}\|^p_{L^p(w)} \leq C\|\chi_E\|^p_{L^p(w)}.$$

That is,

$$\|\sum_{k\geq 2}\chi_{I_k}W_{2,n}(\chi_E\chi_{J_{k2}})\|_{L^p_*(w)} \leq C\|\chi_E\|_{L^p(w)}.$$

c3) Let $k \geq 2$ and $x \in I_k$. By (14), Hölder's inequality for Lorentz spaces and (4), it follows

$$|H((1-t^2)\chi_E\chi_{J_{k3}}Q_nw, x)| \leq$$
$$\leq C2^k\int_{-1}^{1}(1-t^2)\chi_E(t)\chi_{J_{k3}}(t)Q_n(t)w(t)dt \leq$$
$$\leq C2^k\|(1-t)\chi_E\chi_{J_{k3}}Q_n\|_{L^1(w)} \leq$$
$$\leq C2^k\|\chi_E\|_{L^p(w)}\|(1-t)(1-t+n^{-2})^{-(2\alpha+3)/4}\chi_{J_{k3}}\|_{L^q_*((1-t)^\alpha)}.$$

By lemma 8 and (8),

$$\|(1-t)(1-t+n^{-2})^{-(2\alpha+3)/4}\chi_{J_{k3}}\|_{L^q_*((1-t)^\alpha)} \leq C(2^{-k})^{1-(\alpha+1)/p}(2^{-k}+n^{-2})^{(2\alpha+1)/4},$$

what, together with (11), implies

$$|H((1-t^2)\chi_E\chi_{J_{k3}}Q_nw, x)| \leq C(1-x)^{-(\alpha+1)/p}(1-x+n^{-2})^{(2\alpha+1)/4}\|\chi_E\|_{L^p(w)}$$

if $x \in I_k$, with a constant C which does not depend on x, E, k, n. Since the I_k are non-overlapping, we have

$$|\sum_{k\geq 2}\chi_{I_k}(x)W_{2,n}(\chi_E\chi_{J_{k3}}, x)| \leq C\chi_{[3/4,1)}(1-x)^{-(\alpha+1)/p}\|\chi_E\|_{L^p(w)}$$

and, by lemma 7,

$$\|\sum_{k\geq 2}\chi_{I_k}W_{2,n}(\chi_E\chi_{J_{k3}})\|_{L^p_*(w)} \leq C\|\chi_E\|_{L^p(w)}.$$

This concludes the proof of the lemma. ∎

§4. Proof of theorem 3.

The weak boundedness

$$\|uS_nf\|_{L^p_*(w)} \leq C\|uf\|_{L^p(w)}$$

implies the following conditions (see [2], theorem 1, with the appropriate changes):

$$u \in L^p_*(w)$$

$$u^{-1} \in L^q(w)$$

$$u(x)w(x)^{-1/2}(1-x^2)^{-1/4} \in L^p_*(w)$$

$$u(x)^{-1}w(x)^{-1/2}(1-x^2)^{-1/4} \in L^q(w).$$

With the weight $u(x) = (1-x)^a(1+x)^b$ and having in mind that $\alpha, \beta \geq -1/2$, this means

$$-\frac{1}{4} \leq a + (\alpha+1)(\frac{1}{p} - \frac{1}{2}) < \frac{1}{4},$$

$$-\frac{1}{4} \leq b + (\beta+1)(\frac{1}{p} - \frac{1}{2}) < \frac{1}{4}.$$

Therefore, we only need to show that the equality cannot occur in the left hand side of these equations. Assume, for example,

$$-\frac{1}{4} = a + (\alpha+1)(\frac{1}{p} - \frac{1}{2}). \tag{15}$$

Let us consider again Pollard's decomposition of the partial sums S_nf. As we mentioned at the beginning of the previous section, the proofs of lemmas 9 and 10 essentially show that there exists a constant C such that for all $f \in L^p(w)$ and every $n \in \mathbb{N}$

$$\|uW_{1,n}f\|_{L^p_*(w)} \leq C\|uf\|_{L^p(w)}$$

and

$$\|uW_{3,n}f\|_{L^p(w)} \leq C\|uf\|_{L^p(w)}$$

(notice that in the case $\alpha, \beta \geq -1/2$, the polynomials P_n satisfy the estimate analogous to (5), what simplifies the proofs). Under our hypothesis, this implies that there exists also a constant C such that for all $f \in L^p(u^pw)$ and every $n \in \mathbb{N}$

$$\|uW_{2,n}f\|_{L^p_*(w)} \leq C\|uf\|_{L^p(w)},$$

that is,

$$\|uP_{n+1}Hg\|_{L^p_*(w)} \leq C\|u(x)(1-x^2)^{-1}Q_n(x)^{-1}w(x)^{-1}g\|_{L^p(w)}.$$

Applying (3), we have

$$\|uP_{n+1}\chi_I\|_{L^p_*(w)} \left(\int_{-1}^{1} \frac{u(x)^{-q}(1-x^2)^q|Q_n(x)|^q w(x)}{(|I|+|x-x_I|)^q}dx\right)^{1/q} \leq C$$

for every interval $I \subseteq [-1,1]$, with a constant $C>0$ independent of n and I; now, by lemma 6

$$\|u(x)(1-x^2)^{-1/4}w(x)^{-1/2}\chi_I\|_{L^p_*(w)}\left(\int_{-1}^1 \frac{u(x)^{-q}(1-x^2)^{q/4}w(x)^{1-q/2}}{(|I|+|x-x_I|)^q}dx\right)^{1/q} \le C.$$

Taking $I=[1-\varepsilon,1]$, it follows

$$\|x^{a-\alpha/2-1/4}\chi_{[0,\varepsilon]}\|_{L^p_*(x^\alpha)}\left(\int_0^1 \frac{x^{-aq+q/4+\alpha(1-q/2)}}{(\varepsilon+|x-\varepsilon/2|)^q}dx\right)^{1/q} \le C. \tag{16}$$

Now, by lemma 7 and (15)

$$\|x^{a-\alpha/2-1/4}\chi_{[0,\varepsilon]}\|_{L^p_*(x^\alpha)} = K \tag{17}$$

and

$$\int_0^1 \frac{x^{-aq+q/4+\alpha(1-q/2)}}{(\varepsilon+|x-\varepsilon/2|)^q}dx = \int_0^1 \frac{x^{1/(p-1)}}{(\varepsilon+|x-\varepsilon/2|)^q}dx \ge C\int_\varepsilon^1 x^{1/(p-1)-q}dx = C|\log\varepsilon|,$$

which, together with (17), leads to a contradiction in (16). Therefore, (15) cannot be true and the theorem is proved.

REFERENCES

[1] J. García Cuerva and J. L. Rubio de Francia, *"Weighted norm inequalities and related topics"*, North-Holland, Amsterdam, 1985.

[2] J. J. Guadalupe, M. Pérez and J. L. Varona, *Weak behaviour of Fourier-Jacobi series*, J. Approx. Theory **61** (1990), 222–238.

[3] R. Hunt, B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251.

[4] A. Máté, P. Nevai and V. Totik, *Asymptotics for the ratio of the leading coefficients of orthogonal polynomials on the unit circle*, Constr. Approx. **1** (1985), 63–69.

[5] A. Máté, P. Nevai and V. Totik, *Necessary conditions for weighted mean convergence of Fourier series in orthogonal polynomials*, J. Approx. Theory **46** (1986), 314–322.

[6] B. Muckenhoupt, *Mean convergence of Jacobi series*, Proc. Amer. Math. Soc. **23** (1969), 306–310.

[7] B. Muckenhoupt and R. L. Wheeden, *Two weight function norm inequalities for the Hardy-Littlewood maximal function and the Hilbert transform*, Studia Math. **55** (1976), 279–294.

[8] C. J. Neugebauer, *Inserting A_p weights*, Proc. Amer. Math. Soc. **87** (1983), 644–648.

[9] H. Pollard, *The mean convergence of orthogonal series. II*, Trans. Amer. Math. Soc. **63** (1948), 355–367.

[10] E. A. Rahmanov, *On the asymptotics of the ratio of orthogonal polynomials. II*, Math. USSR. Sb. **46** (1983), 105–117.

[11] E. M. Stein and G. Weiss, *"Introduction to Fourier analysis on euclidean spaces"*, Princeton Univ. Press, New Jersey, 1975.