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Pythagorean triangles with legs less than n

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Abstract

We obtain asymptotic estimates for the number of Pythagorean triples (a,b,c) such that $a<n,\ b<n$. These estimates (considering the triple (a,b,c) different from (b,a,c)) is $(4\pi^{-2}\log(1+\sqrt{2}))n+\mathrm{O}(\sqrt{n})$ in the case of primitive triples, and $(4\pi^{-2}\log(1+\sqrt{2}))n\log n+\mathrm{O}(n)$ in the case of general triples. Furthermore, we derive, by a self-contained elementary argument, a version of the first formula which is weaker only by a log-factor. Also, we tabulate the number of primitive Pythagorean triples with both legs less than n, for selected values of $n \leqslant 1\,000\,000\,000$, showing the excellent precision obtained. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction and main results

A triple of strictly positive integers (a,b,c) is called a Pythagorean triple if it satisfies $a^2+b^2=c^2$. The corresponding triangle with legs a, b and hypotenuse c is a Pythagorean triangle. A Pythagorean triple (or triangle) is called primitive if and only if a, b, c are coprime.

If we take a positive parameter n, we can count the number of Pythagorean triangles (primitive or not) with some property or characteristic bounded by the parameter n; for instance, area, hypotenuse, etc. Thus, we have constructed a function depending on n. In the literature, several asymptotic estimates for this kind of functions related to Pythagorean triples have been studied.

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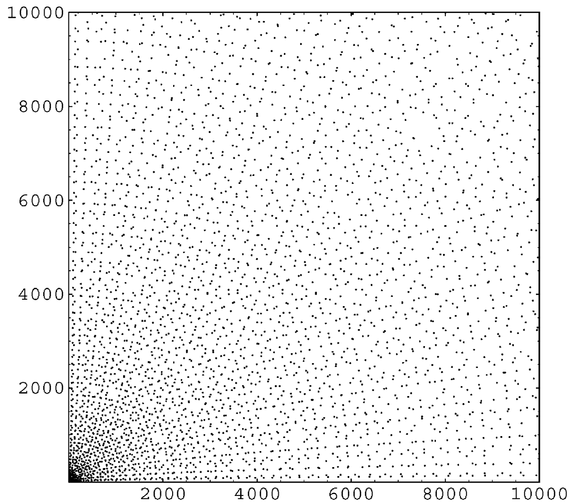

Fig. 1. Legs of primitive Pythagorean triples of square $(0, 10\,000) \times (0, 10\,000)$.

In this way, estimates for the number of primitive Pythagorean triangles with hypotenuse, perimeter and area less than n are studied in [6–8,13]. Asymptotics for the number of general Pythagorean triangles with hypotenuse less than n are shown in [5,10–12]. Finally, in [2], we can find estimates for the number of Pythagorean triangles with a common hypotenuse, leg-sum, etc., less than n.

However, so far as we know, the problem of estimating the number of Pythagorean triangles with both legs less than n has not been studied. This is the aim of this paper, i.e., to find asymptotic estimates for the number of Pythagorean triangles with legs less than n. We study it for both cases: primitive and general triangles.

In Fig. 1, we show, in the ab-plane, the pairs (a,b) for primitive triples (a,b,c) verifying the condition $a < n$, $b < n$ (considering (a,b) different from (b,a)) for $n = 10\,000$. Similarly, Fig. 2 shows the general Pythagorean triples. In this way, we can say that we are counting (and estimating) the number of Pythagorean triangles in the square $(0,n) \times (0,n)$.

Let us denote by $P(n)$ the number of primitive Pythagorean triples (a,b,c) with legs $a < n$, $b < n$; and by $T(n)$ the number of general Pythagorean triples with $a < n$, $b < n$. In both $P(n)$ and $T(n)$, we consider the triple (a,b,c) to be the same than (b,a,c). If we consider (a,b,c) and (b,a,c) to be different triples, we denote the corresponding numbers of primitive and general Pythagorean triples by $\tilde{P}(n)$ and $\tilde{T}(n)$, respectively. It is clear that $\tilde{P}(n) = 2P(n)$ and $\tilde{T}(n) = 2T(n)$.

For primitive Pythagorean triples (a,b,c), the following parametrization due to Diophantus is well known:

$$a = x^2 - y^2, \quad b = 2xy, \quad c = x^2 + y^2$$

with x, y coprime (positive) integers of opposite parity. This formula generates all the primitive Pythagorean triples for odd a and even b. The number of such triples is, clearly, $P(n)$.

So, the problem of finding the number $P(n)$ is reduced to the problem of counting the points (x,y) with integer coordinates, which are coprime and of opposite parity, in the region of the xy-plane

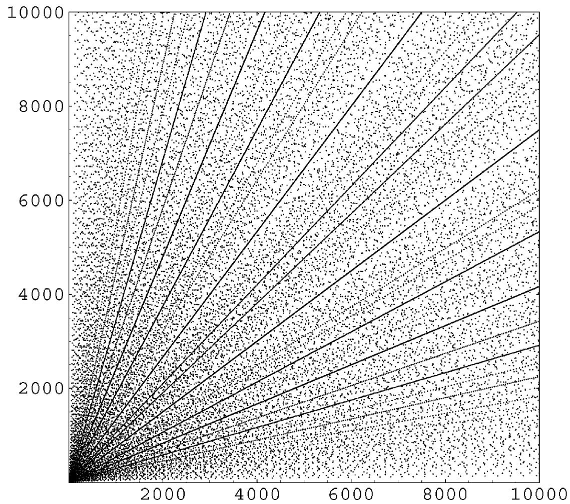

Fig. 2. Legs of Pythagorean triples in the square $(0, 10\,000) \times (0, 10\,000)$.

defined by

$$\begin{aligned} &x^2 - y^2 < n, \\ &2xy < n, \\ &x > y > 0. \end{aligned} \tag{1}$$

Of course, this will be an essential issue in the arguments in this paper.

The main results of this article are:

Theorem 1. *The number $\tilde{P}(n)$ of primitive Pythagorean triples (a, b, c) such that $a < n$ and $b < n$ (considering the triple (a, b, c) different from (b, a, c)) is*

$$\tilde{P}(n) = \frac{4 \log(1 + \sqrt{2})}{\pi^2} n + O(\sqrt{n}).$$

As a consequence,

Corollary 2. *The number $\tilde{T}(n)$ of Pythagorean triples (a, b, c) such that $a < n$ and $b < n$ (considering the triple (a, b, c) different from (b, a, c)) is*

$$\tilde{T}(n) = \frac{4 \log(1 + \sqrt{2})}{\pi^2} n \log n + O(n).$$

We have computed the exact numbers $\tilde{P}(n)$ and $\tilde{T}(n)$ for some values of n, and compared these values with the estimates (rounding to the nearest integer) given by Theorem 1 and Corollary 2. This is shown in Table 1.

Table 1
Exact values of $\tilde{P}(n)$ and $\tilde{T}(n)$ and their estimates

n	$\tilde{P}(n)$	$\frac{4\log(1+\sqrt{2})}{\pi^2}\,n$	$\tilde{T}(n)$	$\frac{4\log(1+\sqrt{2})}{\pi^2}\,n\log n$
10	2	4	4	8
50	16	18	52	70
1 00	36	36	124	165
5 00	180	179	910	1 110
1 000	358	357	2 064	2 468
5 000	1 780	1 786	13 228	15 212
10 000	3 576	3 572	28 942	32 900
50 000	17 856	17 860	173 494	193 245
100 000	35 722	35 721	371 720	411 250
500 000	178 600	178 604	2 145 994	2 343 702
1 000 000	357 200	357 207	4 539 566	4 935 001
5 000 000	1 786 016	1 786 036	25 572 200	27 549 518
10 000 000	3 572 022	3 572 073	53 619 836	57 575 008
50 000 000	17 860 382	17 860 363	296 845 152	316 620 185
100 000 000	35 720 710	35 720 726	618 449 498	658 000 090
200 000 000	71 441 356	71 441 452	1 286 418 190	1 365 519 621
300 000 000	107 162 112	107 162 178	1 973 076 850	2 091 729 956
400 000 000	142 882 968	142 882 904	2 671 874 926	2 830 078 125
500 000 000	178 603 536	178 603 630	3 379 697 288	3 577 451 904
600 000 000	214 324 350	214 324 356	4 094 712 042	4 332 018 235
700 000 000	250 045 106	250 045 082	4 815 708 888	5 092 565 894
800 000 000	285 765 804	285 765 808	5 541 827 134	5 858 234 014
900 000 000	321 486 520	321 486 534	6 272 419 600	6 628 378 925
1 000 000 000	357 207 278	357 207 260	7 006 991 998	7 402 501 013

We can see that the accuracy of the estimate for $\tilde{P}(n)$ using $(4\log(1+\sqrt{2})/\pi^2)\,n$ is excellent. The corresponding values seem to be closer than suggested by the error term $O(\sqrt{n})$ that appears in Theorem 1. Then, we conjecture that the theorem can be improved by finding an asymptotic lower bound for the error.

2. Several lemmas

Given a positive integer n, let us denote by P_n and Q_n the following sets:

$$P_n = \{(x, y) \text{ verifying (1)}:\ \gcd(x, y) = 1,\ \text{opposite parity}\},$$

$$Q_n = \{(x, y) \text{verifying (1)}:\ \gcd(x, y) = 1\}$$

(of course, x and y being positive integer numbers). As we saw in the previous section, $P(n) = \#P_n$; similarly, let us define $Q(n) = \#Q_n$.

In this way, we have

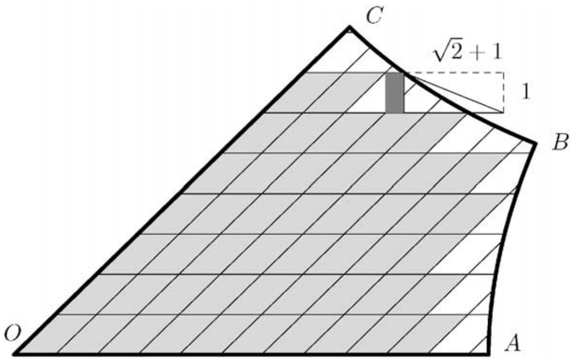

Fig. 3. Region of the xy-plane bounded by $x^2 - y^2 < t^2$, $2xy < t^2$, $x > y > 0$.

Lemma 3. *According to the previous notation*,

$$P(n) = \sum_{k \geqslant 0} (-1)^k Q\left(\frac{n}{2^k}\right).$$

Proof. If $(x, y) \in Q_n$, there are two possibilities: (i) x and y have opposite parity; or (ii) x and y are odd numbers. In case (i), $(x, y) \in P_n$. Let us analyze (ii): for any such (x, y), let (a, b, c) be its corresponding Pythagorean triple according to Diofantus's parametrization.

It is easy to check that, if x and y are coprime odd numbers, then $\gcd(a, b, c) = 2$; conversely, if $\gcd(a, b, c) = 2$, then there exist x and y coprime odd numbers such that $a = x^2 - y^2$, $b = 2xy$, and $c = x^2 + y^2$. Thus, in case (ii), (x, y) generates a Pythagorean triple (a, b, c) with $a < n$, $b < n$, $c < n$ and $\gcd(a, b, c) = 2$. Then, $(a/2, b/2, c/2)$ is a primitive Pythagorean triple with $a/2 < n/2$, $b/2 < n/2$, $c/2 < n/2$; that is, $(x, y) \in P_{n/2}$.

In this way, we have proved $Q(n) = P(n) + P(n/2)$. Now, it is clear that $P(n) = Q(n) - Q(n/2) + Q(n/2^2) - \cdots$ and so on. □

For convenience, let us rewrite (1) taking $n = t^2$, $t \in [1, \infty)$. Then, it becomes

$$\begin{aligned} &x^2 - y^2 < t^2, \\ &2xy < t^2, \\ &x > y > 0. \end{aligned} \tag{2}$$

For $t = 1$, we denote by R the region in the xy-plane bounded by Eqs. (2). For general t, the regions Rt (homothetic to R), are obtained via the similarity of center $(0, 0)$ and ratio t. In Fig. 3, the dark line shows the border of Rt.

Now, let us use $L(Rt)$ to denote the number of points with integer coordinates (x, y) in the region Rt; in Fig. 3, these points correspond to the intersection between the horizontal lines and the sloping ones. Also, let us denote by $L'(Rt)$ the number of points of Rt whose coordinates are coprime. We establish the following:

Lemma 4. *We have the relations*

$$L(Rt) = \sum_{d \geqslant 1} L'\left(R\frac{t}{d}\right) = \sum_{t \geqslant d \geqslant 1} L'\left(R\frac{t}{d}\right) \tag{3}$$

and

$$L'(Rt) = \sum_{d \geqslant 1} \mu(d) L\left(R\frac{t}{d}\right) = \sum_{t \geqslant d \geqslant 1} \mu(d) L\left(R\frac{t}{d}\right), \tag{4}$$

being $\mu(d)$ *the Möbius function.*

Proof. First, note that $L(R\lambda) = L'(R\lambda) = 0$ for $\lambda < 1$, and thus finite and infinite sums are equivalent.

With a small abuse of notation, let us use L both to denote the set and its cardinal; and the same with L'. Let $(x, y) \in L(Rt)$ with $\gcd(x, y) = d$. Then, $x = x_1 d$, $y = y_1 d$ with $\gcd(x_1, y_1) = 1$; that is, $(x_1, y_1) \in L'(Rt/d)$. Therefore, there are as many points in $L(Rt)$ whose gcd is d, as points in $L'(Rt/d)$. So, (3) follows.

To establish (4), we only have to apply the Möbius inversion formula (see, for instance [3, Theorem 268]). □

Finally, a little more notation. We will use $M(Rt)$ to denote the area of the region Rt. Thus, we get the following:

Lemma 5. *According to the previous notation, we have*

$$0 \leqslant M(Rt) - L(Rt) < (\sqrt{2} + \tfrac{1}{2})t + 3. \tag{5}$$

In particular, $M(Rt) - L(Rt) = \mathrm{O}(t)$.

Proof. Let us illustrate the proof with Fig. 3. Here, in the xy-plane, the line OA is the axis $y = 0$, and OC is the line $y = x$. Similarly, the arcs AB and BC represent, respectively, $x^2 - y^2 = t^2$ and $2xy = t^2$. Thus, the border of Rt is the dark line (remember that $M(Rt)$ is, by definition, its inner area).

In the figure, we also show the lines $y = j$ (horizontal lines) and $y = x - i$ (slope lines) for positive integer numbers j and i. With their intersections, we form a lattice; $L(Rt)$ is the number of points in the lattice. With these points, we form (disjoint) parallelograms. We paint in light grey those contained in the region Rt. It is clear that the area of any of these parallelograms is 1.

Fix one of these parallelograms. We claim that it is included in Rt if and only if its right-upper vertex is in Rt (and so, it is one of the points counted by $L(Rt)$). In this way, we conclude $L(Rt) < M(Rt)$.

Let us prove the claim. Of course, we only need to study the parallelogram on the right for any band between $y = j$ and $j + 1$. In what follows, we do not deal with those just surrounding B (they would need a more detailed explanation). For any parallelogram with its right-upper vertex above B, the claim is clear. When the vertex is below B, it is also true, because the slope in the arc AB is always greater than 1 (the slope for $y = x - i$).

Now, let us prove the upper bound for $M(Rt) - L(Rt)$ in (5). We have to estimate the white area in the figure. For any horizontal band between $y = j$ and $j + 1$, let us bound the size of its white part. First, let us study a band above B.

In the figure, for one of such bands, we have split the white area into three parts: the dark rectangle in the middle (with left-upper corner in the lattice and right-upper corner in the arc BC), the triangle on the left (that is half a parallelogram) and the triangle with a curved side on the right (with the arc BC being one of its sides). It is clear that the area of the dark rectangle is less or equal to 1, and that the area of the triangle on the left is always $\frac{1}{2}$. Let us also find a bound for the area of the curved triangle on the right. For that, taking into account the slope in the arc AB, it is easy to check that the curved triangle is always included in a right-angled triangle with legs 1 and $1+\sqrt{2}$, as is also shown in the figure. And the area of this triangle is $(1+\sqrt{2})/2$, a fixed number. Then, the area of the white part for this band is less than

$$1+\frac{1}{2}+\frac{1+\sqrt{2}}{2}=2+\frac{\sqrt{2}}{2}. \tag{6}$$

For a band below B, or for the band that contains point B, we can use easier arguments and find closer bounds. In particular, bound (6) for the area of its white part also holds.

Finally, note that we have, at most, $t/\sqrt{2}+1$ horizontal bands, because the point C is $C=(t/\sqrt{2},t/\sqrt{2})$. Then, (5) follows. (Actually, using a more accurate bound for the white part in the bands below B, a better upper estimate for $M(Rt)-L(Rt)$ can be found; but, again, it is also $O(t)$.) □

By integration, we obtain that the area $M(Rt)$ is

$$M(Rt)=\frac{\log(1+\sqrt{2})}{2}t^2$$

and hence, by Lemma 5, we can write

$$L(Rt)=\frac{\log(1+\sqrt{2})}{2}t^2+O(t). \tag{7}$$

We will also use a more precise estimate for $L(Rt)$. To obtain it, a more careful lattice point counting argument must be used. For instance, Vinogradov's method for estimating sums of fractional parts can be applied to estimate the white part in Fig. 3. It is cumbersome, but can be done adapting the arguments in [4, Sections 6.11 and 6.12 (in particular, Theorem 11.3)]. In this way, we can get

$$L(Rt)=\frac{\log(1+\sqrt{2})}{2}t^2-\frac{2+\sqrt{2}}{4}t+O(t^{2/3}\log t).$$

But it is easier to obtain this kind of results by using the recent paper [9]. Our present lattice point problem essentially coincides with the case of exponent 2 in this cited article. Actually, in it, the region is as ours but with some axial reflections; also, the points on the axis (for instance, on OA, i.e., $x=0$) are included. In this way, making the corresponding adjustments we get the following:

Lemma 6. *The number of points with integer coordinates in the region Rt is*

$$L(Rt)=\frac{\log(1+\sqrt{2})}{2}t^2-\frac{2+\sqrt{2}}{4}t+O(t^{46/73}\log^{315/146}(t)). \tag{8}$$

3. Proofs of the main results

Proof of Theorem 1. By using (4) and (8), we get

$$\begin{aligned}
L'(Rt) &= \sum_{t\geqslant d\geqslant 1} \mu(d) L\left(R\frac{t}{d}\right) \\
&= \sum_{t\geqslant d\geqslant 1} \mu(d)\left(\frac{\log(1+\sqrt{2})}{2}\frac{t^2}{d^2} - \frac{2+\sqrt{2}}{4}\frac{t}{d} + \mathrm{O}\left(\left(\frac{t}{d}\right)^{46/73}\log^{315/146}\left(\frac{t}{d}\right)\right)\right) \\
&= \frac{\log(1+\sqrt{2})}{2}\sum_{t\geqslant d\geqslant 1}\mu(d)\frac{t^2}{d^2} - \frac{2+\sqrt{2}}{4}\sum_{t\geqslant d\geqslant 1}\mu(d)\frac{t}{d} + \sum_{t\geqslant d\geqslant 1}\mathrm{O}\left(\left(\frac{t}{d}\right)^{\alpha}\right)
\end{aligned}$$

for any α such that $46/73 < \alpha < 1$. Let us analyze the three summands in this expression.

For the first one, let us apply the formula

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}, \quad s > 1 \tag{9}$$

(see, for instance, [3, Theorem 287] or [1, Section 11.4]); in our case, for $s=2$, we also have $\zeta(2)=\pi^2/6$. Then,

$$\sum_{t\geqslant d\geqslant 1}\mu(d)\frac{t^2}{d^2} = t^2\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} - t^2\sum_{d=t+1}^{\infty}\frac{\mu(d)}{d^2} = \frac{6}{\pi^2}t^2 + \mathrm{O}(t).$$

For the second summand, let us take into account that $\sum_{t\geqslant d\geqslant 1}\mu(d)(t/d)=\mathrm{O}(t)$ (see [1, Theorem 3.13]). Finally, for the third one, let us use that, for $\alpha > 0$, $\alpha \neq 1$,

$$\sum_{t\geqslant d\geqslant 1}\left(\frac{t}{d}\right)^{\alpha} = \frac{t}{1-\alpha} + \zeta(\alpha)t^{\alpha} + \mathrm{O}(1)$$

(see [1, Theorem 3.2 (b)]). That is, again, $\sum_{t\geqslant d\geqslant 1}\mathrm{O}((t/d)^{\alpha}) = \mathrm{O}(t)$ for any value of α.

With these facts, it is clear that

$$L'(Rt) = \frac{3\log(1+\sqrt{2})}{\pi^2}t^2 + \mathrm{O}(t).$$

Undoing the change $n=t^2$, we get

$$Q(n) = \frac{3\log(1+\sqrt{2})}{\pi^2}n + \mathrm{O}(\sqrt{n}).$$

Then, by Lemma 3,

$$\begin{aligned}
P(n) &= \sum_{k\geqslant 0}(-1)^k Q\left(\frac{n}{2^k}\right) = \frac{3\log(1+\sqrt{2})}{\pi^2}n\sum_{k\geqslant 0}\left(\frac{-1}{2}\right)^k + \mathrm{O}(\sqrt{n}) \\
&= \frac{2\log(1+\sqrt{2})}{\pi^2}n + \mathrm{O}(\sqrt{n})
\end{aligned}$$

and, remembering that $\tilde{P}(n)=2P(n)$, the result follows. □

Remark 7. A weaker version of Theorem 1 can be obtained if we apply (7) instead of (8). Actually, in this way we obtain the following result: The number $\tilde{P}(n)$ of primitive Pythagorean triples (a,b,c) such that $a<n$ and $b<n$ (considering the triple (a,b,c) different from (b,a,c)) is

$$\tilde{P}(n)=\frac{4\log(1+\sqrt{2})}{\pi^2}n+\mathrm{O}(\sqrt{n}\log n). \tag{10}$$

This result is, of course, less precise than the one in Theorem 1, but we include its proof for competeness, to get a more self-contained paper (note that (7) is completely proved in this article). Thus, let us check (10).

By using (4) and (7), we get

$$L'(Rt)=\sum_{t\geqslant d\geqslant 1}\mu(d)L\left(R\frac{t}{d}\right)=\sum_{t\geqslant d\geqslant 1}\mu(d)\left(\frac{\log(1+\sqrt{2})}{2}\frac{t^2}{d^2}+\mathrm{O}\left(\frac{t}{d}\right)\right).$$

The Möbius function μ takes values 0 or ± 1. Moreover, it is easy to check that $\sum_{t\geqslant d\geqslant 1}(t/d)=\mathrm{O}(t\log t)$ and $\sum_{d>t}(t^2/d^2)=\mathrm{O}(t)$ (for instance, estimating with integrals). Finally, we apply (9). With these facts, we get

$$L'(Rt)=\frac{\log(1+\sqrt{2})}{2}t^2\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}+\mathrm{O}(t\log t)=\frac{3\log(1+\sqrt{2})}{\pi^2}t^2+\mathrm{O}(t\log t)$$

and so

$$Q(n)=\frac{3\log(1+\sqrt{2})}{\pi^2}n+\mathrm{O}(\sqrt{n}\log n).$$

Then, by Lemma 3,

$$P(n)=\sum_{k\geqslant 0}(-1)^kQ\left(\frac{n}{2^k}\right)=\frac{3\log(1+\sqrt{2})}{\pi^2}n\sum_{k\geqslant 0}\left(\frac{-1}{2}\right)^k+\mathrm{O}(\sqrt{n}\log n)$$

$$=\frac{2\log(1+\sqrt{2})}{\pi^2}n+\mathrm{O}(\sqrt{n},\log n)$$

and, since $\tilde{P}(n)=2P(n)$, result (10) follows.

Finally, let us see the proof of Corollary 2. We obtain it as a consequence of (10). If, instead, we use the more powerful result that appears in Theorem 1, we get the same error term.

Proof of Corollary 2. Let (a,b,c) be a Pythagorean triple with $a<n$, $b<n$ and $\gcd(a,b,c)=d$. Then, we have $a=a_1d$, $b=b_1d$, $c=c_1d$, and (a_1,b_1,c_1) is a primitive Pythagorean triple with $a_1<n/d$, $b_1<n/d$. In this way, we get the relation $\tilde{T}(n)=\sum_{d=1}^{n}\tilde{P}(n/d)$. Thus, we can apply (10) and so

$$\tilde{T}(n)=\sum_{d=1}^{n}\tilde{P}\left(\frac{n}{d}\right)=\sum_{d=1}^{n}\left(\frac{4\log(1+\sqrt{2})}{\pi^2}\left(\frac{n}{d}\right)+\mathrm{O}\left(\left(\frac{n}{d}\right)^{1/2}\log\frac{n}{d}\right)\right)$$

$$=\frac{4\log(1+\sqrt{2})}{\pi^2}n\sum_{d=1}^{n}\frac{1}{d}+\sum_{d=1}^{n}\mathrm{O}\left(\left(\frac{n}{d}\right)^{1/2}\log\frac{n}{d}\right).$$

Now, we will use the well-known estimate $\sum_{d=1}^{n} 1/d = \log n + \gamma + O(1/n)$ and, also, $\sum_{d=1}^{n} (n/d)^{1/2} \log(n/d) = O(n)$ (this can be checked by estimating with an integral). Then,

$$\tilde{T}(n) = \frac{4\log(1+\sqrt{2})}{\pi^2}\, n\log n + O(n). \qquad \square$$

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