Content Dictionaries for Algebraic Topology*

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Abstract. Kenzo is a Symbolic Computation System devoted to Algebraic Topology that works with the main mathematical structures used in this discipline. In this paper, we present OpenMath Content Dictionaries for each mathematical structure in Algebraic Topology the Kenzo system works with. Besides, how using them to interoperate with a particular Theorem Prover and to obtain certified calculations from Kenzo is explained.

1 Introduction

Kenzo [2] is a Common Lisp system devoted to Symbolic Computation in Algebraic Topology. It was developed in 1997 under the direction of F. Sergeraert and has been successful, in the sense that it has been capable of computing homology groups unreachable by any other means.

Up to now, the mathematical structures Kenzo works with have not been represented in OpenMath [1]. In order to communicate Kenzo with other systems, like GAP [3] or ACL2 [6], we have tackled the goal of developing these OpenMath Content Dictionaries, previous works in these directions are [7] and [4].

2 Description of Kenzo Content Dictionaries

The Kenzo system works with the main mathematical structures used in Simplicial Algebraic Topology, [5]. Figure 1 shows the Kenzo class diagram where each class corresponds to the respective mathematical structure. For each one of these mathematical structures, an OpenMath Content Dictionary has been defined (these Content Dictionaries are available at `http://www.unirioja.es/cu/joheras/cdgroup.html`).

Now, we are going to focus on the simplicial sets Content Dictionary; the rest are based on the same ideas.

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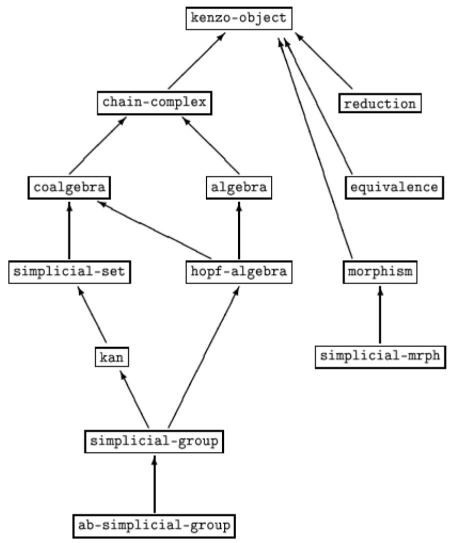

Fig. 1. Class diagram of the Kenzo system.

Definition 1. A *simplicial set* K*,* [5], is a disjoint union $K = \bigcup_{q\geq 0} K^q$*, where the* K^q are sets, together with functions

$$\begin{array}{l}\partial_i^q : K^q \rightarrow K^{q-1},\, q > 0,\, i = 0, \ldots, q, \\ \eta_i^q : K^q \rightarrow K^{q+1},\, q \geq 0,\, i = 0, \ldots, q,\end{array}$$

subject to relations

$$\begin{array}{rll}\partial_i^{q-1}\partial_j^q &= \partial_{j-1}^{q-1}\partial_i^q, & i < j \\ \eta_i^{q+1}\eta_j^q &= \eta_j^{q+1}\eta_{i-1}^q, & i > j \\ \partial_i^{q+1}\eta_j^q &= \eta_{j-1}^{q-1}\partial_i^q, & i < j \\ \partial_i^{q+1}\eta_i^q &= \partial_{i+1}^{q+1}\eta_i^q, & \textit{identity} \\ \partial_i^{q+1}\eta_j^q &= \eta_j^{q-1}\partial_{i-1}^q, & i > j+1\end{array}$$

The functions ∂ and η are called the *face operators* and the *degeneracy operators* respectively.

To define a simplicial set, we must provide the disjoint sets $\{K^q\}_{q\geq 0}$ and the face and degeneracy operators. The sets $\{K^q\}_{q\geq 0}$ can be seen as a graded set so it is possible to consider its characteristic function which, from an element x and a degree g, determines if the element x belongs to the set K^g. To be precise, an *invariant* function can be used in order to encode the characteristic function of the graded set $\{K^q\}_{q\geq 0}$.

Based on the previous way of representation, the following signature has been defined for simplicial sets.

```
inv  : u  nat      -> bool
```

```
face : u   nat   nat -> u
deg  : u   nat   nat -> u
```

where `u` denotes the Universe, of Lisp objects in this case.

The OpenMath signature of Figure 2 corresponds with the previous one:$\coprod$

```
<Signature name="simplicial-set">
    <OMOBJ xmlns="http://www.openmath.org/OpenMath">
        <OMA>
            <OMS name="mapsto" cd="sts"/>
            <OMA id="inv">
                <OMS cd="sts" name="mapsto"/>
                <OMV name="Simplicial-Set-Element"/>
                <OMV name="PositiveInteger"/>
                <OMS cd="setname2" name="boolean"/>
            </OMA>
            <OMA id="face">
                <OMS cd="sts" name="mapsto"/>
                <OMV name="Simplicial-Set-Element"/>
                <OMV name="PositiveInteger"/>
                <OMV name="PositiveInteger"/>
                <OMV name="Simplicial-Set-Element"/>
            </OMA>
            <OMA id="degeneracy">
                <OMS cd="sts" name="mapsto"/>
                <OMV name="Simplicial-Set-Element"/>
                <OMV name="PositiveInteger"/>
                <OMV name="PositiveInteger"/>
                <OMV name="Simplicial-Set-Element"/>
            </OMA>
            <OMV name="Simplicial-Set"/>
        </OMA>
    </OMOBJ>
</Signature>
```

Fig. 2. Signature of simplicial-set.

The formal mathematical properties of the simplicial sets are given in the `<FMP>` tag of the `simplicial-set` definition. Note that, besides the simplicial properties, some invariance properties of face and degeneracy operators and the relations among them must be added. All of them have also been included in natural language by using the `<CMP>` tags. For instance, the face operator invariance has been represented as in Figure 3.

Finally, an example of simplicial set has been included. Namely, the simplicial set with only one element, `nil`, belonging to each set K^q and with each face operation of degree q returning the element of degree $q-1$ has been considered, a piece of this example can be seen in Figure 4.

```
<CMP> The face operator is invariant </CMP>
<FMP>
...
<OMA>
    <OMS cd="logic1" name="implies"/>
    <OMA>
        <OMV name="inv"/>
        <OMV name="x"/>
        <OMV name="q"/>
    </OMA>
    <OMA>
        <OMV name="inv"/>
        <OMA>
            <OMV name="face"/>
            <OMV name="x"/>
            <OMV name="i"/>
            <OMV name="q"/>
        </OMA>
        <OMA>
            <OMS cd="arith1" name="minus"/>
            <OMV name="q"/>
            <OMI>1</OMI>
        </OMA>
    </OMA>
</OMA>
...
</FMP>
```

Fig. 3. A Formal Mathematical Property of Simplicial Sets.

```
<example>
...
    <OMBIND>
        <OMS name="face"/>
        <OMBVAR>
            <OMV name="x"/>
            <OMV name="i"/>
            <OMV name="q"/>
        </OMBVAR>
        <OMS cd="list" name="nil"/>
    </OMBIND>
...
</example>
```

Fig. 4. Fragment of the example for Simplicial Sets.

More complicated examples than the previous one can be included without difficulties. The reason to provide such a simple example, like the previous one, is that we have used our OpenMath Content Dictionaries to integrate Kenzo with a

Theorem Prover, namely with ACL2 [6] in order to obtain certified computations from Kenzo.

The ACL2 tool used for dealing with axiomatic structures is that of *encapsulate*. An encapsulate has a list of function signatures and some properties of the encapsulated functions. In addition, ACL2 demands giving a *witness* for the set of functions, that is, an instance satisfying the properties. This ensures that the encapsulate has at least one *model* which avoids the appearance of inconsistencies in the ACL2 logic.

Obviously, there exists a relationship between each Content Dictionary and the respective encapsulate for the same mathematical structure. In this line, an interpreter to extract the encapsulate from each Content Dictionary has been developed. Now, a fragment of an encapsulate for Simplicial Sets is shown in Figure 5.

```
(encapsulate

 (((inv * *) => *)
  ((face * * *) => *)
  ((degeneracy * * *) => *)
  )

 (local (defun inv (x q)
          (declare (IGNORE q))
          (equal x nil)))

 (local (defun face (x i q)
          (declare (IGNORE x i q))
          nil))

 (local (defun deg (x i q)
          (declare (IGNORE x i q))
          nil))

 ; ... lines skipped

 (defthm prop5
     (implies (and (inv x q) (< i j))
              (equal (face (deg x j q) i (+ q 1))
                     (deg (face x i q) (- j 1) (- q 1)))))

 )
```

Fig. 5. Encapsulate of Simplicial Sets.

In this way, besides representing some of the main concepts used in Algebraic Topology, our Content Dictionaries allow us to interact with the ACL2 Theorem Prover.

Moreover, some mathematical structures (such as spheres, Moore spaces, loop spaces and so on) are predefined objects in the Kenzo system. These objects have been included in the corresponding Content Dictionary in a descriptive way. For instance, spheres are Simplicial Sets, and its Kenzo representation is given by means of a function with a natural number as argument (constructing the corresponding Simplicial Set). We show an example in Figure 6.

```
<CDDefinition>
        <Name>sphere</Name>
        <Description>
            This symbol is a function with one argument, which should
            be a natural number n between 1 and 14. When applied to
            n it represents the sphere of dimension n.
        </Description>
        ...
        <FMP>
            <OMOBJ xmlns="http://www.openmath.org/OpenMath">
                <OMA>
                    <OMS cd="logic1" name="implies"/>
                    <OMA>
                        <OMS cd="Simplicial-Sets" name="sphere"/>
                        <OMV name="n"/>
                    </OMA>
                    <OMA>
                        <OMS cd="logic1" name="and"/>
                        <OMA>
                            <OMS cd="set1" name="in"/>
                            <OMV name="n"/>
                            <OMS name="N" cd="setname1"/>
                        </OMA>
                        <OMA>
                            <OMS cd="relation1" name="leq"/>
                            <OMI>1</OMI>
                            <OMV name="n"/>
                        </OMA>
                        <OMA>
                            <OMS cd="relation1" name="leq"/>
                            <OMV name="n"/>
                            <OMI>14</OMI>
                        </OMA>
                    </OMA>
                </OMA>
            </OMOBJ>
        </FMP>
</CDDefinition>
```

Fig. 6. Representation of spheres.

By using the encapsulate obtained from the simplicial sets Content Dictionary it is possible to prove that each Kenzo sphere is really a Simplicial Set.

3 Conclusions

In this paper, we have presented some OpenMath Content Dictionaries where the main mathematical structures used in Simplicial Algebraic Topology have been defined. The definitions given in these Content Dictionaries include the axiomatic parts and have been used, for example, to interoperate with deduction systems. In this way, a gate has been opened not only to communicate with other systems which work with the same mathematical structures but also to prove the correctness of some constructions or calculations carried out by the Kenzo system. For instance, when Kenzo builds an object belonging to the simplicial-set class, that it is really a simplicial set can be proved. In this way, some calculations with certificates can be carried out.

References

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