From Homological Perturbation to Spectral Sequences: a Case Study[1]

A. Romero

Universidad de La Rioja, Departamento de Matemáticas y Computación,
ana.romero@dmc.unirioja.es

Abstract

In this paper, a program computing spectral sequences is reported. The theoretical algorithm supporting this program is based on effective homology and homological perturbation techniques. We illustrate the fundamental ideas of this algorithm by means of an example related to the famous Serre spectral sequence.

Keywords: Symbolic computation, spectral sequences, Serre spectral sequence, constructive algebraic topology, homological perturbation.

1 Introduction

Homological methods are important in the field of formal integrability of PDE systems. In particular, Spencer cohomology is a tool which can be used to determine the involutivity of a system (see [1] and [2]). Recently, Sergeraert has developed some programs computing the Koszul homology of polynomial ideals[2], a notion closely connected to Spencer cohomology. This program is based on his theory of effective homology and, in particular, uses intensively homological perturbation techniques.

In parallel, the author of the present paper has obtained algorithms computing spectral sequences of filtered chain complexes, even when the

[1] Partially supported by Comunidad Autónoma de La Rioja, project ANGI-2005/19, and by Ministerio de Educación y Ciencia, project MTM2006-06513.

[2] A pdf presentation for several talks about this subject can be found in
`www-fourier.ujf-grenoble.fr/~sergerar/Papers/Koszul.pdf`

original chain complex is not of finite type, on condition that this chain complex is an object *with effective homology*. The aim of this paper is to illustrate, by means of a significant example, the corresponding computer program which has been written down to implement this algorithm, in order to facilitate its further application to reach more knowledge in the case of the Koszul homology and the Spencer cohomology.

The organization of the paper is the following. In Section 2 we introduce an example of application of the Serre spectral sequence, showing the non-constructive nature of this spectral sequence and comparing this method with the effective homology one. Then, in the next section, some necessary definitions and results about spectral sequences and effective homology are explained. Sections 4 and 5 show two important examples of application of the effective homology technique, including as a particular case the example presented before. In Section 6 we explain how the effective homology method can also be used to compute spectral sequences, and we illustrate it again by means of our particular case study. Finally, the paper ends with a section of conclusions and further work.

2 An example of spectral sequence

A Spectral Sequence is a family of "pages" $\{E^r_{p,q}, d^r_{p,q}\}$ of differential bigraded modules, each page being made of the homology groups of the preceding one. If we know the stage r in the spectral sequence (E^r, d^r) we can build the bigraded module at the stage $r+1$, E^{r+1}, but this cannot define the next differential d^{r+1} which therefore must be independently defined too.

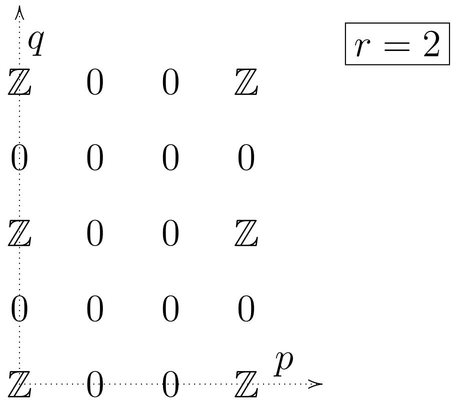
One of the first examples of spectral sequence is due to Serre (using previous work of Leray), involving the fibrations $G \hookrightarrow E \to B$, where G is the *fiber* space, B the *base* space and E the *total* space. The three spaces were initially topological spaces, but this notion of fibration can be generalized to many other situations, in particular the case where B is a simplicial set and F is a simplicial group. The total space E can be considered as a *twisted product* of B and G, and the underlying twisting operator $\tau : B \to G$ explains how the twisted product $E = B \times_\tau G$ is different from the trivial product $B \times G$. The definition of the associated spectral sequence is given by the following theorem, and it can be found in [3].

Theorem. *Let $G \hookrightarrow E \to B$ be a fibration with a base space B simply connected. Then a first quadrant spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}_{r\geq 2}$ is defined with $E^2_{p,q} = H_p(B, H_q(G))$ and $E^r_{p,q} \Rightarrow H_{p+q}(E)$.*

It is frequently thought this spectral sequence is a process allowing one to compute the groups $H_*(E)$ when the groups $H_*(B)$ and $H_*(G)$ are known. But in general this is false, because the differentials $d^r_{p,q}$ are unknown and in many cases we do not have the necessary information to compute them. And even if we know all the differentials $d^r_{p,q}$ and we can reach the limit groups $E^\infty_{p,q}$, we must deal with a (not always solvable) extension problem to determine the homology groups $H_*(E)$. This means that the Serre spectral sequence is not an algorithm that allows us to compute the homology groups of the total space of the fibration, but in fact it is a (rich and interesting) set of relations between the groups $H_*(G)$, $H_*(E)$ and $H_*(B)$. Moreover, we must emphasize here that in many cases this spectral sequence can not be determined. To illustrate this non-constructive nature, we include here one of the initial examples of Serre, considering the beginning of his calculations.

The computation of sphere homotopy groups is known as a difficult problem in algebraic topology. It is not hard to prove that $\pi_n(S^k) = 0$ for $n < k$ and $\pi_k(S^k) = \mathbb{Z}$. Furthermore, in 1937 Freudenthal proved that $\pi_4(S^2) = \mathbb{Z}_2$, and at the beginning of the fifties Serre computed many sphere homotopy groups, being his famous spectral sequence the main tool to obtain these calculations. In particular, Serre proved $\pi_6(S^3)$ has 12 elements, but he was unable to choose between the two possible options $\mathbb{Z}_{12}$ and $\mathbb{Z}_2 + \mathbb{Z}_6$.

For instance, how can we use the Serre spectral sequence to determine the homotopy groups of S^3? It is well-known that $\pi_n(S^3) = 0$ for $n < 3$ and $\pi_3(S^3) = \mathbb{Z}$. To compute $\pi_4(S^3)$, we consider a fibration $F_2 \hookrightarrow X_4 \to S^3$, where $F_2 = K(\mathbb{Z}, 2)$ is an Eilenberg-MacLane space. The beginning of the spectral sequence (that is, the groups $E^2_{p,q}$) is determined by means of the well-known homology groups of S^3 and F_2; the result is shown in the next figure.



We observe that all the arrows $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$ are necessarily null so that the groups $E^3_{p,q}$ are equal to the corresponding $E^2_{p,q}$. But problems arise when trying to determine the differential maps $d^3_{p,q}$. The arrow $d^3_{3,0}$ must be an isomorphism, but to know the arrows $d^3_{3,2q}$ some other (extra) information than which is given by the spectral sequence itself is necessary. In this particular case, a specific tool (the multiplicative structure of the cohomology) gives the solution, the arrow $d^3_{3,2q} : \mathbb{Z} \to \mathbb{Z}$ is the multiplication by $q + 1$. This implies the $E^3_{3,2q}$ die and $E^r_{0,2q} = \mathbb{Z}_q$ for $4 \leq r \leq \infty$ and $q \geq 2$. In this way, the Serre spectral sequence entirely gives the homology groups $H_0(X_4) = \mathbb{Z}$, $H_{2n}(X_4) = \mathbb{Z}_n$ for $n \geq 2$ and the other $H_n(X_4)$ are null. In particular, the Hurewicz theorem and the long exact sequence of homotopy imply that $\pi_4(S^3) = \pi_4(X_4) = H_4(X_4) = \mathbb{Z}_2$, a result known by Freudenthal.

Then, a new fibration $F_3 \hookrightarrow X_5 \to X_4$ is considered to determine $\pi_5(S^3)$, where $F_3 = K(\mathbb{Z}_2, 3)$ is chosen because $\pi_4(X_4) = \mathbb{Z}_2$. In this case Serre was also able to obtain all the necessary ingredients to compute the maps $d^r_{p,q}$ which play an important role in the beginning of the associated spectral sequence. The main tool (extra information) are the multiplicative structure in cohomology and more generally the module structure with respect to the Steenrod algebra $\mathcal{A}_2$. The final groups $E^\infty_{p,q}$ (with $p + q \leq 8$) of this spectral sequence are showed in the following figure.

$r = \infty$

q								
$\mathbb{Z}_2$	0							
$\mathbb{Z}_2$	0	0						
$\mathbb{Z}_2$	0	0	0					
0	0	0	0	0				
0	0	0	0	0	$\mathbb{Z}_2$			
0	0	0	0	0	0	0		
0	0	0	0	0	0	0	0	
$\mathbb{Z}$	0	0	0	0	0	$\mathbb{Z}_3$	0	0 p

Again the Hurewicz theorem and the long homotopy exact sequence imply $\pi_5(S^3) = \pi_5(X_4) = \pi_5(X_5) = H_5(X_5) = \mathbb{Z}_2$; it was the first important result obtained by Serre.

These two examples illustrate the fact that the computation of the Serre spectral sequence is not an easy task and in some situations some other information than which is given by the spectral sequence itself is needed. In other cases, the computation of the Serre spectral sequence is in fact *not possible*, since some differentials $d^r_{p,q}$ can not be determined by any other mean (we do not have the necessary extra information). Therefore, as we have said, the Serre spectral sequence is not an algorithm that allows us to compute $H_*(E)$ if $H_*(B)$ and $H_*(G)$ are known.

On the contrary, the method based on the notion of *object with effective homology* provides real algorithms for the computation of homology groups of many complicated spaces. In particular, this technique can be applied to compute the homology groups of the total space of fibrations when the base and fiber spaces are objects with effective homology, replacing in this way the Serre spectral sequence.

Based on the effective homology method, the Kenzo system [4] was developed. Kenzo is a Common Lisp program devoted to Symbolic Computation in Algebraic Topology that works with rich and complex algebraic structures (chain complexes, differential graded algebras, simplicial sets, simplicial groups, morphisms between these objects, etc). It implements the effective homology method for the computation of homology groups of different spaces. As an example, we show in the following lines how this program can be used to compute the homology groups of the space X_5 introduced in this section.

First of all, the object X_5 must be built, and we can do it by means of the following instructions. We do not explain the Lisp functions that appear here but most of them are self-explanatory.

```
> (setf s3 (sphere 3))
[K1 Simplicial-Set]
> (setf f3 (z-whitehead s3 (chml-clss s3 3)))
[K37 Fibration K1 -> K25]
> (setf x4 (fibration-total f3))
[K43 Simplicial-Set]
> (setf f4 (z2-whitehead x4 (chml-clss x4 4)))
[K292 Fibration K43 -> K278]
> (setf x5 (fibration-total f4))
[K298 Simplicial-Set]
```

The result of the last evaluation is the object K_{298}, which is an instance of the class `Simplicial-Set`, and is located through the symbol `x5`. We can ask for the <u>ef</u>fective <u>h</u>o<u>m</u>ology of X_5:

```
> (efhm x5)
[K608 Homotopy-Equivalence K298 <= K598 => K594]
```

We will see in the following section what a homotopy equivalence is. The homology groups of X_5 are then easily computable using its effective homology, for example, in degrees 5 and 6 the known results $H_5(X_5) = \mathbb{Z}_2$ and $H_6(X_5) = \mathbb{Z}_6$ are obtained.

```
> (homology x5 5)
Homology in dimension 5 :
Component Z/2Z
---done---
> (homology x5 6)
Homology in dimension 6 :
Component Z/6Z
---done---
```

But although the effective homology method allows us to compute the homology groups of the total space of a fibration (in our example, X_5), the structure of the Serre spectral sequence can also give useful informations about the involved construction, sometimes even more interesting than the final homology groups. In fact, both techniques can be combined and it can be seen the effective homology method can also be applied to obtain an algorithm that compute, as a by-product, the relevant spectral sequence (with the whole set of its components).

This algorithm combining both spectral sequence and effective homology methods has been concretely implemented as an extension of the Kenzo program, allowing the user to compute in an easy way spectral sequences associated with filtered complexes, and as a particular case, Serre spectral sequences. For example, as we will see in Section 6, with these programs all the ingredients of the spectral sequence associated with the fibration $F_3 \hookrightarrow X_5 \rightarrow X_4$ introduced in this section (groups, differential maps, convergence...) are easily obtained by means of simple instructions, without needing any extra information.

3 Preliminaries

In this section, some necessary concepts about spectral sequences and effective homology are presented. First of all, we include here some basic definitions and results of Algebraic Topology that can be found, for instance, in [5].

Definition 1. A *chain complex* is a pair (C, d) where $C = \{C_n\}_{n\in\mathbb{Z}}$ is a graded Abelian group and $d = \{d_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ (*the differential map*) is a graded group homomorphism of degree -1 such that $d_{n-1}d_n = 0\ \forall n \in \mathbb{Z}$. The graded *homology group* of the chain complex C is $H(C) = \{H_n(C)\}_{n\in\mathbb{N}}$, with

$$H_n(C) = \text{Ker } d_n/\text{Im } d_{n+1}$$

A *chain complex homomorphism* $f : (A, d_A) \to (B, d_B)$ between two chain complexes (A, d_A) and (B, d_B) is a graded group homomorphism (degree 0) such that $fd_A = d_Bf$.

Definition 2. A *filtration* F of a chain complex (C, d) is a family of sub-chain complexes $F_pC \subset C$ such that

$$\cdots \subset F_{p-1}C_n \subset F_pC_n \subset F_{p+1}C_n \subset \cdots \quad \forall n \in \mathbb{Z}$$

Note 1. A filtration F on C induces a filtration on the graded homology group $H(C)$. Let $i_p : F_pC \hookrightarrow C$ the p-injection, then $F_p(H(C)) = H(i_p)(H(F_p(C)))$.

Definition 3. A filtration F of a chain complex C is said to be *bounded* if for each degree n there are integers $s = s(n) < t = t(n)$ such that $F_sC_n = 0$ and $F_tC_n = C_n$.

Definition 4. A $\mathbb{Z}$*-bigraded module* is a family of $\mathbb{Z}$-modules $E = \{E_{p,q}\}_{p,q\in\mathbb{Z}}$. A *differential* $d : E \to E$ of bidegree $(-r, r-1)$ is a family of homomorphisms of $\mathbb{Z}$-modules $d_{p,q} : E_{p,q} \to E_{p-r,q+r-1}$ for each $p, q \in \mathbb{Z}$, with $d_{p,q} \circ d_{p+r,q-r+1} = 0$. The *homology* of E under this differential is the bigraded module $H(E) \equiv H(E, d) = \{H_{p,q}(E)\}_{p,q\in\mathbb{Z}}$ with $H_{p,q}(E) = \text{Ker } d_{p,q}/ \text{ Im } d_{p+r,q-r+1}$

Definition 5. A *spectral sequence* $E = \{E^r, d^r\}$ is a family of $\mathbb{Z}$-bigraded modules E^1, E^2,..., each provided with a differential $d^r = \{d^r_{p,q}\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$, $r = 1, 2, \ldots$

Definition 6. A spectral sequence $E = \{E^r, d^r\}$ is said to be *convergent* if for every $p, q \in \mathbb{Z}$ there exists $r_{p,q} \in \mathbb{N}$ such that $d^r_{p,q} = 0 = d^r_{p+q,q-r+1}$ for all $r \geq r_{p,q}$.

If $E = \{E^r, d^r\}$ is convergent, then $E^r_{p,q} = E^{r_{p,q}}_{p,q}$ $\forall r \geq r_{p,q}$. We define $E^\infty_{p,q} = E^{r_{p,q}}_{p,q}$, which can be seen as the "limit" of the groups $E^r_{p,q}$ when $r \to \infty$.

Definition 7. A spectral sequence (E^r, d^r) is said to *converge* to a graded module H (denoted by $E^1 \Rightarrow H$) if there is a filtration F of H and for each p isomorphisms $E^\infty_p \cong F_pH/F_{p-1}H$ of graded modules.

Theorem 1. (Theorem 3.1, Chapter XI, in [5]) *Each filtration F of a chain complex (C, d) determines a spectral sequence (E^r, d^r), defined by*

$$E^r_{p,q} = \frac{Z^r_{p,q} \cup F_{p-1}C_{p+q}}{dZ^{r-1}_{p+r-1,q-r+2} \cup F_{p-1}C_{p+q}}$$

where $Z^r_{p,q}$ is the submodule $[a|\ a \in F_pC_{p+q}, d(a) \in F_{p-r}C_{p+q-1}]$, and the differential map $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is the homomorphism induced on these subquotients by the differential on C, $d : C \to C$.
If F is bounded, $E^1 \Rightarrow H(C)$; more explicitly,

$$E^\infty_{p,q} \cong F_p(H_{p+q}C)/F_{p-1}(H_{p+q}C)$$

(with $F_p(HC)$ induced by the filtration F, as explained in Note 1).

On the other hand, we also include here some definitions and fundamental ideas about the effective homology method. More details can be found in [6].

Definition 8. A *reduction* $\rho : D \Rightarrow C$ between two chain complexes is a triple (f, g, h) where

a) the components $f : D \to C$ and $g : C \to D$ are chain complex morphisms;

b) the component $h : D \to D$ is a graded group homomorphism of degree $+1$;

c) the following relations are satisfied
$fg = \mathrm{id}_C$; $gf + d_Dh + hd_D = \mathrm{id}_D$; $fh = 0$; $hg = 0$; $hh = 0$

Remark 1. These relations express that D is the direct sum of C and a contractible (acyclic) complex. This decomposition is simply $D = \text{Ker } f \oplus \text{Im } g$, with $\text{Im } g \cong C$ and $H_*(\text{Ker } f) = 0$. In particular, this implies that the graded homology groups $H_*(D)$ and $H_*(C)$ are canonically isomorphic.

Definition 9. A *(strong chain) equivalence* between the complexes C and E (denoted by $C \Longleftrightarrow E$) is a triple (D, ρ, ρ') where D is a chain complex, and ρ and ρ' are reductions from D over C and E respectively:

$$C \overset{\rho}{\Longleftarrow} D \overset{\rho'}{\Longrightarrow} E$$

Note 2. An effective chain complex is essentially a free chain complex C where each group C_n is finitely generated, and there is an algorithm that returns a $\mathbb{Z}$-base in each degree n (for details, see [6]).

Definition 10. An *object with effective homology* is a triple (X, HC, ε) where HC is an effective chain complex and ε is an equivalence between a free chain complex canonically associated with X and HC.

Note 3. It is important to understand that in general the HC component of an object with effective homology is *not* made of the homology groups of X; this component HC is a free $\mathbb{Z}$-chain complex of finite type, in general with a non-null differential, allowing to *compute* the homology groups of X; the justification is the equivalence ε.

In this way, the notion of object with effective homology makes it possible to compute homology groups of complicated spaces by means of homology groups of effective complexes (which can easily be obtained using some elementary operations).

The next theorem is a very useful tool that will be considered to obtain the effective homology of several spaces. In particular, it is one of the main ingredients for the proof of the effective homology version of the Serre spectral sequence, explained in Section 5. The general idea of this theorem is that given a reduction, if we *perturb* the *big* complex then it is possible to perturb the *small* one so that we obtain a new reduction between the perturbed complexes. A reference where this theorem can be found is [7].

Theorem 2 (Basic Perturbation Lemma, BPL). *Let $\rho = (f, g, h)$ be a reduction $\rho : C \Rightarrow D$ and δ a* perturbation *of d_C, that is, an operator defined on C of degree -1 satisfying the relation $(d_C + \delta) \circ (d_C + \delta) = 0$. Furthermore, the composite function $h \circ \delta$ is assumed* locally nilpotent*, that is, $\forall x \in C$, there exists $n \in \mathbb{N}$ such that $(h \circ \delta)^n x = 0$. Then a new reduction $\rho' : C' \Rightarrow D'$, $\rho' = (f', g', h')$, can be constructed where:*

a) C' is the chain complex obtained from C by replacing the old differential d_C by $(d_C + \delta)$,

b) the new chain complex D' is obtained from the chain complex D by replacing the old differential d_D by $(d_D + \bar{\delta})$, with $\bar{\delta} = f \circ \delta \circ \phi \circ g = f \circ \psi \circ \delta \circ g$,

c) $f' = f \circ \psi = f \circ (Id - \delta \circ \phi \circ h)$,

d) $g' = \phi \circ g$,

e) $h' = \phi \circ h = h \circ \psi$,

where the operators ϕ and ψ are defined by

$$\phi = \sum_{i=0}^{\infty} (-1)^i (h \circ \delta)^i; \quad \psi = \sum_{i=0}^{\infty} (-1)^i (\delta \circ h)^i = Id - \delta \circ \phi \circ h,$$

(the series are convergent thanks to the locally nilpotency of $h \circ \delta$)

In Sections 4 and 5, we present two important examples of application of this theorem.

4 Effective homology of a bicomplex

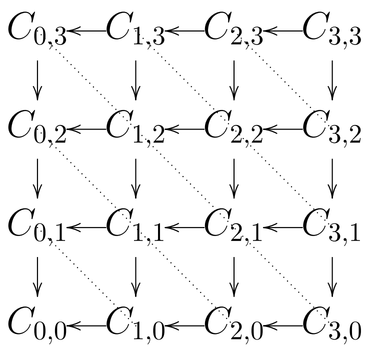
The computation of the effective homology of a bicomplex (double complex) is a very simple example where the BPL can be applied. First of all, let us recall the definition of a bicomplex.

Definition 11. A *bicomplex* (or *double complex*) is a bigraded module $C = \{C_{p,q}\}_{p,q\in\mathbb{Z}}$ provided with morphisms $d'_{p,q} : C_{p,q} \to C_{p-1,q}$ and $d''_{p,q} : C_{p,q} \to C_{p,q-1}$ satisfying $d'd' = 0, d''d'' = 0$ and $d'd'' + d''d' = 0$. Then, we define the *totalization* $(T(C), d)$ of the bicomplex C as the chain complex given by

$$T_n(C) = \bigoplus_{p+q=n} C_{p,q}$$

and differential map $d = d' + d''$.

This notion is easy to understand by means of the following diagram, where the horizontal arrows are the maps $d'_{p,q}$ and the vertical arrows are the differentials $d''_{p,q}$. The totalization is represented by the diagonals.



From now on, we consider C to be a *first quadrant* bicomplex, that is, such that $C_{p,q} = 0$ if $p < 0$ or $q < 0$.

The identity $d''d'' = 0$ implies that for a fixed $i \in \mathbb{N}$ the column $C^i = \{C_{i,n}\}_{n\in\mathbb{N}}$ is a chain complex, so it makes sense to look for the relation between the homologies of the columns C^i and that of the totalization $T(C)$. Let us suppose that the columns C^i are objects with effective homology, in particular such that there exist reductions $C^i \Rightarrow HC^i$ with HC^i an effective complex for all $i \in \mathbb{N}$. Then we are going to construct a new effective complex HC which provides us the effective homology of the totalization $T(C)$.

As a first step, we build a chain complex $(T(C), d')$ totalization of the bicomplex C, but where only the vertical arrows are considered. Using the reductions of each C^i over HC^i, it is easy to construct a reduction of

$(T(C), d')$ over a new chain complex $(T(HC), \bar{d}')$ which is the totalization of a new double complex with columns HC^i (and where all the horizontal arrows are null). It is clear that in each degree the component $T_i(HC)$ is a sum of finite type groups, so that the chain complex $T(HC)$ is an effective complex.

The reduction $(T(C), d') \Rightarrow (T(HC), \bar{d}')$ is the first ingredient for the application of the BPL. Then, we also need a perturbation of the differential d', which is defined by the horizontal arrows, $\delta = d''$. It is not difficult to see that the composition $h \circ \delta$ is locally nilpotent, so that the conditions of the BPL are satisfied. In this way, we deduce a reduction from the complex $(T(C), d)$ (the initial one, where now all the arrows are considered) over a finite type complex, obtaining the looked-for effective homology of $(T(C), d)$.

A natural generalization of double complexes are multicomplexes, where, in addition to horizontal and vertical arrows, morphisms $d^r_{p,q} : C_{p,q} \rightarrow C_{p-r,q+r-1}$ are considered for each $r \in \mathbb{N}$. The totalization is obtained in the same way, with differential map defined as the sum of all the components, $d = \sum d^r$.

Again, if for each column C^i there exists a reduction $C^i \Rightarrow HC^i$ where the HC^i are effective complexes, then using the BPL as before it is possible to construct a new effective complex HC and a reduction $T(C) \Rightarrow HC$ that provides us the effective homology of the multicomplex C.

5 Effective homology of a fibration

Given a fibration

$$G \hookrightarrow E \rightarrow B$$

with fiber G and base B, where G and B are objects with effective homology, in this section we explain how to determine the effective homology of the total space $E = B \times_\tau G$.

From now on, all the chain complexes canonically associated with simplicial sets are *normalized*, that is, only the non-degenerate n-simplices of

X are considered to be generators of $C_n(X)$.

Let us suppose there exist two homotopy equivalences

$$C(G) \Leftarrow\!\!\Leftarrow DG \Rightarrow\!\!\Rightarrow HG \qquad C(B) \Leftarrow\!\!\Leftarrow DB \Rightarrow\!\!\Rightarrow HB$$

where HG and HB are effective complexes. How can we obtain a new equivalence between $C(B \times_\tau G)$ and an effective chain complex?

The starting point is the Eilenberg-Zilber reduction $C(B \times G) \Rightarrow C(B) \otimes C(G)$ (see [8]), that relates the cartesian product of two simplicial sets with the tensorial product of the associated chain complexes. In our case, we must also take account of the torsion τ, that does not change the underlying graded group, only the differential is modified (by a perturbation $\delta(b, g) = (\partial_0 b, \partial_0 g \cdot \tau(b)) - (\partial_0 b, \partial_0 g)$). We could try to apply the BPL; for this, the nilpotency condition must be satisfied.

In both chain complexes $C(B \times G)$ and $C(B) \otimes C(G)$ it is possible to define the following filtrations. First of all, $C(B \times G)$ is filtered through the degeneracy degree with respect to the base space: a generator $(x_n, y_n) \in C_n(B \times G)$ has a filtration degree less or equal to q if $\exists \bar{x}_q \in B_q$ such that $x_n = \eta_{i_{n-q}} \cdots \eta_{i_1} \bar{x}_q$. On the other hand, the filtration on $C(B) \otimes C(G)$ is defined through the dimension of the base component,

$$F_p(C(B) \otimes C(G)) = \bigoplus_{m \leq p} C(B)_m \otimes C(F)$$

It is not difficult to see that the three operators involved in the Eilenberg-Zilber reduction (that is, the three components f, g and h) are compatible with these filtrations. On the contrary, the perturbation δ decreases the filtration degree on $C(B \times G)$ by one unit, so that the composition $h \circ \delta$ is locally nilpotent and the hypothesis of the BPL are satisfied. In this way, a new reduction $C(B \times_\tau G) \Rightarrow C(B) \otimes_t C(G)$ is obtained, where the symbol $\otimes_t$ represents a twisted (perturbed) tensor product, induced by τ.

On the other hand, with the effective homologies of B and G, it is easy to build a new equivalence

$$
\begin{array}{ccccc}
 & & DB \otimes DG & & \\
 & \swarrow\!\!\swarrow & & \searrow\!\!\searrow & \\
C(B) \otimes C(G) & & & & HB \otimes HG
\end{array}
$$

Let us consider now the necessary perturbation $\bar{\delta}$ of $C(B) \otimes C(G)$ to obtain the twisted cartesian product $C(B) \otimes_t C(G)$ (this perturbation $\bar{\delta}$ has been obtained when applying the BPL to the Eilenberg-Zilber reduction). If the base space B is 1-reduced then it can be seen that $\bar{\delta}$ decreases the filtration degree at least by 2. This perturbation can be transferred to the top chain complex $DB \otimes DG$, obtaining a twisted product $DB \otimes_t DG$, modified by a perturbation on $DB \otimes DG$ with the same property about the filtration degree. Finally, the homotopy operator of the reduction $DB \otimes DG \Rightarrow HB \otimes HG$ increases the filtration degree at most by one, and therefore the Basic Perturbation Lemma can be applied to the right reduction and an equivalence is obtained as follows.

$$
\begin{array}{ccccc}
 & & DB \otimes_t DG & & \\
 & \swarrow\!\!\swarrow & & \searrow\!\!\searrow & \\
C(B) \otimes_t C(G) & & & & HB \otimes_t HG
\end{array}
$$

The chain complex $HB \otimes_t HG$ is an effective complex, so that the composition of the two equivalences

$$
\begin{array}{ccccccccc}
 & & C(B \times_\tau G) & & & & DB \otimes_t DG & & \\
 & \overset{Id}{\swarrow\!\!\swarrow} & & \searrow\!\!\searrow & & \swarrow\!\!\swarrow & & \searrow\!\!\searrow & \\
C(B \times_\tau G) & & & & C(B) \otimes_t C(G) & & & & HB \otimes_t HG
\end{array}
$$

is the effective homology of $B \times_\tau G$.

We consider now our particular example X_5 introduced in Section 2, which is the total space of the fibration $F_3 \hookrightarrow X_5 \rightarrow X_4$, where $F_3 = K(\mathbb{Z}_2, 3)$. The object X_4 is again the total space of a fibration $F_2 \hookrightarrow X_4 \rightarrow S^3$ with $F_2 = K(\mathbb{Z}, 2)$.

To compute the effective homology of X_5, we need the effective homologies of the fiber and base spaces, F_3 and X_4. First, the simplicial group $F_3 = K(\mathbb{Z}_2, 3)$ is of finite type and therefore its effective homology is trivial. To compute the effective homology of X_4, which is the total space of another fibration, we must apply again the same method, so that the effective homologies of F_2 and S^3 are necessary. On one hand, the simplicial set S^3 is already of finite type and therefore its effective homology presents no problem. And finally, the difficult part is the computation of the effective homology of $F_2 = K(\mathbb{Z}, 2)$.

The computation of the effective homology of Eilenberg-MacLane spaces $K(\pi, n)$'s is in general a difficult problem (especially in what regards to the algorithmic complexity), but in the case $\pi = \mathbb{Z}$ it can be solved as follows. First, the space $K(\mathbb{Z}, 0)$ is considered to be the simplicial group with all the components equal to $\mathbb{Z}$ and with the identity map as faces and degeneracies. Then, the simplicial group $K(\mathbb{Z}, n)$ can be recursively defined as the classifying space of $K(\mathbb{Z}, n-1)$, that is, $K(\mathbb{Z}, n) = \mathcal{W}(K(\mathbb{Z}, n-1))$. In our case, $F_2 = K(\mathbb{Z}, 2) = \mathcal{W}(K(\mathbb{Z}, 1)) = \mathcal{W}(\mathcal{W}(K(\mathbb{Z}, 0)))$.

It is well known $K(\mathbb{Z}, 1)$ has the homotopy type of the circle S^1. Moreover, although we are not going to give the details about this construction, it can be seen that there exist a mechanism (similar to that of the fibration) for the computation of the effective homology of the classifying space of a simplicial group with effective homology. Therefore we can apply this method to compute the effective homology of F_2.

In this way, we have the necessary ingredients to obtain the effective homology of X_4, and recursively, that of X_5. With this effective homology we can easily compute, as we have seen in Section 2, the homology groups of this space.

As we have showed in this section, the effective homology method applied to a fibration $G \hookrightarrow E \to B$ (with base space B 1-reduced) gives in particular to its user an algorithm to compute the homology groups of the total space E, replacing in this way the Serre spectral sequence technique. But anyway, even if the homology groups of E are known, this spectral sequence has a great interest by itself and therefore it also would be interesting to compute the whole set of its elements. As we see in the next section, the effective homology method can also be useful for this task.

6 Computing spectral sequences

The next theorem combines both spectral sequence and effective homology concepts and is the main result that allows us to use the effective homology method to compute spectral sequences of filtered complexes. The proof is straightforward and is not included here.

Theorem 3. *Let C be a filtered chain complex with effective homology (HC, ε), with $\varepsilon = (D, \rho, \rho')$, $\rho = (f, g, h)$, and $\rho' = (f', g', h')$. Let us suppose that filtrations are also defined on the chain complexes HC and D. If the maps f, f', g, and g' are morphisms of filtered complexes (i.e., they are compatible with the filtrations) and both homotopies h and h' have order $\leq t$ (i.e. they increase the filtration degree at most by t), then the spectral sequences of the complexes C and HC are isomorphic for $r > t$:*

$$E(C)^r_{p,q} \cong E(HC)^r_{p,q} \quad \forall r > t$$

This theorem provides us an algorithm to compute spectral sequences of (complicated) filtered complexes with effective homology. If a filtered complex is effective, then its spectral sequence (that of Theorem 1) can be computed by means of elementary operations with matrices (in a similar way to the computation of homology groups); otherwise, the effective homology is needed to compute the $E^r_{p,q}$ by means of an analogous spectral sequence deduced of an appropriate filtration on the associated effective complex, which is isomorphic to the spectral sequence of the initial complex after some level r. In particular, we can apply this result to compute the Serre spectral sequence, as we explain in the following paragraph.

The Serre spectral sequence associated with a fibration $G \hookrightarrow E \to B$ can be defined as the spectral sequence of the total space E, with the natural filtration of cartesian products. The space E is not effective in most situations, so in general it is not possible to compute directly its spectral sequence. However, as we have seen in Section 5, provided that the spaces B and G are spaces with effective homology (and B is 1-reduced) we can also build the effective homology of the total space E, which allows us to determine the homology groups of E. Moreover, the natural filtration of tensor products can be defined on the effective complex and we have already seen that all the homotopies involved in the equivalence have order ≤ 1. Applying Theorem 3, the spectral sequence of E and that of the effective complex are isomorphic after level $r = 2$, and in this way we can compute the Serre spectral sequence associated with the fibration by means of the spectral sequence of an effective complex (which can easily be computed).

Using these results, we have developed a set of programs enhancing the Kenzo system that allow computations of spectral sequences of filtered complexes when the effective homology of this complex is available. The programs determine not only the groups, but also the differential maps d^r in the spectral sequence, as well as the stage r on which the convergence has been reached and the filtration of the homology groups by the spectral sequence. As a particular case, the computation of Serre spectral sequences where the base and the fiber spaces are objects with effective homology is possible.

We consider again the example X_5 introduced in Section 2, total space of the fibration $F_3 \hookrightarrow X_5 \to X_4$. As we have seen, the effective homology of X_5 can be determined by means of the effective homology of F_3 and X_4, and in fact the Kenzo program implements this computation and uses it to determine the homology groups $H_*(X_5)$. This effective homology is also necessary to compute the corresponding Serre spectral sequence with our new programs.

First of all, the space X_5 and its effective equivalent object must be filtered with the natural filtrations of cartesian products and tensor products respectively, as follows.

```
>(change-chcm-to-fltrchcm x5 fbrt-flin ‘(fbrt-flin))
[K298 Filtered-Simplicial-Set]
>(change-chcm-to-fltrchcm (rbcc (efhm x5)) tnpr-flin
‘(tnpr-flin))
[K594 Filtered-Chain-Complex]
```

Then, the whole set of the Serre spectral sequence can easily be obtained. We show here the computation of some groups.

```
>(print-spct-sqn-cmpns x5 2 6 0)
Spectral sequence E^2_{6,0}
Component Z/3Z
>(print-spct-sqn-cmpns x5 4 8 0)
Spectral sequence E^4_{8,0}
Component Z/4Z
>(print-spct-sqn-cmpns x5 4 4 3)
Spectral sequence E^4_{4,3}
Component Z/2Z
```

The differential maps $d^r_{p,q}$ can also be determined for every r. For example, $d^4_{8,0} : E^4_{8,0} = \mathbb{Z}_4 \rightarrow E^4_{4,3} = \mathbb{Z}_2$ sends the generator of $E^4_{8,0} = \mathbb{Z}_4$ to the generator of $E^4_{4,3} = \mathbb{Z}_2$.

```
>(spct-sqn-dffr x5 4 8 0 ’(1))
(1)
```

The convergence level of the spectral sequence for $p + q = 8$ is $r = 9$.

```
>(spct-sqn-cnvg-level x5 8)
9
```

And finally, we can determine the filtration of the homology groups by the spectral sequence. For instance, for $H_6(X_5) \equiv H_6 = \mathbb{Z}_6$, we obtain $F_0H_6 = F_1H_6 = \ldots = F_5H_6 = \mathbb{Z}_2, F_6H_6 = H_6 = \mathbb{Z}_6$.

```
>(homology-fltr x5 6 0)
Filtration F_0 H_6
Component Z/2Z
>(homology-fltr x5 6 5)
Filtration F_5 H_6
Component Z/2Z
>(homology-fltr x5 6 6)
Filtration F_6 H_6
Component Z/6Z
```

7 Conclusions and further work

In this paper, a program computing spectral sequences of filtered complexes has been presented. It is based on the effective homology method and in particular allows its user to compute the Serre spectral sequence associated with a fibration where the base and fiber space are objects with effective homology. For a better understanding of the fundamental ideas on which this program is based, we have considered a particular example of application.

At this point, new goals appear. First of all, one of our next aims is the application of our program to the computation of the Koszul homology and the Spencer cohomology ([2]), enriching in this way the program developed by Sergeraert. On the other hand, we are planning to extend our programs to the case of spectral sequences which are not necessarily associated with a filtered complex. Concretely, we focus on the Bousfield-Kan spectral sequence [9], used to compute homotopy groups of simplicial sets.

References

[1] Guillemin, V.W. and Sternberg, S.: An algebraic model of transitive differential geometry, *Bull. Amer. Math. Soc.*, 70, 16-47, 1964.

[2] Lambe, L. A. and Seiler, W. M.: Differential Equations, Spencer Cohomology, and Computing Resolutions, *Georgian Math. J.*, 9, 723-772, 2002.

[3] Serre, J.P.: Homologie singulière des espaces fibrés, *Ann. of Math*, 54, 425-505, 1951.

[4] Dousson, X., Rubio, J., Sergeraert, F. and Siret, Y.: *The Kenzo program*, Institut Fourier, Grenoble, 1999.
`http://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo/`
[5] Mac Lane, S.: *Homology*, Springer, 1994.
[6] Rubio, J. and Sergeraert, F.: Constructive Algebraic Topology, *Bulletin des Sciences Mathématiques*, 126, 389-412, 2002.
[7] Brown, R.: The twisted Eilenberg–Zilber theorem, *Celebrazioni Arch. Secolo XX Simp. Top.*, 34-37, 1967.
[8] Eilenberg, S. and Zilber, J.A.: On products of complexes, *Am. J. Math.*, 75, 200-204, 1953.
[9] Bousfield, A. K. and Kan, D. M.: *Homotopy limits, completions and localizations, Lecture Notes in Mathematics*, 304, Springer-Verlag, 1972.