# $L^p$-BOUNDEDNESS OF THE KERNELS RELATIVE TO GENERALIZED JACOBI WEIGHTS

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*A.M.S. Classification*: 42C10.

**Resumen**

Sea $w$ un peso de Jacobi generalizado sobre el intervalo $[-1,1]$, es decir, $w(x) = h(x)(1-x)^\alpha(1+x)^\beta \prod_{i=1}^N |x-x_i|^{\gamma_i}$, con $\alpha, \beta, \gamma_i > -1$ y ciertas condiciones de continuidad sobre $h$. Mediante el uso de la teoría de pesos $A_p$, se puede demostrar la convergencia de la serie de Fourier de polinomios ortonormales con respecto a $w$ cuando $\gamma_i \geq 0\ \forall i$. En este trabajo obtenemos acotaciones de las normas en $L^p$ de los núcleos relativos a $w$, que permiten extender el resultado de la convergencia en media al caso general.

Let $w$ be a generalized Jacobi weight, that is,

$$w(x) = h(x)(1-x)^\alpha(1+x)^\beta \prod_{i=1}^N |x-x_i|^{\gamma_i}, \qquad x \in [-1,1],$$

where:

(a) $\alpha, \beta, \gamma_i > -1$, $t_i \in (-1,1)$, $t_i \neq t_j\ \forall i \neq j$;

(b) $h$ is a positive, continuous function on $[-1,1]$ and $\omega(h,\delta)\delta^{-1} \in L^1(0,2)$, $\omega(h,\delta)$ being the modulus of continuity of $h$.

This paper were published in “Polinomios Ortogonales y Aplicaciones” (Actas del VI Simposium, Gijón, 1989), 168–178, Universidad de Oviedo, Gijón, 1990.

Let $d\mu = w(x)\,dx$ on $[-1,1]$ and let $S_n$ $(n \geq 0)$ be the $n$-th partial sum of the Fourier series in the orthonormal polynomials with respect to $d\mu$. The study of the boundedness

$$\|S_n f\|_{L^p(u^p\,d\mu)} \leq C\|f\|_{L^p(v^p\,d\mu)}, \tag{1}$$

where

$$u(x) = (1-x)^a(1+x)^b \prod_{i=1}^{N} |x-t_i|^{g_i}$$

and

$$v(x) = (1-x)^A(1+x)^B \prod_{i=1}^{N} |x-t_i|^{G_i}$$

was done by Badkov ([B]) in the case $u = v$ by means of a direct estimation of the kernels $K_n(x,y)$ associated with the polynomials orthogonal with respect to $d\mu$. Later, Varona ([V]) considered the same problem, with $u$ and $v$ not necessarily equal; his method consists of an appropriate use of the theory of $A_p$ weights. He found conditions for (1) which generalized those obtained for $u = v$ by Badkov. However, this result, which we state below, follows only in the case $\gamma_i \geq 0$ $\forall i$:

**Theorem 1.** *Let $\gamma_i \geq 0$ $\forall i$ and $1 < p < \infty$. If the inequalities*

$$\begin{cases} A + (\alpha+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{1}{4}, \\ B + (\beta+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{1}{4}, \\ G_i + (\gamma_i+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{1}{2} \quad \forall i, \end{cases} \tag{2}$$

$$\begin{cases} A + (\alpha+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{\alpha+1}{2}, \\ B + (\beta+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{\beta+1}{2}, \\ G_i + (\gamma_i+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{\gamma_i+1}{2} \quad \forall i, \end{cases} \tag{3}$$

$$\begin{cases} a + (\alpha+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) > -\dfrac{1}{4}, \\ b + (\beta+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) > -\dfrac{1}{4}, \\ g_i + (\gamma_i+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) > -\dfrac{1}{2} \quad \forall i, \end{cases} \tag{4}$$

$$
(5) \qquad \begin{cases} a+(\alpha+1)\left(\dfrac{1}{p}-\dfrac{1}{2}\right) > -\dfrac{\alpha+1}{2}, \\ b+(\beta+1)\left(\dfrac{1}{p}-\dfrac{1}{2}\right) > -\dfrac{\beta+1}{2}, \\ g_i+(\gamma_i+1)\left(\dfrac{1}{p}-\dfrac{1}{2}\right) > -\dfrac{\gamma_i+1}{2} \quad \forall i, \end{cases}
$$

$$
(6) \qquad A \le a, \qquad B \le b, \qquad G_i \le g_i \quad \forall i,
$$

*hold, then there exists* $C > 0$ *such that*

$$
\|S_n f\|_{L^p(u^p\,d\mu)} \le C\|f\|_{L^p(v^p\,d\mu)} \quad \forall f \in L^p(v^p\,d\mu), \quad \forall n \in \mathbb{N}.
$$

The objective of this paper is to eliminate the restriction $\gamma_i \ge 0$, by studying the norms of the kernels $K_n(x,y)$. We are going to use the following notation: $\{P_n(x)\}$, $\{k_n\}$ and $\{K_n(x,y)\}$ will be, respectively, the orthonormal polynomials, their leading coefficients and the kernels relatives to $d\mu$; if $c \in (-1,1)$, $\{P_n^c(x)\}$, $\{k_n^c\}$ and $\{K_n^c(x,y)\}$ will be the corresponding to $(x-c)^2\,d\mu$. Then, it is not difficult to establish the following relations, $\forall n \ge 1$:

$$
(7) \qquad K_n(x,y) = (x-c)(y-c)K_{n-1}^c(x,y) + \frac{K_n(x,c)K_n(c,y)}{K_n(c,c)}\,,
$$

$$
(8) \qquad K_n(x,c) = \frac{k_n}{k_n^c}\,P_n(c)P_n^c(x) - \frac{k_{n-1}^c}{k_{n+1}}\,P_{n+1}(c)P_{n-1}^c(x).
$$

It can be also shown (see [MNT], theorems 10 and 11, and [R], page 212) that

$$
(9) \qquad \lim_{n\to\infty}\frac{k_n}{k_n^c} = \lim_{n\to\infty}\frac{k_{n-1}^c}{k_{n+1}} = \frac{1}{2}.
$$

If we define

$$
d(x,n) = (1-x+n^{-2})^{-(2\alpha+1)/4}(1+x+n^{-2})^{-(2\beta+1)/4}\prod_{i=1}^{N}(|x-t_i|+n^{-1})^{-\gamma_i/2},
$$

it is known ([B]) that there exists a constant $C$ such that $\forall x \in [-1,1]$, $\forall n$,

$$
(10) \qquad |P_n(x)| \le C d(x,n).
$$

There are also some well-known estimates for the kernels, one of them being this ([N], page 4 and page 119, theorem 25): if $c \in (-1,1)$ and the factor $|x-c|$ is present in $w$ with an exponent $\gamma$, there exist some positive constants $C_1$ and $C_2$, depending on $c$, such that, $\forall n$,

$$C_1 n^{\gamma+1} \le K_n(c,c) \le C_2 n^{\gamma+1}. \tag{11}$$

From now on, all constants will be denoted $C$, so by $C$ we will mean a constant, possibly different in each occurrence. Using (8), (9) and (10), we get the following result:

**Proposition 2.** *Let $1 < p < \infty$, $1/p + 1/q = 1$. Suppose the inequalities (4) and (5) hold. Let $-1 < c < 1$ and let $\gamma$ and $g$ be the exponents of $|x-c|$ in $w$ and $u$, respectively. Then, there exists a positive constant $C$ such that, $\forall n \ge 0$,*

$$\|K_n(x,c)\|_{L^p(u^p w)} \le \begin{cases} Cn^{(\gamma+1)/q - g}, & \text{if} \quad g < (\gamma+1)(1/2 - 1/p) + 1/2, \\ Cn^{\gamma/2}(\log n)^{1/p}, & \text{if} \quad g = (\gamma+1)(1/2 - 1/p) + 1/2, \\ Cn^{\gamma/2}, & \text{if} \quad (\gamma+1)(1/2 - 1/p) + 1/2 < g. \end{cases}$$

*Proof.* From (10) it follows that $|P_n(c)| \le Cn^{\gamma/2}$. Since $\{P_n^c\}$ is the sequence associated with $(x-c)^2\, d\mu$, it also follows from (10) that

$$|P_n^c(x)| \le C(|x-c| + n^{-1})^{-1} d(x,n).$$

Now, from (8) and (9) we get

$$|K_n(x,c)| \le Cn^{\gamma/2}(|x-c| + n^{-1})^{-1} d(x,n). \tag{12}$$

Let us take $\varepsilon > 0$ such that $|t_i - c| > \varepsilon$ $\forall i$ for all $t_i \ne c$. We can write

$$\begin{aligned} &\|K_n(x,c)\|^p_{L^p(u^p w)} \\ &\qquad = \int_{|x-c|>\varepsilon} |K_n(x,c)|^p u(x)^p w(x)\, dx + \int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x)\, dx. \end{aligned}$$

Using (12), for the first term we obtain

$$\begin{aligned} &\int_{|x-c|>\varepsilon} |K_n(x,c)|^p u(x)^p w(x)\, dx \\ &\qquad \le Cn^{p\gamma/2} \int_{|x-c|>\varepsilon} (|x-c| + n^{-1})^{-p} d(x,n)^p u(x)^p w(x)\, dx \\ &\qquad \le C_1 n^{p\gamma/2} \int_{-1}^{1} d(x,n)^p u(x)^p w(x)\, dx. \end{aligned}$$

It is easy to deduce from (4) and (5) that this last integral is bounded by a constant which does not depend on $n$, so

$$\int_{|x-c|>\varepsilon} |K_n(x,c)|^p u(x)^p w(x)\,dx \le C n^{p\gamma/2}. \tag{13}$$

Let us take now the second term; since for $|x-c|<\varepsilon$ there exists a constant $C$ such that, $\forall n$, $d(x,n) \le C(|x-c|+n^{-1})^{-\gamma/2}$, $u(x) \le C|x-c|^g$ and $w(x) \le C|x-c|^\gamma$, we have

$$\begin{aligned}
&\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x)\,dx \\
&\quad \le C n^{p\gamma/2} \int_{|x-c|<\varepsilon} (|x-c|+n^{-1})^{-p} d(x,n)^p u(x)^p w(x)\,dx \\
&\quad \le C_1 n^{p\gamma/2} \int_{|x-c|<\varepsilon} (|x-c|+n^{-1})^{-p(1+\gamma/2)} |x-c|^{gp+\gamma}\,dx \\
&\quad \le C_2 n^{p\gamma/2} \int_0^1 (y+n^{-1})^{-p(1+\gamma/2)} y^{gp+\gamma}\,dy \\
&\quad = C_2 n^{p\gamma/2+p(1+\gamma/2)-gp-\gamma-1} \int_0^1 (ny+1)^{-p(1+\gamma/2)} (ny)^{gp+\gamma} n\,dy \\
&\quad = C_2 n^{p\gamma/2+p(1+\gamma/2)-gp-\gamma-1} \int_0^n (r+1)^{-p(1+\gamma/2)} r^{gp+\gamma}\,dr.
\end{aligned}$$

Taking into account that $p(1+\gamma/2)-gp-\gamma-1 = p[(\gamma+1)(1/2-1/p)-g+1/2]$ and there exist some constants $C_1$ and $C_2$ such that $C_1 \le r+1 \le C_2$ on $[0,1]$ and $C_1 r \le r+1 \le C_2 r$ on $[1,n]$, we finally get the inequality

$$\begin{aligned}
&\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x)\,dx \\
&\quad \le C n^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]} \int_0^1 r^{gp+\gamma}\,dr \\
&\qquad + C n^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]} \int_1^n r^{-p[(\gamma+1)(1/2-1/p)-g+1/2]-1}\,dr.
\end{aligned} \tag{14}$$

Since (5) implies $gp+\gamma>-1$, the first term can be bounded in this way:

$$C n^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]} \int_0^1 r^{gp+\gamma}\,dr \le C n^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]}. \tag{15}$$

Let us consider separately the three cases in the statement:

(a) If $g < (\gamma+1)(1/2-1/p)+1/2$, then $-p[(\gamma+1)(1/2-1/p)-g+1/2]-1 < -1$; so:

$$\int_1^n r^{-p[(\gamma+1)(1/2-1/p)-g+1/2]-1}\, dr \leq C.$$

In this case, (14) and (15) imply

$$\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x)\, dx \leq C n^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]}.$$

Since $p[(\gamma+1)(1/2-1/p)-g+1/2] > 0$, from this inequality and (13) we obtain

$$\begin{aligned}\|K_n(x,c)\|^p_{L^p(u^p w)} &\leq C n^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]}\\ &= C n^{p[(\gamma+1)(1-1/p)-g]} = C n^{p[(\gamma+1)/q-g]},\end{aligned}$$

as we had to prove.

(b) If $(\gamma+1)(1/2-1/p)+1/2 < g$, then $-p[(\gamma+1)(1/2-1/p)-g+1/2]-1 > -1$; therefore:

$$\int_1^n r^{-p[(\gamma+1)(1/2-1/p)-g+1/2]-1}\, dr \leq C n^{-p[(\gamma+1)(1/2-1/p)-g+1/2]}.$$

From (14) and (15), we get now $\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x)\, dx \leq C n^{p\gamma/2}$ and, finally,

$$\|K_n(x,c)\|^p_{L^p(u^p w)} \leq C n^{p\gamma/2}.$$

(c) If $g = (\gamma+1)(1/2-1/p)+1/2$,

$$\int_1^n r^{-p[(\gamma+1)(1/2-1/p)-g+1/2]-1}\, dr \leq C \log n;$$

hence,

$$\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x)\, dx \leq C n^{p\gamma/2} \log n$$

and

$$\|K_n(x,c)\|^p_{L^p(u^p w)} \leq C n^{p\gamma/2} \log n.$$

This concludes the proof of the proposition. □

**Corollary 3.** *Let $1 < p < \infty$, $1/p + 1/q = 1$. Suppose the inequalities (2) and (3) hold. Let $-1 < c < 1$ and let $\gamma$ and $G$ be the exponents of $|x-c|$ in $w$ and $v$, respectively. Then, there exists a positive constant $C$ such that, $\forall n \geq 0$,*

$$\|K_n(x,c)\|_{L^q(v^{-q}w)} \leq \begin{cases} C n^{\gamma/2}, & \text{if } \ G < (\gamma+1)(1/2-1/p)-1/2,\\ C n^{\gamma/2}(\log n)^{1/q}, & \text{if } \ G = (\gamma+1)(1/2-1/p)-1/2,\\ C n^{(\gamma+1)/p+G}, & \text{if } \ (\gamma+1)(1/2-1/p)-1/2 < G.\end{cases}$$

*Proof.* Just apply proposition 2 to the weight $v^{-1}$ and keep in mind the equality $1/2 - 1/p = 1/q - 1/2$. □

The following result is just what we need to extend theorem 1 to the general case $\gamma_i > -1$:

**Corollary 4.** *Let $1 < p < \infty$, $1/p + 1/q = 1$. Suppose the inequalities (2), (3), (4), (5) and (6) hold. Let $-1 < c < 1$. Then, there exists a positive constant $C$ such that, $\forall n \geq 0$,*

$$\|K_n(x,c)\|_{L^p(u^p w)} \|K_n(x,c)\|_{L^q(v^{-q} w)} \leq C K_n(c,c).$$

*Proof.* It is a simple consequence of proposition 2, corollary 3 and the estimate (11). The only thing we must do is to consider each case in these results separately. □

**Note.** Although it will not be used in what follows, corollary 4 also holds when $c = \pm 1$. The proof is similar: starting from other expressions for $K_n(x, \pm 1)$, analogous results to proposition 2 and corollary 3 can be obtained, and then corollary 4 follows.

We are now ready to extend theorem 1 to the general case $\gamma_i > -1$:

**Theorem 5.** *Let $1 < p < \infty$. If the inequalities*

$$\begin{cases} A + (\alpha+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{1}{4}, \\ B + (\beta+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{1}{4}, \\ G_i + (\gamma_i+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{1}{2} \quad \forall i, \end{cases} \qquad \begin{cases} A + (\alpha+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{\alpha+1}{2}, \\ B + (\beta+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{\beta+1}{2}, \\ G_i + (\gamma_i+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) < \dfrac{\gamma_i+1}{2} \quad \forall i, \end{cases}$$

$$\begin{cases} a + (\alpha+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) > -\dfrac{1}{4}, \\ b + (\beta+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) > -\dfrac{1}{4}, \\ g_i + (\gamma_i+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) > -\dfrac{1}{2} \quad \forall i, \end{cases} \qquad \begin{cases} a + (\alpha+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) > -\dfrac{\alpha+1}{2}, \\ b + (\beta+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) > -\dfrac{\beta+1}{2}, \\ g_i + (\gamma_i+1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right) > -\dfrac{\gamma_i+1}{2} \quad \forall i, \end{cases}$$

$$A \leq a, \qquad B \leq b, \qquad G_i \leq g_i \quad \forall i,$$

*hold, then there exists $C > 0$ such that*

$$\|S_n f\|_{L^p(u^p\,d\mu)} \leq C \|f\|_{L^p(v^p\,d\mu)} \quad \forall f \in L^p(v^p\,d\mu), \quad \forall n \in \mathbb{N}.$$

*Proof.* By induction on the number of negative exponents $\gamma_i$. If $\gamma_i \geq 0\ \forall i$, the result is true, as we saw before (theorem 1).

Suppose there exist $k$ negative exponents $\gamma_i$, with $k > 0$, and the result is true for $k-1$. Let $c \in (-1,1)$ be a point with a negative exponent $\gamma$. Let us remember the formula (7):

$$K_n(x,y) = (x-c)(y-c)K_{n-1}^c(x,y) + \frac{K_n(x,c)K_n(c,y)}{K_n(c,c)}.$$

We define the operators

$$T_nf(x) = \int_{-1}^1 \frac{K_n(x,c)K_n(c,y)}{K_n(c,c)} f(y)w(y)\,dy,$$
$$R_nf(x) = \int_{-1}^1 (x-c)(y-c)K_{n-1}^c(x,y)f(y)w(y)\,dy.$$

Then, $S_n = T_n + R_n$. We are going to study firstly the operators $T_n$:

$$T_nf(x) = \int K_n(x,c)K_n(c,c)\int_{-1}^1 K_n(c,y)f(y)w(y)\,dy,$$

thus

$$\begin{aligned}
\|T_nf\|_{L^p(u^pw)} &\leq \frac{\int_{-1}^1 |K_n(c,y)|v(y)^{-1}|f(y)|v(y)w(y)\,dy}{K_n(c,c)}\|K_n(x,c)\|_{L^p(u^pw)} \\
&\leq \frac{\|K_n(x,c)\|_{L^p(u^pw)}\|K_n(x,c)v(x)^{-1}\|_{L^q(w)}}{K_n(c,c)}\|fv\|_{L^p(w)} \\
&= \frac{\|K_n(x,c)\|_{L^p(u^pw)}\|K_n(x,c)\|_{L^q(v^{-q}w)}}{K_n(c,c)}\|f\|_{L^p(v^pw)}.
\end{aligned}$$

From corollary 4 it follows now that $\|T_nf\|_{L^p(u^p\,d\mu)} \leq C\|f\|_{L^p(v^p\,d\mu)}\ \forall f \in L^p(v^p\,d\mu)$, $\forall n \in \mathbb{N}$. So, we only have to proof the same bound for the operators $R_n$.

But, if we denote by $S_n^c$ the partial sums of the Fourier series with respect to the measure $(x-c)^2w(x)\,dx$, it turns out that

$$R_nf(x) = (x-c)\int_{-1}^1 (y-c)K_{n-1}^c(x,y)f(y)w(y)\,dy = (x-c)S_{n-1}^c\left(\frac{f(y)}{y-c}, x\right),$$

whence

$$\begin{aligned}
&\|R_nf\|_{L^p(u^pw)} \leq C\|f\|_{L^p(v^pw)} \quad \forall f \in L^p(v^pw) \\
&\iff \left\|(x-c)S_{n-1}^c\left(\tfrac{f(y)}{y-c}, x\right)\right\|_{L^p(u^pw)} \leq C\|f\|_{L^p(v^pw)} \quad \forall f \in L^p(v^pw) \\
&\iff \|(x-c)S_{n-1}^cg(x)\|_{L^p(u^pw)} \leq C\|(x-c)g\|_{L^p(v^pw)} \quad \forall g \in L^p(|x-c|^pv^pw) \\
&\iff \|S_{n-1}^cg\|_{L^p(|x-c|^pu^pw)} \leq C\|g\|_{L^p(|x-c|^pv^pw)} \quad \forall g \in L^p(|x-c|^pv^pw) \\
&\iff \|S_{n-1}^cg\|_{L^p(\tilde{u}^p(x-c)^2w)} \leq C\|g\|_{L^p(\tilde{v}^p(x-c)^2w)} \quad \forall g \in L^p(\tilde{v}^p(x-c)^2w),
\end{aligned}$$

where $\tilde{u}(x) = |x-c|^{1-2/p}u(x)$ and $\tilde{v}(x) = |x-c|^{1-2/p}v(x)$.

Therefore, we must prove the boundedness of the partial sums $S_n^c$ with the weights $(\tilde{u}, \tilde{v})$. But the Fourier series we are considering now corresponds to the Jacobi generalized weight $(x-c)^2 w(x)$, which has only $k-1$ negative exponents $\gamma_i$, since on the point $c$ the exponent is $\gamma + 2 > 1$. By hypothesis, the theorem holds in this case and we only have to see that the conditions in the statement hold for the weights $(x-c)^2 w(x)$, $|x-c|^{1-2/p}u(x)$ and $|x-c|^{1-2/p}v(x)$.

Except for the point $c$, these weights have the same exponents as $w$, $u$ and $v$; thus, those conditions are the same and, therefore, they are satisfied. At the point $c$, the exponents are, respectively,

$$\gamma + 2, \qquad g + 1 - 2/p, \qquad G + 1 - 2/p.$$

So, the inequalities we have to check are the following:

$$\begin{aligned}
(G + 1 - 2/p) + (\gamma + 2 + 1)(1/p - 1/2) &< 1/2, \\
(G + 1 - 2/p) + (\gamma + 2 + 1)(1/p - 1/2) &< (\gamma + 2 + 1)/2, \\
(g + 1 - 2/p) + (\gamma + 2 + 1)(1/p - 1/2) &> -1/2, \\
(g + 1 - 2/p) + (\gamma + 2 + 1)(1/p - 1/2) &> -(\gamma + 2 + 1)/2, \\
G + 1 - 2/p &\le g + 1 - 2/p.
\end{aligned}$$

It is easy to see that all of them are satisfied, from our hypothesis. Consequently, we get

$$\|S_{n-1}^c g\|_{L^p(\tilde{u}^p (x-c)^2 w)} \le C \|g\|_{L^p(\tilde{v}^p (x-c)^2 w)} \quad \forall g \in L^p(\tilde{v}^p (x-c)^2 w), \quad \forall n \in \mathbb{N};$$

thus

$$\|R_n f\|_{L^p(u^p w)} \le C \|f\|_{L^p(v^p w)} \quad \forall f \in L^p(v^p w), \quad \forall n \in \mathbb{N}.$$

And, finally,

$$\|S_n f\|_{L^p(u^p \, d\mu)} \le C \|f\|_{L^p(v^p \, d\mu)} \quad \forall f \in L^p(v^p \, d\mu), \quad \forall n \in \mathbb{N}.$$

That is, the result is true for $k$ negative exponents $\gamma_i$. By induction, it is true in general and the theorem is proved. □

**Note.** It can be shown that the converse is also valid, that is, the boundedness of the partial sums $S_n$ implies the five conditions of the theorem.

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