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1 Previous definitions and results

Definition. *A matrix A is said to be in* reduced row echelon form up to the column k *if:*

1. *All rows consisting only of 0's up to the position k appear at the bottom of the matrix.*
2. *In any nonzero row up to the position k, the first nonzero entry is a 1. This entry is called a leading entry.*
3. *For any two consecutive nonzero rows up to the position k, the leading entry of the lower row is to the right of the leading entry of the upper row.*
4. *Any column that contains a leading entry has 0's in all other positions.*

Definition. *A matrix A is said to be in* reduced row echelon form *if it is in* reduced row echelon form up to the last column*. Equivalently, a matrix A is said to be in* reduced row echelon form *if:*

1. *All rows consisting only of 0's appear at the bottom of the matrix.*
2. *In any nonzero row, the first nonzero entry is a 1. This entry is called a leading entry.*
3. *For any two consecutive rows, the leading entry of the lower row is to the right of the leading entry of the upper row.*
4. *Any column that contains a leading entry has 0's in all other positions.*

Theorem. *Let $\mathbb{F}$ be a field. Given any matrix $A \in \mathbb{F}^{n \times m}$, there exists an invertible matrix $P \in \mathbb{F}^{n \times n}$ such that $G = P * A$, where G is in reduced row echelon form (G is the reduced row echelon form of A).*

2 Demonstration

Theorem. *Given an invertible matrix A of dimension n ($A \in \mathbb{F}^{n \times n}$), its reduced row echelon form is the identity matrix.*

Proof. From here on, we abbreviate *reduced row echelon form* as *rref*. Let G be the *rref* of A. G is invertible because it is a product of invertible matrices (there exists an invertible matrix P such that $G = P * A$ holds).

To make easier the proof, we reformulate the result. We will prove that G is an identity matrix up to any position (k, k). In particular, when $k = n$ the whole matrix will be the identity.

We make the proof by induction on k. Suppose that G contains the identity matrix up to the position (k, k) included:

$$
\begin{array}{c}
\\ \\ \\ k \\ k+1 \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} & & k & k+1 & & & \end{array} \\
\left(\begin{array}{ccc|c|ccc}
1 & 0 & 0 & * & * & \dots & * \\
0 & 1 & 0 & * & \dots & \dots & * \\
0 & 0 & 1 & * & \dots & \dots & * \\
\cline{1-3}
* & \dots & \dots & * & \dots & \dots & * \\
\cline{1-4}
* & \dots & \dots & \dots & \dots & \dots & * \\
* & \dots & \dots & \dots & \dots & \dots & *
\end{array}\right)
\end{array}
$$

We want to prove that G contains the identity matrix up to the position $(k + 1, k + 1)$.

We know that G is in *rref*, so particularly it is in *rref* up to the columns k and $k + 1$. We have to demonstrate two facts:

1. $G_{jj} = 1$ where $j \leq k + 1$
2. $G_{ij} = 0$ where $i \neq j$, $i \leq k + 1$ and $j \leq k + 1$.

We start with the first fact and applying cases on j:

- If $j \leq k$ the result is trivially proved by induction hypothesis.
- If $j = k + 1$, then the row k is the greatest nonzero row up to the position k (included). Hence, the rows below it contain zeroes up to the position k (included) due to the first condicion of *rref* up to k (G is in *rref* up to the column k):

$$
\begin{array}{c}
\\ \\ \\ k \\ k+1 \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} & & k & k+1 & & & \end{array} \\
\left(\begin{array}{ccc|c|ccc}
1 & 0 & 0 & * & * & \dots & * \\
0 & 1 & 0 & * & \dots & \dots & * \\
0 & 0 & 1 & * & \dots & \dots & * \\
0 & \dots & 0 & * & \dots & \dots & * \\
0 & \dots & 0 & * & \dots & \dots & * \\
0 & \dots & 0 & * & \dots & \dots & *
\end{array}\right)
\end{array}
$$

  Then, why $G_{k+1,k+1} = 1$? We proceed by *reductio ad absurdum*. We suppose that $G_{k+1,k+1} \neq 1$ and we apply once again cases.

  - If $G_{k+1,k+1} \neq 0$ then we obtain a contradiction using the second condition of *rref* up to $k + 1$ (the row $k + 1$ would have a leading entry different to 1).
  - If $G_{k+1,k+1} = 0$ then we have:

$$
\begin{array}{r}
\\ \\ \\ k \\ k{+}1 \\ \\ \\
\end{array}
\left(\begin{array}{ccc|c|ccc}
1 & 0 & 0 & * & * & \dots & * \\
0 & 1 & 0 & * & \dots & \dots & * \\
0 & 0 & 1 & * & \dots & \dots & * \\
\hline
0 & \dots & 0 & 0 & \dots & \dots & * \\
\hline
* & \dots & \dots & \dots & \dots & \dots & * \\
* & \dots & \dots & \dots & \dots & \dots & *
\end{array}\right)
$$

and thanks to the first condition of *rref* up to $k+1$ we know that:

$$
\begin{array}{r}
\\ \\ k \\ k{+}1 \\ \\ \\
\end{array}
\left(\begin{array}{ccc|c|ccc}
1 & 0 & 0 & * & * & \dots & * \\
0 & 1 & 0 & * & \dots & \dots & * \\
0 & 0 & 1 & * & \dots & \dots & * \\
0 & \dots & 0 & 0 & \dots & \dots & * \\
0 & \dots & 0 & 0 & \dots & \dots & * \\
0 & \dots & 0 & 0 & \dots & \dots & *
\end{array}\right)
$$

But then we could write the column $k+1$ as a linear combination of the previous ones, which is a contradiction because G won't be invertible.

Thus $G_{jj} = 1$. Finally we have to prove the second fact ($G_{ij} = 0$ where $i \neq j$, $i \leq k+1$ and $j \leq k+1$). Applying cases:

- If $i \leq k$ and $j \leq k$, the result is trivially proved by induction hypothesis.
- If $i = k+1$ and $j \leq k$ then the result has been already demonstrated. In the previous fact, we have proved that all rows below the $k-th$ one contain zeroes up to the position k (included).
- If $i \leq k$ and $j = k+1$, then $G_{ij} = 0$ making use of the fourth condition of *rref* up to $k+1$.

Then:

$$
\begin{array}{r}
\\ \\ k \\ k{+}1 \\ \\ \\
\end{array}
\left(\begin{array}{ccc|c|ccc}
1 & 0 & 0 & 0 & * & \dots & * \\
0 & 1 & 0 & 0 & * & \dots & * \\
0 & 0 & 1 & 0 & * & \dots & * \\
0 & 0 & 0 & 1 & * & \dots & * \\
0 & \dots & 0 & 0 & * & \dots & * \\
0 & \dots & 0 & 0 & * & \dots & *
\end{array}\right)
$$

□